Subshifts Defined by Nondeterministic and Alternating Plane-Walking Automata

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Subshifts over the plane

Let Σ be an alphabet.

- any $x \in \Sigma^{\mathbb{Z}^2}$ is a *configuration*
- $L \subset \Sigma^{\mathbb{Z}^2}$ is a *subshift*
- for $P \subset \mathbb{Z}^2$, $p \in \Sigma^P$ is a *pattern*

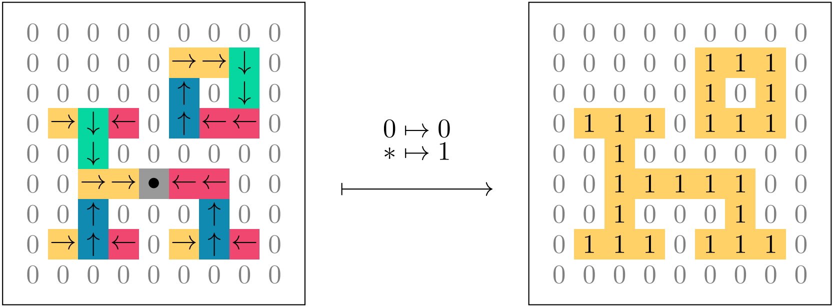
Subshifts are invariant by translation. A Subshifts of Finite Type (SFT) is a subshift which contains all the configurations which avoid a set of forbidden finite patterns $\mathcal{F}$.

$$\mathcal{F} = \{11, \tfrac{1}{1}\}$$

0	0	0	0	0	0	0	0	0	1
0	0	1	0	0	0	0	0	0	0
0	0	0	0	0	1	0	1	0	0
1	0	1	0	0	0	1	0	0	0
0	0	0	0	0	0	0	0	0	0
1	0	1	0	0	1	0	1	0	0
0	0	0	0	1	0	1	0	0	0
0	0	0	0	0	0	0	0	0	1
1	0	0	0	0	0	0	0	0	0
0	0	0	0	1	0	0	0	0	0

Sofic subshifts

A sofic subshift is an SFT combined with a projection map on the symbols.



Allowed (all rotations and symmetries) : $\{0 \rightarrow, \rightarrow\rightarrow, 0 \uparrow, \rightarrow\uparrow, 0\bullet, \rightarrow \bullet, 00\}$

Planewalking automata (Salo and Törmä, 2014)

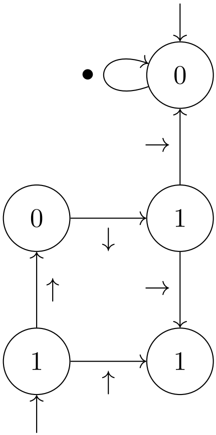
Definition

A PWA is a labelled directed graph $A = (V, E, \Sigma, D, I)$ where

- $E \subset V^2 \times \{\rightarrow, \leftarrow, \uparrow, \downarrow, \bullet\}$,
- $D : V \rightarrow \Sigma$,
- $I \in V$, with D a bijection from I to Σ.

A run rejects from a position p if it is finite. A configuration $x \in \mathbb{Z}^2$ is accepted by A if it accepts from all positions $p \in \mathbb{Z}^2$.

$L \subset \Sigma^{\mathbb{Z}^2}$ recognized by a PWA $\Leftrightarrow L \in \Delta_1$.



Alternating planewalking automata

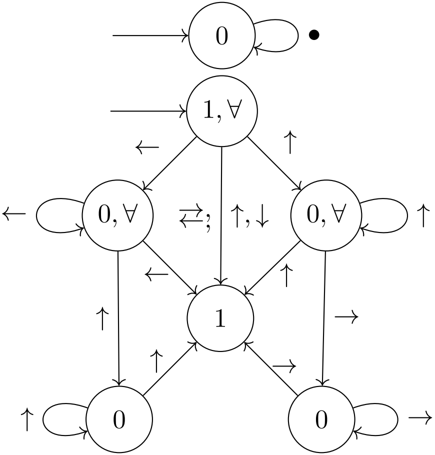
Definition

An alternating PWA is a labelled directed graph $A = (V, E, \Sigma, D, I, Q)$ where

- $E \subset V^2 \times \{\rightarrow, \leftarrow, \uparrow, \downarrow, \bullet\}$,
- $D : V \rightarrow \Sigma$,
- $I \in V$, with D a bijection from I to Σ,
- $Q : V \rightarrow \{\exists, \forall\}$.

