A Deterministic Approach to Shortest Path Restoration in Edge Faulty Graphs

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Jena, Germay

Keerti Choudhary, **Rishabh Dhiman**

Department of Computer Science and Engineering
Indian Institute of Technology Delhi

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Overview

1. Preliminaries
2. Known Results
3. Our Results
4. Open Questions

Preliminaries

Fault Tolerant Distance Preserver

Input: Graph $G = (V, E)$, pairs $P \subseteq V \times V$, integer f.
Output: Subgraph H of G, such that for all $F \subseteq E$, $|F| \leq f$

$$\text{dist}(s, t, G \setminus F) = \text{dist}(s, t, H \setminus F), \text{ for all } (s, t) \in P.$$

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Source-Wise Distance Preserver: $P = S \times V$
Subset Distance Preserver: $P = S \times S$

Known Results

Bodwin et al. (2017)

For every directed or undirected unweighted graph $G = (V, E)$, integer $f \geq 1$, one can construct in time $O(fmn)$ an f-fault tolerant $S \times V$ preserver of size $\tilde{O}(f|S|^{1/2^f} n^{2-1/2^f})$.

Known Results

Restoration Lemma, Afek et al. (2002)

For graph $G = (V, E)$, $s, t \in V$ and failing edge e, $s \rightsquigarrow t$ shortest path in $G \setminus e$ can be represented as concatenation of two shortest paths in the original G, $s \rightsquigarrow x$ and $x \rightsquigarrow t$.

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Concatenation of *any* shortest paths. Hard to trivially generate a small canonical path family.

Bodwin and Parter (2021)

There's a randomized algorithm to generate a canonical path family of size 2 for all pair of nodes.

Our Results

Canonical Path Family

There's an $O(mn)$ deterministic algorithm to generate a canonical path family of size 2 for all pair of nodes.

New Structure

Fault Tolerant Distance Preserver

Input: Graph $G = (V, E)$, pairs $P \subseteq V \times V$, integer f.
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$(f, 1)$-preserver

Input: Graph $G = (V, E)$, pairs $P \subseteq V \times V$, integer f.
Output: Subgraph H of G, such that for all $F \subseteq E$, $|F| \leq f$, $(s, t) \in P$ and edge e such that $\text{dist}(s, t, G \setminus F) = \text{dist}(s, t, G \setminus (F \cup e))$, then

$$\text{dist}(s, t, G \setminus (F \cup e)) = \text{dist}(s, t, H \setminus (F \cup e)).$$

Our Results

$(f,1)$-sourcewise preserver computation

For $f \geq 1$, for an undirected, unweighted n-vertex graph $G = (V, E)$ and source $S \subseteq V$, we can compute an $(f,1)$-$S \times V$-preserver for G in $\tilde{O}(fn^{2-1/2^f}|S|^{1/2^f})$.
For $f = 0$, we can compute a $(0,1)$-fault tolerant $S \times V$ preserver with $O(|S|n)$ edges in $O(m+n)$ time.

This matches the best construction of f-FT $S \times V$ preserver given by Bodwin et al (2017).

Motivation

$(0, 1)$ - $\{s\} \times \{t\}$ preserver is the canonical path family of size 2.

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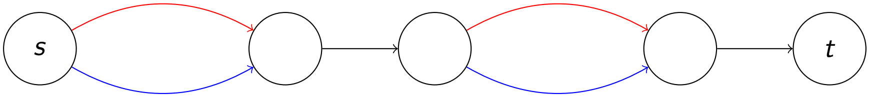
Two paths from s, t which intersect only on distance cut edges.



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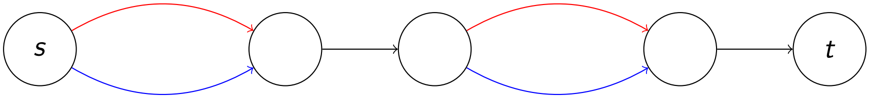

Subset distance preservers

An $(f, 1)$-*sourcewise* distance preserver is an $(f + 1)$-*subset* distance preserver.

Just use the restoration lemma.

$(0, 1)$-preserver Construction

(0, 1)-sourcewise preserver

Given a graph G and source set S, we can compute a $(0, 1)$-sourcewise preserver of size $O(|S|n)$ in $O(|S|m)$ time.

