Canonical labeling of sparse random graphs

Oleg Verbitsky
Humboldt-Universität zu Berlin

Maksim Zhukovskii
University of Sheffield

STACS
March 6, 2025

Canonical labeling

We consider n-vertex graphs.

An injective function $\ell_G : V(G) \to \{1, \ldots, n\}$ is a **canonical labeling** if

$$G \cong H \text{ if and only if } G^{\ell(G)} = H^{\ell(H)},$$

where $G^{\ell(G)}$ is the isomorphic image of G under $\ell(G)$.

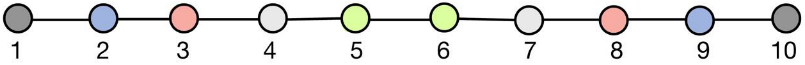
Canonical labeling

We consider n-vertex graphs.

An injective function $\ell_G : V(G) \to \{1, \ldots, n\}$ is a **canonical labeling** if

$$G \cong H \text{ if and only if } G^{\ell(G)} = H^{\ell(H)},$$

where $G^{\ell(G)}$ is the isomorphic image of G under $\ell(G)$.



Canonical labeling of almost all graphs

Theorem (Babai, Erdős, Selkow '80)

$$\ell_G(v) = \left(\deg_G v, \{\!\{\deg_G u\}\!\}_{u \in N(v)}\right)$$

is canonical labeling for **almost all** *graphs* G.

SIAM J. COMPUT.
Vol. 9, No. 3, August 1980

0097–5397/80/0903–0014 $01.00/0

RANDOM GRAPH ISOMORPHISM*

LÁSZLÓ BABAI†, PAUL ERDŐS‡ AND STANLEY M. SELKOW§

- Set $r = \lfloor 3 \log_2 n \rfloor$ and observe that, for almost all graphs, r largest degrees occur exactly once.
- Order vertices in the descending order of their degrees.
- Code every vertex $v > r$ with respect to its adjacencies to $1, \ldots, r$.

the simplicity of our algorithm can hardly be improved on, and it may be worth noting that still, such an algorithm canonizes almost all graphs.

Isomorphism testing

- If G and H are labeled canonically, it takes linear time to decide whether $G \cong H$.

Isomorphism testing

- If G and H are labeled canonically, it takes linear time to decide whether $G \cong H$.
- There exists a linear time algorithm that decides whether $G \cong H$ for **almost all** G and **all** H.

Isomorphism testing

- If G and H are labeled canonically, it takes linear time to decide whether $G \cong H$.
- There exists a linear time algorithm that decides whether $G \cong H$ for **almost all** G and **all** H.
- Babai '16: Graph Isomorphism is solvable in time $\exp(\log^{O(1)} n)$.

Random graphs

$G(n, p)$:

- $\{1, \ldots, n\}$ — set of vertices;
- edges appear independently with probability $p = p(n)$.

Random graphs

$G(n, p)$:

- $\{1, \ldots, n\}$ — set of vertices;
- edges appear independently with probability $p = p(n)$.

$G(n, 1/2)$ — uniformly random graph

BES '80: There exists a linear-time algorithm that **whp** labels canonically $G(n, 1/2)$.

Other p?

Existence of a polynomial-time algorithm that labels canonically $G(n,p)$ whp:

- BES '80 (extension): $\frac{\ln n}{n^{1/5}} \ll p \leq \frac{1}{2}$;
- Bollobás '82: $C_1 \frac{\ln n}{n} \leq p \leq C_2 n^{-11/12}$;
- Czajka, Pandurangan '08: $\frac{\ln^4 n}{n \ln \ln n} \ll p \leq \frac{1}{2}$;
- Linial, Mosheiff '17: $\frac{1}{n} \ll p \leq \frac{1}{2}$.

Other p?

Existence of a polynomial-time algorithm that labels canonically $G(n,p)$ whp:

- BES '80 (extension): $\frac{\ln n}{n^{1/5}} \ll p \le \frac{1}{2}$;
- Bollobás '82: $C_1 \frac{\ln n}{n} \le p \le C_2 n^{-11/12}$;
- Czajka, Pandurangan '08: $\frac{\ln^4 n}{n \ln \ln n} \ll p \le \frac{1}{2}$;
- Linial, Mosheiff '17: $\frac{1}{n} \ll p \le \frac{1}{2}$.

$$p = O(1/n)?$$

Universal algorithm

Theorem (Verbitsky, Z)

There exists a polynomial-time algorithm A such that, for every $p = p(n) \in [0, 1]$, whp A labels canonically $G(n, p)$.[1]

[1] Proved independently by Anastos, Kwan, Moore (STOC '25).

