Approximate Minimum Tree Cover in All Symmetric Monotone Norms Simultaneously

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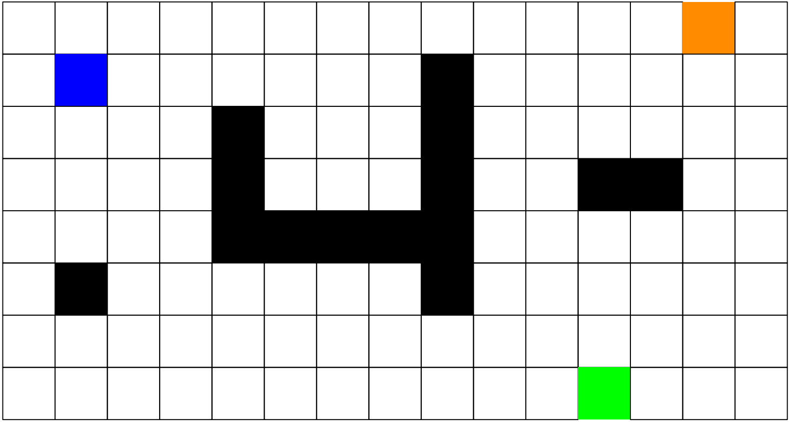
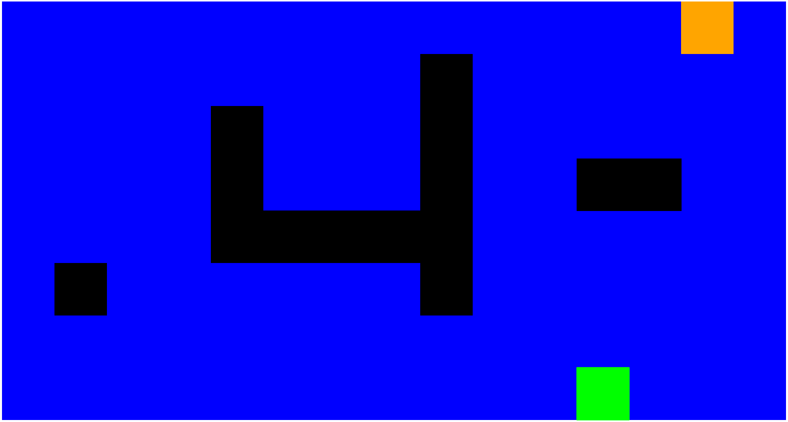
Problem Statement, 1st attempt

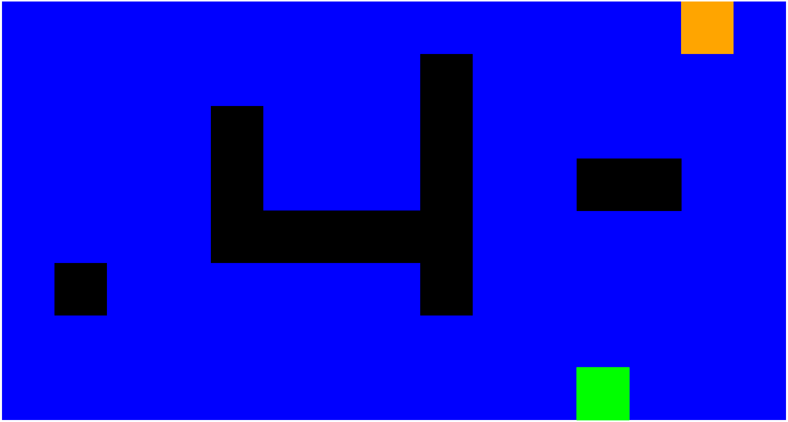
We have some agents and some environment where every location in the environment should be visited by at least one of the agents. The total distance for the agents should be minimal.



Solution, 1st attempt

Just compute a Minimum Spanning Forest where each tree contains one agent.





Perhaps not the fairest solution...

What if we care about fairness?

Minimize the maximum size of a spanning tree in the forest.