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Two rooted tree subgraphs are said to be independent only if they intersect at the cut edges.

Georgiadis and Tarjan (2012)

Given a directed graph $G = (V, E)$ and a designated source r, a pair of independent trees T_1, T_2 rooted at r are computable in $O(m+n)$ time.

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- Construct shortest path DAG $D_s = (V, E_D \subseteq E)$, such that it only contains edges $\text{dist}(s, y, G) = \text{dist}(s, x, G) + 1$.

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- Construct shortest path DAG $D_s = (V, E_D \subseteq E)$, such that it only contains edges $\text{dist}(s, y, G) = \text{dist}(s, x, G) + 1$.
- Compute T_s^1 and T_s^2 rooted at s using the above lemma.
- $\bigcup_{s \in S}(T_s^1 \cup T_s^2)$ is a $(0,1)$-preserver.

$(f, 1)$-sourcewise preserver Construction

Canonical Path Family

For each $(s, t) \in S \times V$, we can implicitly compute shortest paths $P_{s,t}$ and $Q_{s,t}$ which intersect only on (s, t)-distance cut edges in $O(|S|n)$.

Follows directly from $(0, 1)$-sourcewise preserver construction.

$(f, 1)$-sourcewise preserver Construction

Canonical Path Family

For each $(s, t) \in S \times V$, we can implicitly compute shortest paths $P_{s,t}$ and $Q_{s,t}$ which intersect only on (s, t)-distance cut edges in $O(|S|n)$.

Follows directly from $(0, 1)$-sourcewise preserver construction.
Reuse Bodwin et al's construction of f-FT sourcewise preserver but replace each shortest path with the pair of above shortest paths.

$(f, 1)$-sourcewise Preserver Construction

Algorithm 1: Compute-Incident-Edges(G, S, t, f).

if $f = 0$ **then**
  Return $\{\text{LastE}(P_{s,t}), \text{LastE}(Q_{s,t}) \mid s \in S\}$;
end
$L \leftarrow \sqrt{8f\,|S|\,n \log n}$;
$R \leftarrow$ uniformly random subset of $V \setminus \{t\}$ of size L;
for $s \in S$ **do**
  $(P_{s,t}, Q_{s,t}) \leftarrow$ Pair of (s, t)-shortest-paths;
  $W_s \leftarrow \{u \in V \mid 1 \leqslant \text{dist}(u, t, P_{s,t} \cup Q_{s,t}) \leqslant 8nf \log n / L\}$
end
$G' \leftarrow \left(V, E \setminus \bigcup_{s \in S} E(P_{s,t}) \cup E(Q_{s,t})\right)$;
$S' \leftarrow (\bigcup_{s \in S} W_s) \cup R$;
Return Compute-Incident-Edges$(G', S', t, f - 1)$;

$(f, 1)$-sourcewise Preserver Construction

Algorithm 2: Compute-Incident-Edges(G, S, t, f).

if $f = 0$ **then**
  Return $\{\text{LastE}(P_{s,t}), \text{LastE}(Q_{s,t}) \mid s \in S\}$;
end
$L \leftarrow \sqrt{8f\,|S|\,n \log n}$;
$R \leftarrow$ uniformly random subset of $V \setminus \{t\}$ of size L;
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$S' \leftarrow (\bigcup_{s \in S} W_s) \cup R$;
Return Compute-Incident-Edges$(G', S', t, f-1)$;

To Show: Last edge of a shortest path in $G \setminus (F \cup e)$ lies in the output.

$(f, 1)$-sourcewise Preserver Construction

Algorithm 3: Compute-Incident-Edges(G, S, t, f).

if $f = 0$ **then**
  Return $\{\text{LastE}(P_{s,t}), \text{LastE}(Q_{s,t}) \mid s \in S\}$;
end
$L \leftarrow \sqrt{8f\,|S|\,n \log n}$;
$R \leftarrow$ uniformly random subset of $V \setminus \{t\}$ of size L;
for $s \in S$ **do**
  $(P_{s,t}, Q_{s,t}) \leftarrow$ Pair of (s,t)-shortest-paths;
  $W_s \leftarrow \{u \in V \mid 1 \leqslant \text{dist}(u, t, P_{s,t} \cup Q_{s,t}) \leqslant 8nf \log n / L\}$
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$G' \leftarrow \left(V, E \setminus \bigcup_{s \in S} E(P_{s,t}) \cup E(Q_{s,t})\right)$;
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Return Compute-Incident-Edges$(G', S', t, f-1)$;

To Show: Last edge of a shortest path in $G \setminus (F \cup e)$ lies in the output.

