Approximation of Spanning Tree Congestion using Hereditary Bisection

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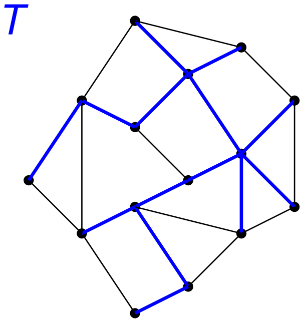
Preliminaries

Definitions

- *spanning tree* of $G = (V, E)$: *subgraph* of G that is a *tree* containing *all vertices* of G
- *cut* of $G = (V, E)$: partition of V into two subsets S and $V \setminus S$
- *cut size*: $|E(S, V \setminus S)|$ = number of edges between S and $V \setminus S$



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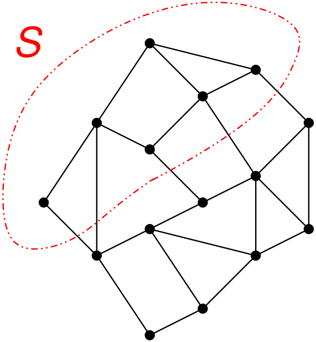

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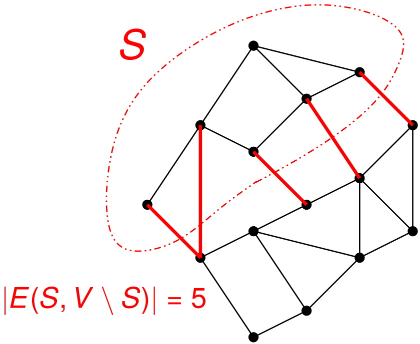


Spanning Tree Congestion

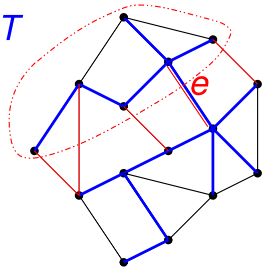
Cuts Induced by a Spanning Tree

- given a spanning tree T of G and an edge $e \in T$, the removal of e defines a cut in G - let $c_T(e)$ denote its size
- *congestion of a span. tree* T of G: $\mathrm{STC}(G, T) = \max_{e \in T} c_T(e)$

Spanning Tree Congestion

- $\mathrm{STC}(G) = \min_{T \in \mathcal{T}} \mathrm{STC}(G, T)$ where $\mathcal{T}$ = all spanning trees of G

$c_T(e) = 5$



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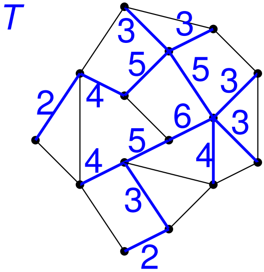
$C(G, T) = 6$



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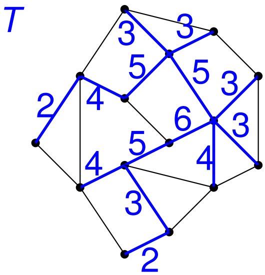
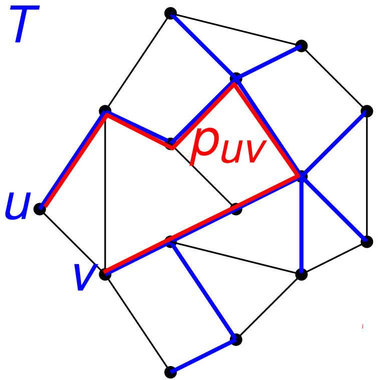
$STC(G) \leq 6$



Spanning Tree Congestion - Alternative View

Simulating G by its Spanning Tree T

- given a spanning tree T of G, for every edge $uv \in E(G)$ there is a unique path p_{uv} in T between u and v
- Claim: for every $e \in E(T)$, $c_T(e) = |\{uv \in E(G) \mid p_{uv} \ni e\}|$.