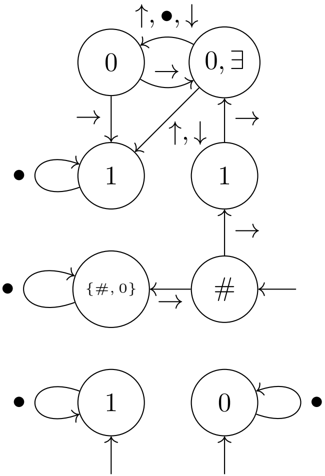
A $\exists$ state accepts iff there exists an accepting successor. A $\forall$ state accepts iff all (> 0) successors are accepting.
Alt is the set of languages associated with alternating PWA.

A ∀ example: the sunny side up



0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0
0	0	0	0	1	0	0	0	0
0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0

A ∃ EXAMPLE: THE CONE LABYRINTH



$$\mathcal{F} = \left\{ \begin{smallmatrix}1\\\#\end{smallmatrix}, \begin{smallmatrix}0\\\#\end{smallmatrix}, \begin{smallmatrix}\#\\0\end{smallmatrix}, \begin{smallmatrix}\#\\1\end{smallmatrix}, 11, 010 \right\}$$

#	#	0	0	0	0	0	#	#
#	#	0	0	0	0	0	#	#
#	#	0	0	0	0	1	#	#
#	#	0	0	0	0	0	#	#
#	#	1	0	0	0	0	#	#
#	#	0	0	0	0	0	#	#
#	#	0	0	0	0	0	#	#
#	#	0	0	0	0	0	#	#
#	#	0	0	0	0	0	#	#

Alternating hierarchy of non-deterministic planewalking automata

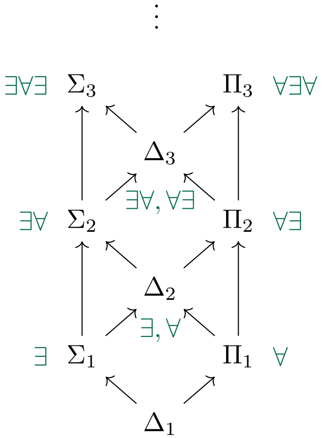
Definition

First,

- $L(A) \in \Delta_1$ iff A is deterministic
- $L(A) \in \Sigma_1$ iff A only has $\exists$ states
- $L(A) \in \Pi_1$ iff A only has $\forall$ states

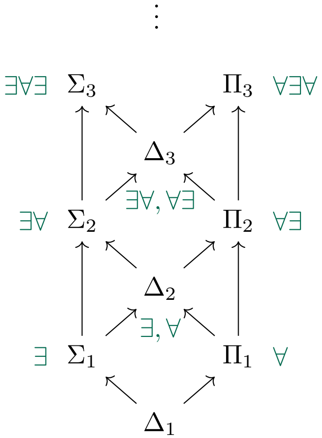
And then,

- $L(A) \in \Delta_k$ iff A can do at most $k-2$ alternations
- $L(A) \in \Sigma_k$ iff A can do at most $k-1$ alternations starting with $\exists$
- $L(A) \in \Pi_k$ iff A can do at most $k-1$ alternations starting with $\forall$

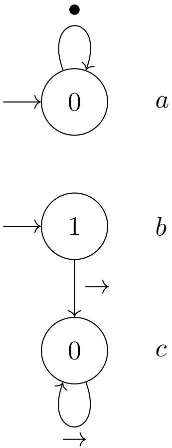


Alternating hierarchy of non-deterministic planewalking automata

- $Alt \subsetneq Sofic$
- $\Pi_1 \not\supseteq \not\subseteq \Sigma_1$



$Alt \subseteq Sofic$



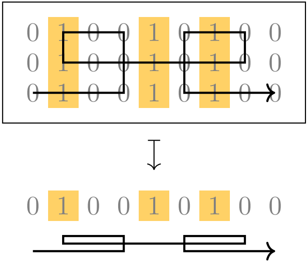
$$\Sigma_S = \Sigma \times 2^V, \text{ where } (s, S) \in \Sigma_S \Leftrightarrow S \cap I \neq \varnothing$$

$\binom{0}{a}$	$\binom{0}{a}$	$\binom{0}{a}$	$\binom{0}{a}$	$\binom{1}{b}$	$\binom{0}{ac}$
$\binom{0}{a}$	$\binom{0}{a}$	$\binom{1}{b}$	$\binom{0}{ac}$	$\binom{0}{ac}$	$\binom{0}{ac}$
$\binom{0}{a}$	$\binom{1}{b}$	$\binom{0}{ac}$	$\binom{0}{ac}$	$\binom{0}{ac}$	$\binom{0}{ac}$