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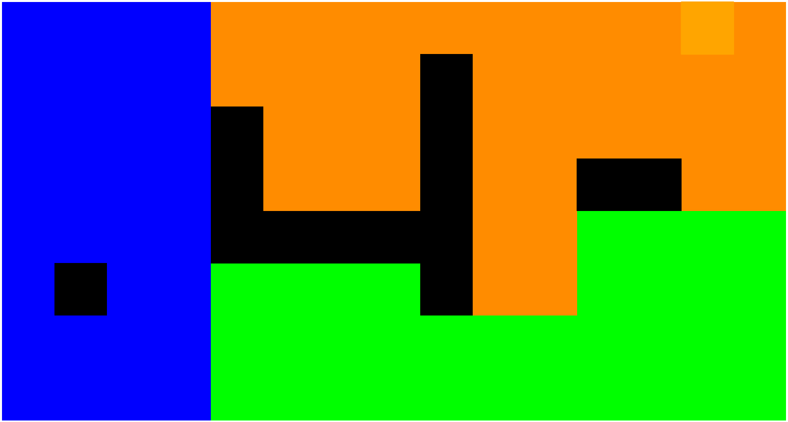
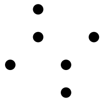
NP-hard, but there exists a 4-approximation in polynomial time.



Much Better

There is such a thing as being too fair



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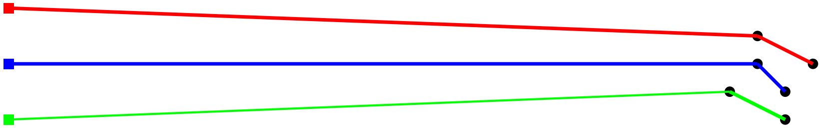

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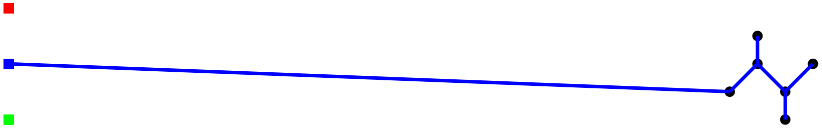

So what do we want?

- The total amount of work should be close to minimal.
- No agents are being treated (too) unfairly.

More Formally

- We are given a metric space (X, d) and agent locations (depots) $A \subseteq X$.

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If we want to minimize both simultaneously, why not try ℓ_2?

Observation

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This still misses some important objective functions, such as top-ℓ-norms.

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A norm $||\cdot||$ is monotone symmetric if

- $x \leq y \implies ||x|| \leq ||y||$.
- $||x|| = ||x'||$ if x' results from x by permuting coordinates.

Observation

We really want to have an algorithm that (approximately) minimizes <u>all</u> monotone symmetric norm <u>simultaneously</u>.

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Problem: It's not even clear such solutions exist.

Our Contribution

Theorem (With Depots)

For any metric space (X, d) *and* $A \subseteq X$ *we can compute in polynomial time a partition* $(X_a)_{a \in A}$ *of* X *such that* $||(w(X_a)_{a \in A})||$ *is minimal up to a universally constant factor* c*, for any monotone symmetric norm* $|| \cdot ||$*.*

Theorem (Without Depots)

For any metric space (X, d) *and* $k \in \mathbb{N}$ *we can compute in polynomial time a partition* $X_1, \ldots, X_k$ *of* X *such that* $||(w(X_1), \ldots w(X_k))||$ *is minimal up to a factor* $c \leq 9$*, for any monotone symmetric norm* $|| \cdot ||$*.*

Key Ideas

- The monotone symmetric norms can be controlled by the top-ℓ-norms [Chakrabarty and Swamy 2019]

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- "good" solutions should be as balanced as possible
- If we are forced to create unbalanced solutions we should find some evidence that balancing the solution would be unreasonably expensive

Very Sketchy Proof Sketch, No Depots

Let's start with (X, d) being a tree, and R as some guess for the optimum solution value wrt. ℓ_∞.

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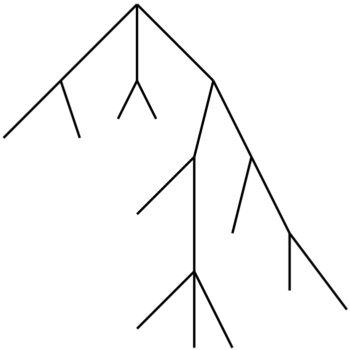
Let's start with (X, d) being a tree, and R as some guess for the optimum solution value wrt. ℓ_∞. We also assume that all edges of the tree have length $\leq R$.

