A Dichotomy Theorem for Ordinal Ranks in MSO

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Area

MSO (regular languages) over the full binary tree with labeled nodes.

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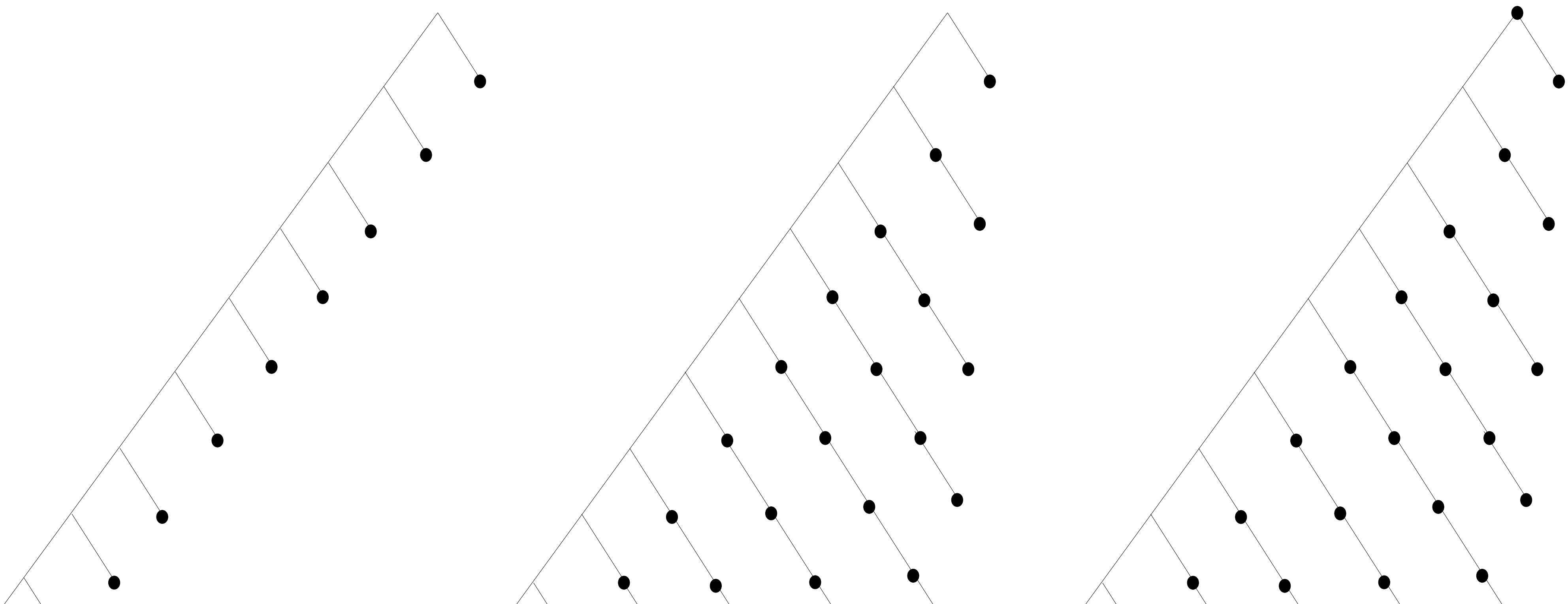
Definition

A set X of nodes is *well-founded* if on every branch there are finitely many nodes from X.