Universal algorithm

Theorem (Verbitsky, Z)

There exists a polynomial-time algorithm A such that, for every $p = p(n) \in [0, 1]$, whp A labels canonically $G(n, p)$.[1]

When $p = O(1/n)$, it takes time $O(n \log n)$ to label canonically $G(n, p)$ whp.

[1] Proved independently by Anastos, Kwan, Moore (STOC '25).

Color refinement

Step 1: $C_1(v) = \deg v$.

Refinement step i: $C_i(v) = \left(C_{i-1}(v), \{\!\{ C_{i-1}(u) \}\!\}_{u \in N(v)} \right)$

Color refinement

Step 1: $C_1(v) = \deg v$.

Refinement step i: $C_i(v) = \left(C_{i-1}(v), \{\!\{ C_{i-1}(u) \}\!\}_{u \in N(v)} \right)$

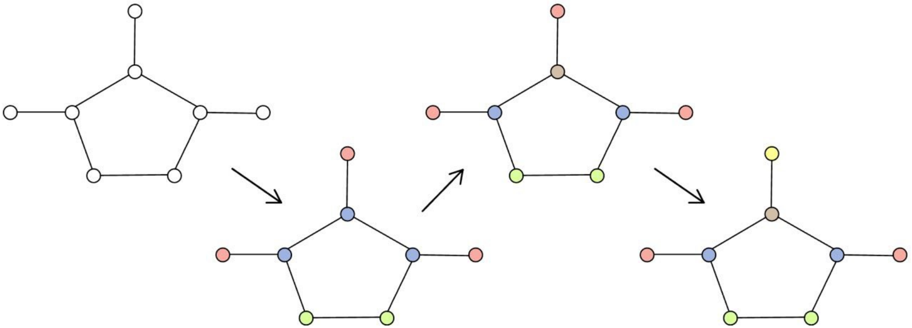
If color classes remain the same after a refinement step, CR terminates and outputs $\{\!\{ C(v), v \in V(G) \}\!\}$.

Color refinement

Step 1: $C_1(v) = \deg v$.

Refinement step i: $C_i(v) = \left(C_{i-1}(v), \{\!\{C_{i-1}(u)\}\!\}_{u \in N(v)}\right)$

If color classes remain the same after a refinement step, CR terminates and outputs $\{\!\{C(v), v \in V(G)\}\!\}$.



Color refinement

G is CR-discrete, if $C(u) \neq C(v)$ for all $u \neq v$.

Color refinement

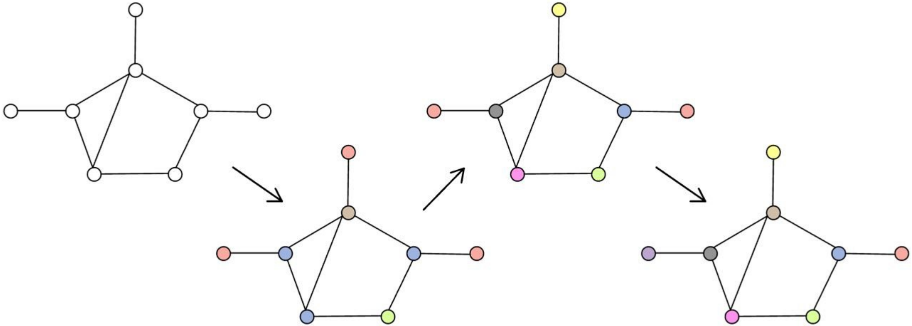
G is CR-discrete, if $C(u) \neq C(v)$ for all $u \neq v$.

If G is CR-discrete, then CR produces a canonical labeling.

Color refinement

G is CR-discrete, if $C(u) \neq C(v)$ for all $u \neq v$.

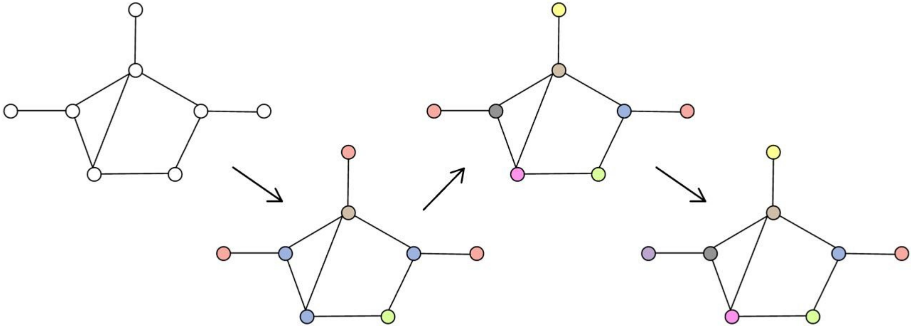
If G is CR-discrete, then CR produces a canonical labeling.



Color refinement

G is CR-discrete, if $C(u) \neq C(v)$ for all $u \neq v$.