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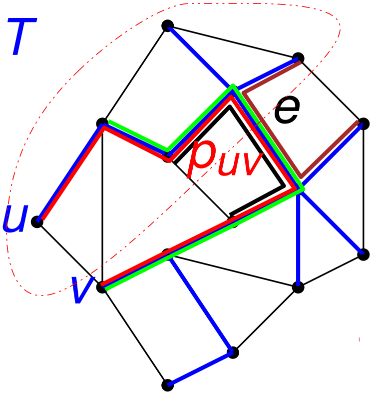

Selected Known Results

Optimization and Decision Versions

- STC: given G, compute STC(G) and find the corresponding tree
- $k-$STC: given G and $k \in \mathbb{N}$, is STC(G) $\leq k$?

Complexity and Approximation

- 1987 - Simonson: problem first studied, under different name
- 2004 - Ostrovskii: STC name, graph-theoretic results
- 2010 - Otachi et al., Löwenstein: NP-hard, even for planar graphs
- 2010 - Otachi et al.:
  - for $k \leq 3$, $k-$STC in P
  - $n/2$-approximation (as STC $\geq m/n$)
- 2010 - Okamoto et al.: exact $O(2^n)$-time algorithm

Selected Known Results, contd.

Complexity and Approximation, contd.

- 2012 - Bodlaender et al.: NP-hard even for graphs with all but one degrees bounded by $O(1)$
  $8-$ STC NP-hard $\Rightarrow$ no c-approx. for $c < 9/8$
  $k-$ STC FPT w.r.t. k and max degree
  $k-$ STC FPT w.r.t. k and treewidth
- 2019 - Chandran et al.: STC $= O(\sqrt{mn})$
  $\Rightarrow O(n/\log n)$-approx. if $\omega(n \log^2 n)$ edges
- 2023 - Luu and Chrobak: $5-$ STC NP-hard, no c-approximation, for $c < 6/5$, unless P=NP
- 2024 - Kolman: $o(n)$-approx. on graphs with polylog degree
- 2024 - Lampis et al.: STC NP-hard for graphs with max degree 8

Observe

- $\Omega(n)$ gap between lower and upper bounds on approximability

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New Results

Algorithm

- $O(\Delta \cdot \log^{3/2} n)$-approximation of STC where Δ is the max degree

Note: An exponential improvement for polylog-degree graphs.

Lower Bound

- $\mathrm{STC}(G) \geq \Omega(hb(G)/\Delta)$ where $hb(G)$ is the hereditary bisection, which is the maximum bisection width over all subgraphs of G

Key Notions

Bisection $b(G)$ of $G = (V, E)$

- $b(G) = \min_{S \subset V} \{|E(S, V \setminus S)| \ : |S| = n/2\}$

Hereditary Bisection $hb(G)$

- $hb(G) = \max\{b(H) \ : \ H \text{ subgraph of } G\}$

c-Balanced Cut , $c \geq 1/2$

- a subset S of V s. t. $|S|, |V \setminus S| \leq c \cdot n$, minimizing $|E(S, V \setminus S)|$

Edge Expansion $\beta(G)$

- $$\beta(G) = \min_{S \subset V} \frac{|E(S, V \setminus S)|}{\min\{|S|, |V \setminus S|\}}$$

Theorem (K., Matoušek, 2004)

Every G contains a subgraph H s.t. $|V(H)| \geq 2n/3$ and $\beta(H) \geq \frac{b(G)}{n}$.

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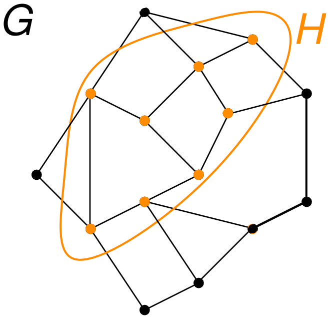
Lemma (Expansion Lower Bound)

For every subgraph H of G, $STC(G) \geq \frac{\beta(H) \cdot k}{\Delta}$ where $k = |V(H)|$.