Sketch: Last edge of $P_{s,t} \cup Q_{s,t}$ will be chosen. So assume $F \cap (P_{s,t} \cup Q_{s,t}) \neq \emptyset$.

$(f, 1)$-sourcewise Preserver Construction

Algorithm 4: Compute-Incident-Edges(G, S, t, f).

if $f = 0$ **then**
  Return $\{\text{LastE}(P_{s,t}), \text{LastE}(Q_{s,t}) \mid s \in S\}$;
end
$L \leftarrow \sqrt{8f\,|S|\,n \log n}$;
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for $s \in S$ **do**
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Return Compute-Incident-Edges$(G', S', t, f-1)$;

To Show: Last edge of a shortest path in $G \setminus (F \cup e)$ lies in the output.

Sketch: Last edge of $P_{s,t} \cup Q_{s,t}$ will be chosen. So assume $F \cap (P_{s,t} \cup Q_{s,t}) \neq \emptyset$.
Consider shortest paths $P_{s,t,F}$ an'd $Q_{s,t,F}$ in $G \setminus F$. WLOG $e \notin P_{s,t,F}$.

$(f, 1)$-sourcewise Preserver Construction

Algorithm 5: Compute-Incident-Edges(G, S, t, f).

if $f = 0$ **then**
  Return $\{\text{LastE}(P_{s,t}), \text{LastE}(Q_{s,t}) \mid s \in S\}$;
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$L \leftarrow \sqrt{8f\,|S|\,n \log n}$;
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Consider shortest paths $P_{s,t,F}$ an'd $Q_{s,t,F}$ in $G \setminus F$. WLOG $e \notin P_{s,t,F}$.

- Find last $\overline{x} \in S'$ on $P_{s,t,F}$,
- $\overline{x} \rightsquigarrow t$ shortest path won't internally intersect $\bigcup_{u \in S}(P_{u,t} \cup Q_{u,t})$.
- $\overline{x} \rightsquigarrow t$ lies in the $(f-1, 1)$-preserver of G'.

Other Applications

Distance Labelling Schemes

For $f \geq 0$, and n-vertex unweighted undirected graph, there is an $(f+1)$-fault-tolerant distance labeling scheme that assigns each vertex a label of $O(fn^{2-1/2^f} \log n)$ bits that can be computed in $O(fmn)$ time each, and $O(|S|m)$ each for $f = 0$.

Subset Replacement Path Problem

The input is a graph $G = (V, E)$ and a set of source vertices S, and for every pair of vertices $s, t \in S$ and failing edge $e \in E$, report $\text{dist}(s, t, G \setminus e)$.
Given a graph $G = (V, E)$ and a set of source vertices S, we can solve the problem in $O(|S|m) + \tilde{O}(|S|^2 n)$ in the word-RAM model.

Open Questions

(f, k)-preserver

Input: Graph $G = (V, E)$, pairs $P \subseteq V \times V$, integer f.
Output: Subgraph H of G, such that for all $F \subseteq E$, $|F| \leq f$, $(s, t) \in P$ and $K \subseteq E$ such that $|K| \leq k$ and $\text{dist}(s, t, G \setminus F) = \text{dist}(s, t, G \setminus (F \cup K))$,

$$\text{dist}(s, t, G \setminus (F \cup K)) = \text{dist}(s, t, H \setminus (F \cup K)), \text{ for all } (s, t) \in P.$$

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$$\text{dist}(s, t, G \setminus (F \cup K)) = \text{dist}(s, t, H \setminus (F \cup K)), \text{ for all } (s, t) \in P.$$

- Is it possible to construct (f, k)-preserver more efficiently than $(f + k)$-preservers?
- For a $(0, k)$-preserver for pair of vertices s, t, is it possible to bound $\#\{u \in V \mid \text{dist}(u, t, H) \leq \alpha\}$ for any $\alpha > 0$?
- Can restoration lemma for k-edge failures be used with (f, k)-preservers?

Thank you!