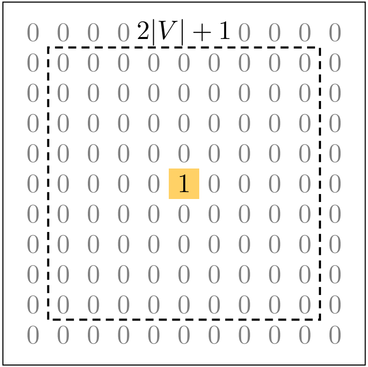
$$\pi : (s, S) \mapsto s$$

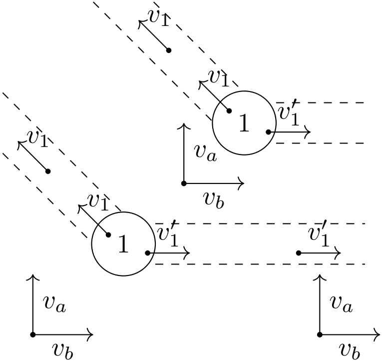
$Alt \neq Sofic$ (argument by Ville Salo)

Let L be a 1D non-sofic language such that the repetition of L in 2D, $\text{lift}(L)$, is sofic.
Assuming $L(A) = \text{lift}(L)$, replacing all $\uparrow, \downarrow$ by $\bullet$ in A gives us a 1D automata recognizing L, a contradiction.

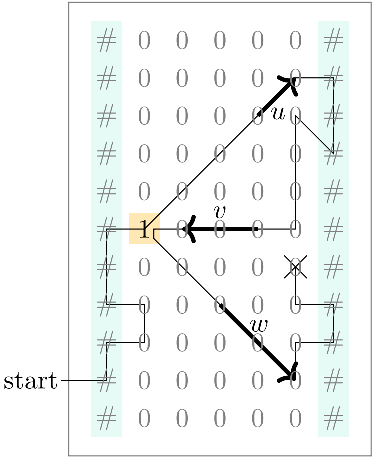


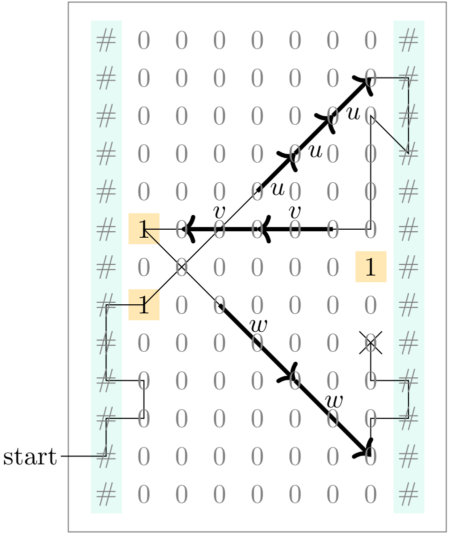
$\Pi_1 \not\subseteq \Sigma_1$





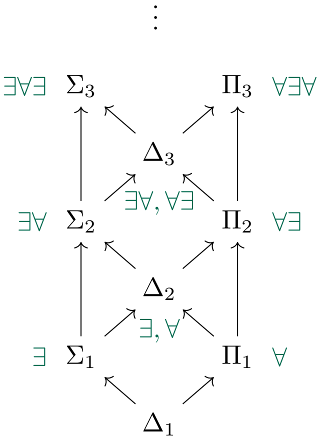
$\Sigma_1 \not\subseteq \Pi_1$





Open questions

- Does $\Sigma_k \not\supseteq \not\subseteq \Pi_k$ hold for all k?
- Is $\Delta_k = \Sigma_k \cap \Pi_k$ true for all k?
- In particular, does $\Delta_1 = \Sigma_1 \cap \Pi_1$?
- $Alt = \bigcup_{k>0} \Delta_k \cup \Sigma_k \cup \Pi_k$?



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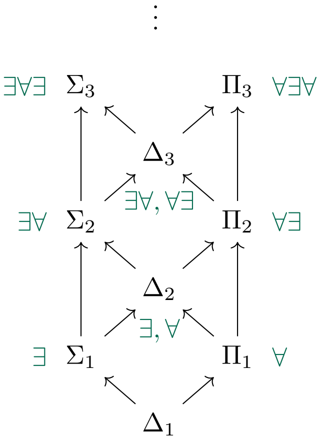

Thank you!