Proof

- T - the optimal spanning tree
- Pick $u \in T$ and a component C of $T \setminus u$ s.t. $\frac{k}{\Delta} \leq |C \cap H| \leq \frac{k}{2}$
- Then $|E(C, V \setminus C)| \geq \beta(H) \frac{k}{\Delta}$
- For each $xy \in E(C, V \setminus C)$, the $x - y$ path in T goes through the edge $Cu \in T$
- Thus, $\mathrm{STC}(G) = \mathrm{STC}(G, T) \geq |E(C, V \setminus C)| \geq \frac{\beta(H) \cdot k}{\Delta}$ $\square$



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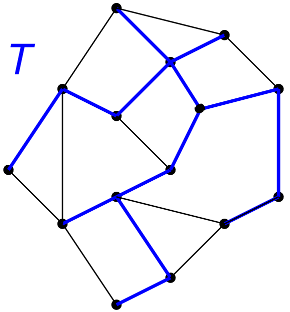

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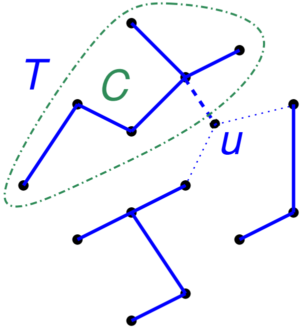

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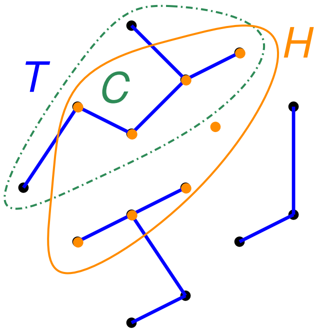

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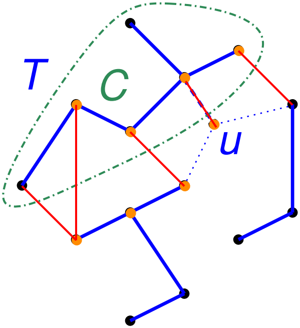

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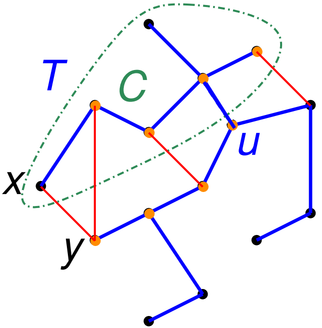

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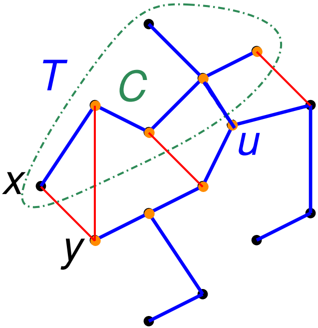

The Main Lower Bound

Theorem (Hereditary Bisection Bound)

For every graph $G = (V, E)$, $STC(G) \geq \Omega\left(\frac{hb(G)}{\Delta}\right)$.

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- Let H be subgraph of G with max bisection, i.e., $b(H) = hb(G)$.
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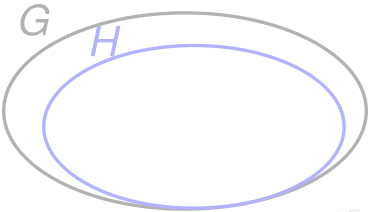

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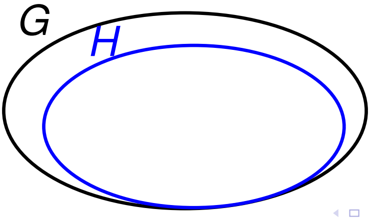

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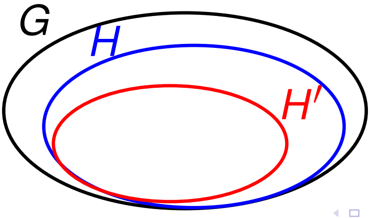