If G is CR-discrete, then CR produces a canonical labeling.



Babai, Erdős, Selkow '80: $G(n, 1/2)$ is CR-discrete whp.

Universal cover

G — connected graph

Universal cover

G — connected graph

H covers G, if $\exists$ a surjective homomorphism $H \xrightarrow{\varphi} G$:

$$\forall v \in V(H) \quad \varphi|_{N(v)} \text{ is a bijection.}$$

Universal cover

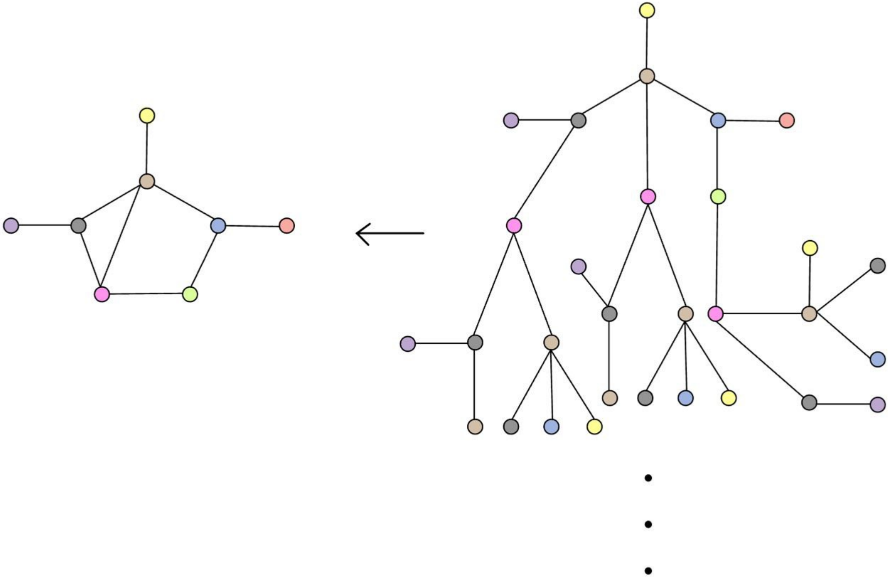
G — connected graph

H covers G, if $\exists$ a surjective homomorphism $H \xrightarrow{\varphi} G$:

$$\forall v \in V(H) \quad \varphi|_{N(v)} \text{ is a bijection.}$$

U^G is a universal cover of G if it covers every connected covering graph of G (equivalently, a tree that covers G)

Universal cover



Universal cover

$G \equiv_{\mathrm{CR}} H$, if $\{\!\{ C(u), u \in V(G) \}\!\} = \{\!\{ C(v), v \in V(H) \}\!\}$

Angluin '80: $G \equiv_{\mathrm{CR}} H \Leftrightarrow U^G \cong U^H$.

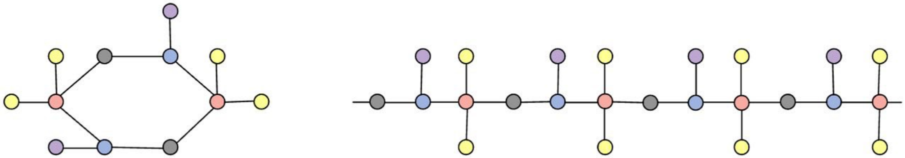
Universal cover

$G \equiv_{\mathrm{CR}} H$, if $\{\!\{ C(u), u \in V(G) \}\!\} = \{\!\{ C(v), v \in V(H) \}\!\}$

Angluin '80: $G \equiv_{\mathrm{CR}} H \Leftrightarrow U^G \cong U^H$.

Corollary

- If T_1, T_2 are trees, then $T_1 \cong T_2 \Leftrightarrow T_1 \equiv_{\mathrm{CR}} T_2$.
- If G_1, G_2 are unicyclic graphs of the same size, then $G_1 \cong G_2 \Leftrightarrow G_1 \equiv_{\mathrm{CR}} G_2$.



Evolution of the random graph

Erdős, Rényi '60; Bollobás '84; Łuczak '90

- If $pn = 1 + \omega(n^{-1/3})$, then whp $G(n, p)$ has one (giant) complex component.
- If $pn = 1 \pm O(n^{-1/3})$, then with a non-vanishing probability $G(n, p)$ has several complex components.
- If $pn = 1 - \omega(n^{-1/3})$, then whp $G(n, p)$ does not have complex components.

Colour Refinement of the random graph

Main Lemma

Let

- $p = O(1/n)$,
- H_n be the union of complex components in $G(n, p)$,
- C_n be the 2-core of H_n.

Then whp $\forall u, v \in V(\mathrm{C}_n)$

$$C(u) = C(v) \Rightarrow u, v \text{ are interchangeable.}$$

interchangeable vertices

Thank you very much!