CMSO-transducing tree-like graph decompositions

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On graphs of bounded treewidth,

- **Thm (Courcelle 90)** every CMSO-definable property can be recognized by a tree automaton over the tree decomposition.
- **Thm (Bodlaender 93)** tree decomposition of bounded width can be obtained in linear-time.
- **Corollary** CMSO model checking can be done in linear time.

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Σ and Γ: two relationnal signatures. *E.g.*,

- **Σ:** the graph vocabulary $\{\text{edge}\}$ — $\text{edge}(x, y) \equiv$ "x–y is an edge"
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universe restriction: subset the universe;
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Theorem (Backward Translation Theorem)

$\tau^{-1}(Y)$ *is* CMSO-*definable* *for* τ, *a* Σ-*to*-Γ CMSO-*transduction* *and* Y, *a* CMSO-*definable set of* Γ-*structures.*

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CMSO-definability = recognizability for graphs of:

- **Thm (Rabin 69)** treewidth 1 (*i.e.*, trees, even infinite)
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One of the consequences of our results

- **Thm** CMSO-definability = recognizability on cographs.

Our results

Theorems

There are CMSO-transductions from graphs to:

- modular decompositions;
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For both undirected and directed graphs.

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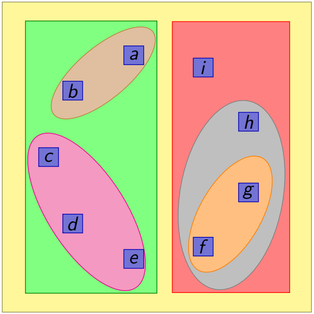

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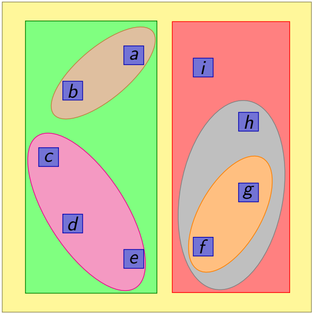

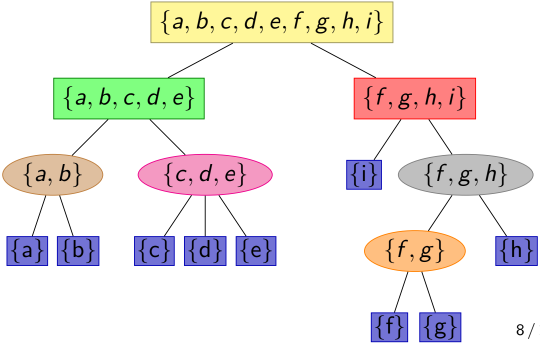

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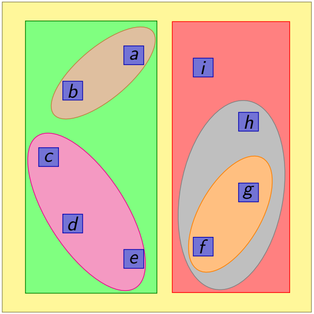

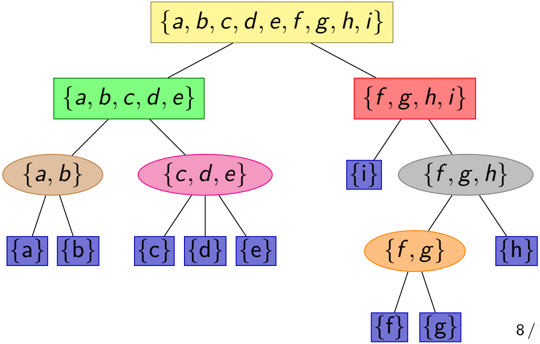

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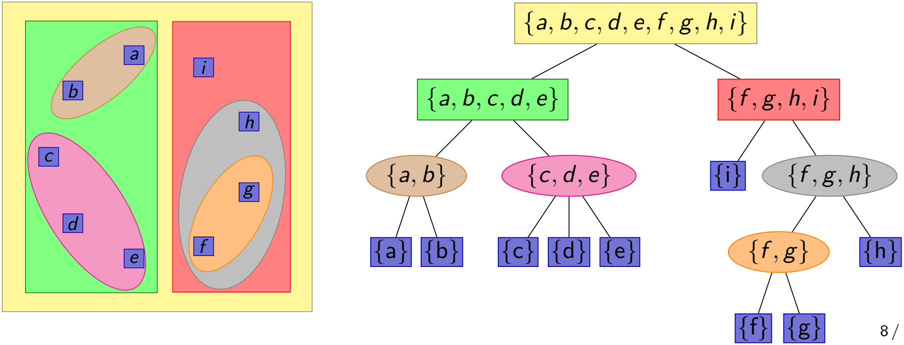