Balanced Partition for a Single Tree



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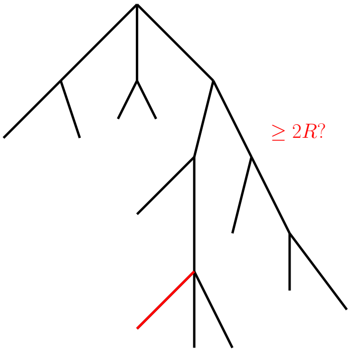

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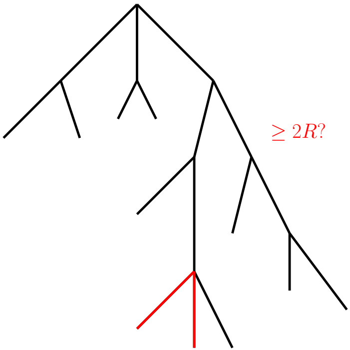

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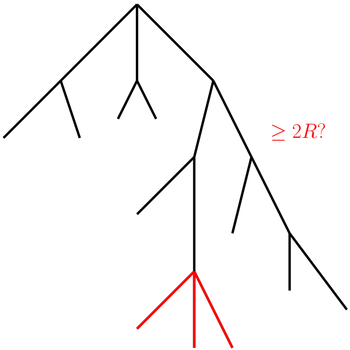

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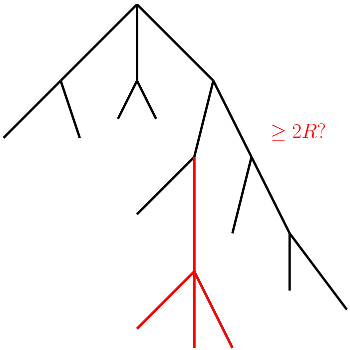

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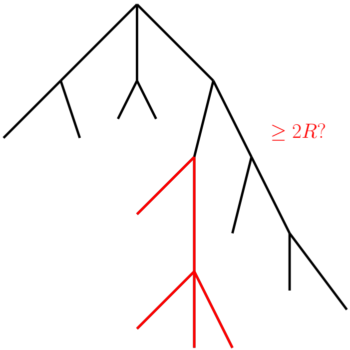

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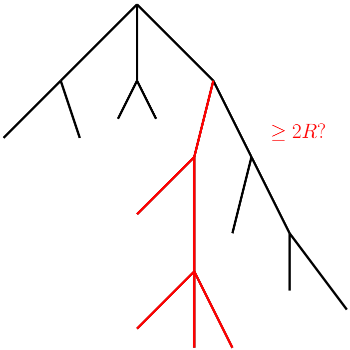

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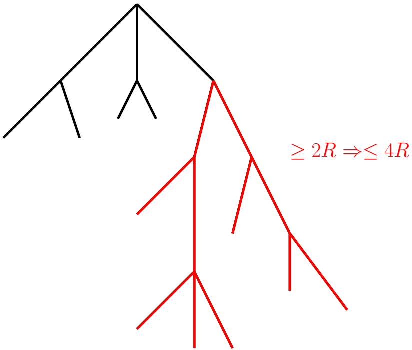

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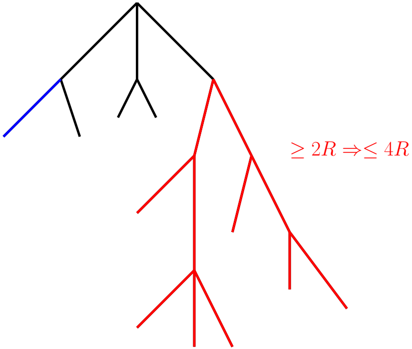

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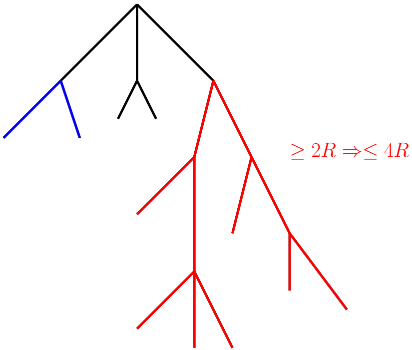