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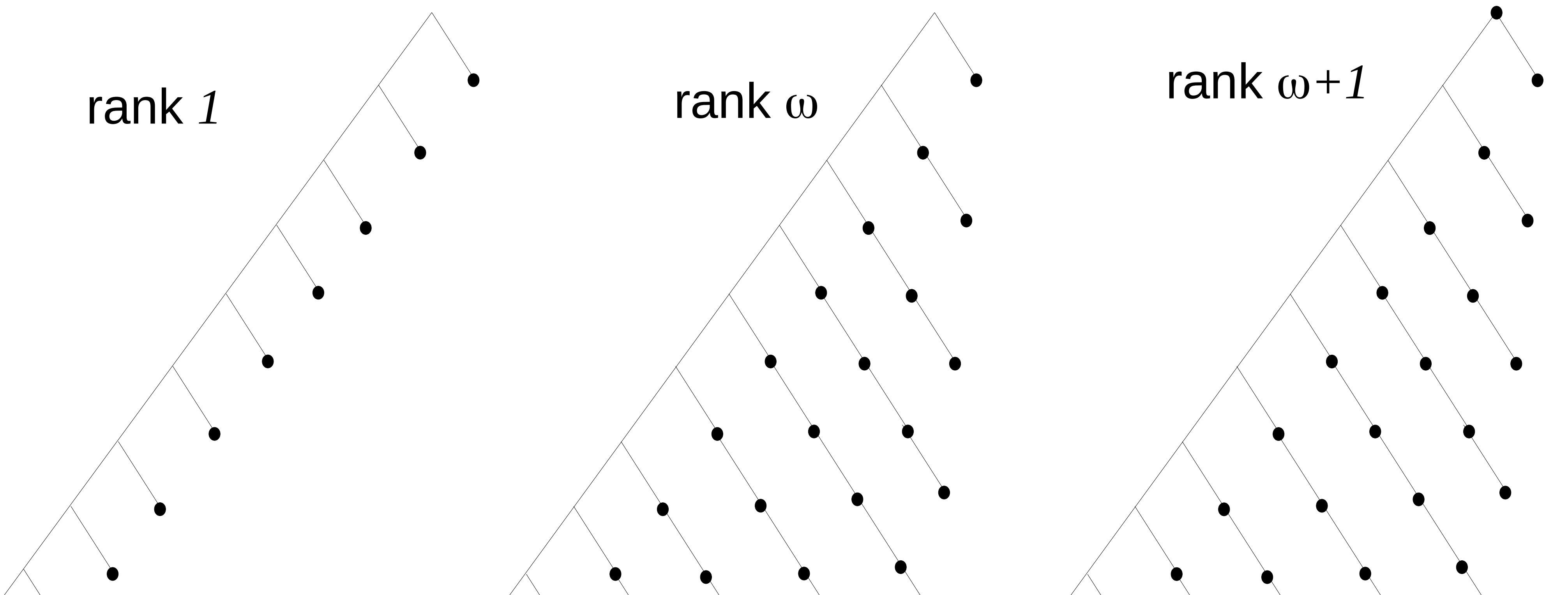
A set X of nodes is *well-founded* if on every branch there are finitely many nodes from X.
Equivalently if there is a function $c:X\rightarrow\eta$, for some ordinal η, such that $c(u)>c(v)$ when u is an ancestor of v.
The minimal such η is the *rank of X*.



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(identify a set X with its characteristic function $t_X \in Tr_{\{0,1\}}$)

The language $WF=\{t_X \in Tr_{\{0,1\}} : X$ is well-founded$\}$ is regular (co-Büchi).

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For a regular language $L \subseteq WF$ consider the ordinals:

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More generally, for a regular relation $\Gamma \subseteq Tr_A \times WF$ consider

$$\eta_\Gamma = \sup_{t \in \pi_A(\Gamma)} \inf_{x \in \Gamma(t)} rank(x)$$

witness for t

a tree having a witness

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Theorem. The ordinal η_Γ is either

- smaller than ω^2, or
- equal to ω_1 (i.e., the first uncountable ordinal).

Moreover, it can be effectively decided which of the cases holds.
In the former case, we can compute a number N such that $\eta_\Gamma < N \cdot \omega$.

Application: mu-calculus

The least fixed point $\mu X.F(X)$ can be computed by iterating *F*.

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Given a model τ, the *closure ordinal* of $\mu X.F(X)$ in τ is the least ordinal η such that $\mu X.F(X)=\mu^{\eta}X.F(X)$.

The *closure ordinal* of $\mu X.F(X)$ is the supremum of these ordinals over all models τ (or ∞ if the supremum does not exist).

Example:

$\mu X.\Diamond X$ → closure ordinal 0

$\mu X.(a \vee \Diamond X)$ → closure ordinal ω

$\mu X.(\Box X)$ → closure ordinal ∞

Application: mu-calculus

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Theorem. Let $F(X)$ be a μ-calculus formula in which the variable X does not occur in scope of any fixed-point operator. Then, the closure ordinal of $\mu X.F(X)$ is either strictly smaller than ω^2, or at least ω_1.
(proved also by Afshari, Barlucchi, Leigh; FICS 2024)

For a regular relation $\Gamma \subseteq Tr_A \times WF$ consider

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Moreover, it can be effectively decided which of the cases holds.
In the former case, we can compute a number N such that $\eta_\Gamma < N \cdot \omega$.

Proof idea: dichotomy game.

A tree $t \in Tr_A$ is *η-challenging* if any witness x in WF, such that $(t,x) \in \Gamma$, has $rank(x) \geq \eta$.

In a dichotomy game

- Eve wins if there is an η-challenging tree for any $\eta < \omega_1$, thus $\eta_\Gamma = \omega_1$;
- Adam wins if any t has a witness x of a uniformly bounded rank.

A bound on the size of Adam's memory yields the desired inequality $rank(x) < N \cdot \omega$.

Proof idea: dichotomy game.