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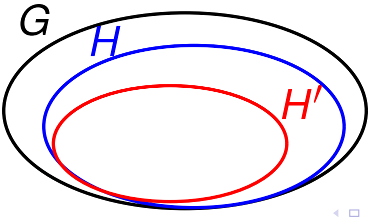
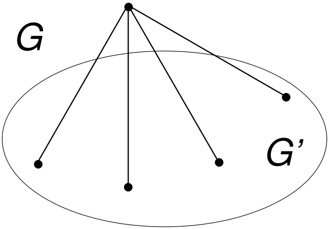


Dependance on the Maximum Degree Δ

Tight Example

Let G' be an $O(1)$-degree expander on $n-1$ vertices, and
let G be G' plus a new node that is connected to all old vertices.

- $b(G') = \Omega(n)$, thus, $hb(G) = \Omega(n)$
- $\Delta(G) = n - 1$
- $\text{STC}(G) = \Omega\left(\frac{hb(G)}{\Delta}\right) = \Omega(1)$
- $\text{STC}(G) = O(1)$ as the star rooted at the new vertex has congestion $O(1)$



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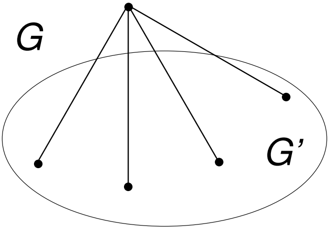
- $\Delta(G) = n - 1$
- $\mathrm{STC}(G) = \Omega\left(\frac{hb(G)}{\Delta}\right) = \Omega(1)$
- $\mathrm{STC}(G) = O(1)$ as the star rooted at the new vertex has congestion $O(1)$



Tight Example

Let G' be an $O(1)$-degree expander on $n-1$ vertices, and let G be G' plus a new node that is connected to all old vertices.

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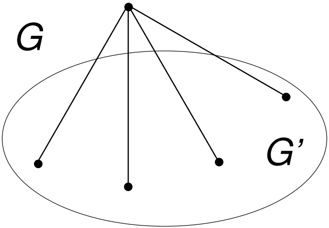

Dependance on the Maximum Degree Δ

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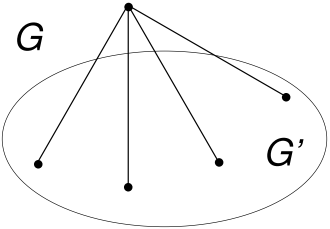

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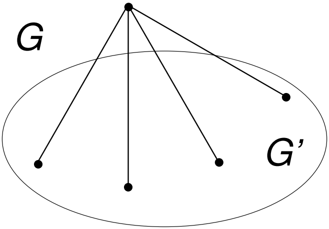

Approximation Algorithm - Sketch

Key Tools

- The new lower bound $OPT = \Omega\left(\frac{hb(G)}{\Delta}\right)$.
- Poly time algorithm (Arora, Rao, Vazirani, 2004) for 2/3-balanced cut of size $O(\sqrt{\log n} \cdot b(G))$.

Key Ideas

- Recursively bisect the graph until each part is small.
- Each level of recursion causes congestion (cf. new lower bound)

$$O(\sqrt{\log n} \cdot hb(G)) = O(\sqrt{\log n} \cdot \Delta \cdot OPT)\ .$$

- Overall congestion

$$O(\log^{3/2} n \cdot hb(G)) = O(\log^{3/2} n \cdot \Delta \cdot OPT)\ .$$

Approximation Algorithm

ConstructST(H)

if $|V(H)| > 1$ **then**
  construct $2/3$-balanced cut $F \subset E(H)$ of H
  for each component C of $H \setminus F$ **do**
    $T_C \leftarrow$ ConstructST(C)
  connect all the spanning trees T_C into a spanning tree T of H
  return T
else
  return H

Theorem

ConstructST is an $O(\Delta \cdot \log^{3/2} n)$-approximation algorithm.

Open Problems

Three Questions

- A better approximation of STC for graphs with large Δ?
- A better lower bound for STC for graphs with large Δ?
- Other usage of hereditary bisection?

Thank you!