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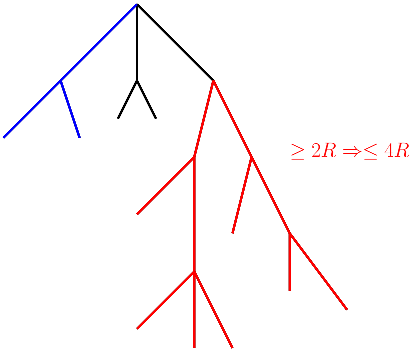

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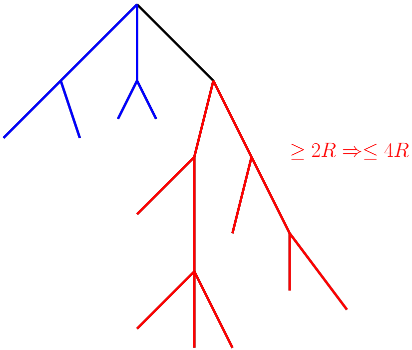

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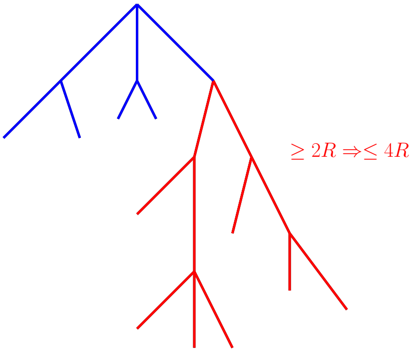

Observation

The smallest R for which this process generates the correct number of trees will also yield a constant factor approximation with respect to any monotone symmetric norm.

Proof.

The ℓ_1 norm of the solution is within a factor 2 of optimal, and all trees have the same size. $\square$

Observation

The smallest R for which this process generates the correct number of trees will also yield a constant factor approximation with respect to any monotone symmetric norm.

Proof.

The ℓ_1 norm of the solution is within a factor 2 of optimal, and all trees (but one) have the same size (up to a constant factor). $\square$

What if there are long edges?

The proof relies on the subtrees growing “continuously”, i.e. by at most $O(R)$.

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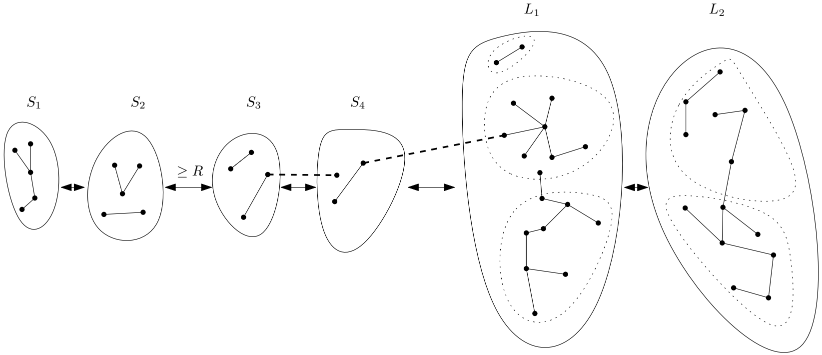
The proof relies on the subtrees growing “continuously”, i.e. by at most $O(R)$. So discard all edges longer than R, and run the partition algorithm in each component.


L_1
L_2
S_1
S_2
S_3
S_4
$\geq R$

With Depots?

Similar, but different.

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All of the edges might be short, but some points can still be far from the depots.

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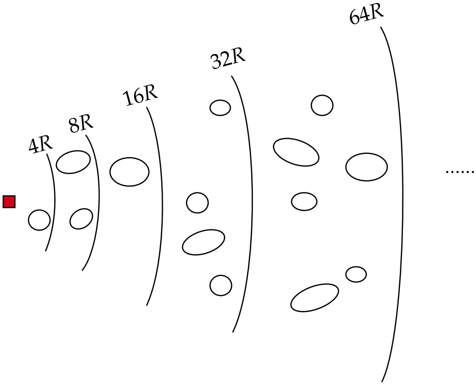
Solution: Solve the problem once at ever "scale".


4R
8R
16R
32R
64R
......

Can You Do Better?

Theorem (Our Work)

ℓ_p-Minimal Tree Cover with prescribed starting points is APX hard for all $p > 1$.

Thank You!