Easier game: Eve wins when for any $\eta < \omega_1$ there is a well-founded tree in L of rank η.

In each round:

Eve proposes a node label $\in \{0,1\}$

Adam chooses a direction

strategy of Eve $\Rightarrow$ a tree t

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In each round:

Eve proposes a node label $\in\{0,1\}$
Eve proposes a transition of A recognizing L

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strategy of Eve $\Rightarrow$ a tree t, a run of A over t

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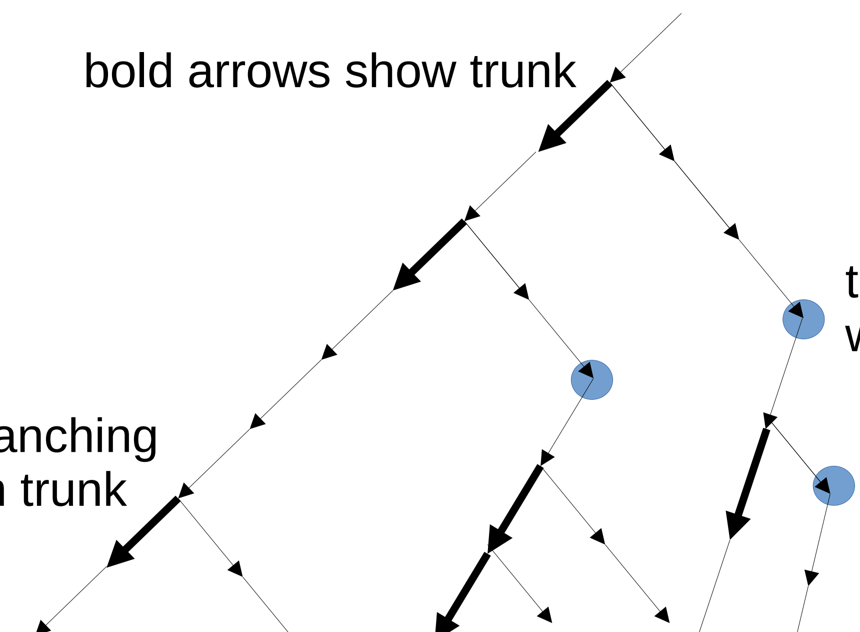
Easier game: Eve wins when for any $\eta<\omega_1$ there is a well-founded tree in L of rank η.

In each round: Adam may say „it's enough"
Eve proposes a node label $\in\{0,1\}$
Eve proposes a transition of A recognizing L
Eve shows „trunk" + „branching points"
Adam chooses a direction

strategy of Eve ⇒ a tree t, a run of A over t, a nested comb structure



Proof idea: dichotomy game.

Goal: Eve wins if there is an η-challenging tree for any $\eta<\omega_1$, thus $\eta_\Gamma=\omega_1$.

Basic ingredient: Eve proposes a node label $\in A$,
Adam chooses a direction

strategy of Eve $\Rightarrow$ a tree

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Basic ingredient: Eve proposes a node label $\in A$,
Adam chooses a direction

strategy of Eve $\Rightarrow$ a tree

additionally: the tree should be η-challenging –
all witnesses x such that $(t,x)\in\Gamma$, should satisfy $rank(x)\geq\eta$.

How Eve can show this?

- Adam provides a witness on the fly? Too convenient for Eve

Proof idea: dichotomy game.

Goal: Eve wins if there is an η-challenging tree for any $\eta<\omega_1$, thus $\eta_\Gamma=\omega_1$.

Position: two sets of states (“trunk” phase, „reach” phase)

In each round:
- Adam may erase some states (i.e., say „it’s enough”)
- Eve proposes a node label $\in\{0,1\}$
- for each state and each transition
  Eve shows „trunk” + „branching points”
- Adam chooses a direction

strategy of Eve ⇒ a tree + a nested comb structure for each run/witness

Can Eve win?
- By definition, there is a nested comb structure for each witness
- In the game: Eve has to play positionally (knowing only the current state, but not knowing the previous and future part of the witness)
- But anyway Eve can win...

Summary

For a regular relation $\Gamma \subseteq Tr_A \times WF$ consider

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Theorem 1. The ordinal η_Γ is either

- smaller than ω^2, or
- equal to ω_1 (i.e., the first uncountable ordinal).

Theorem 2. Let $F(X)$ be a μ-calculus formula in which the variable X does not occur in scope of any fixed-point operator. Then, the closure ordinal of $\mu X.F(X)$ (i.e., the number of iterations needed to reach fixed point) is either strictly smaller than ω^2, or at least ω_1.

In both theorems it can be effectively decided which of the cases holds.