Key construction

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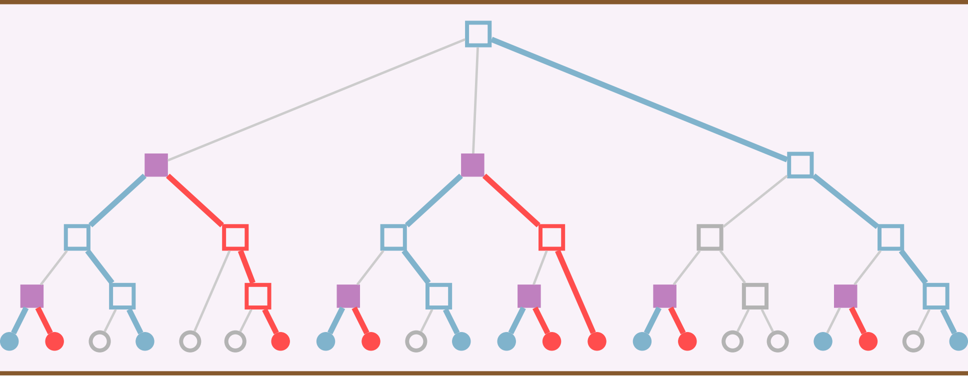

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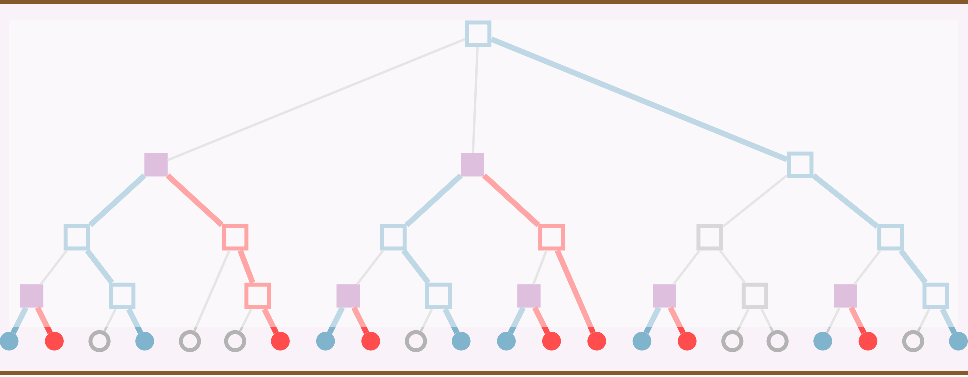

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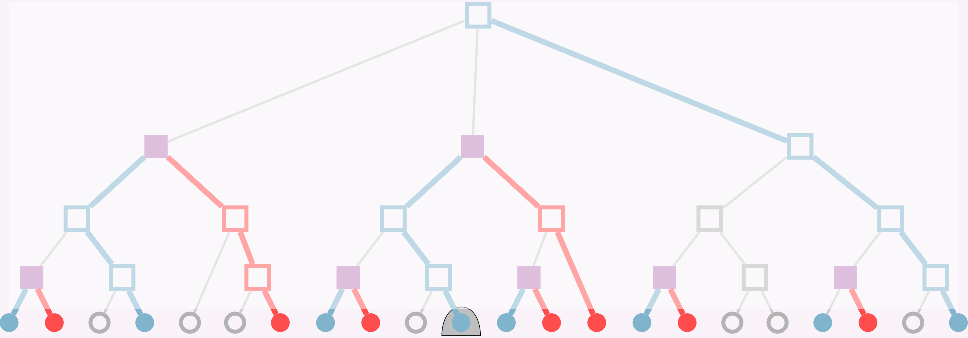
Z is a descendant of X



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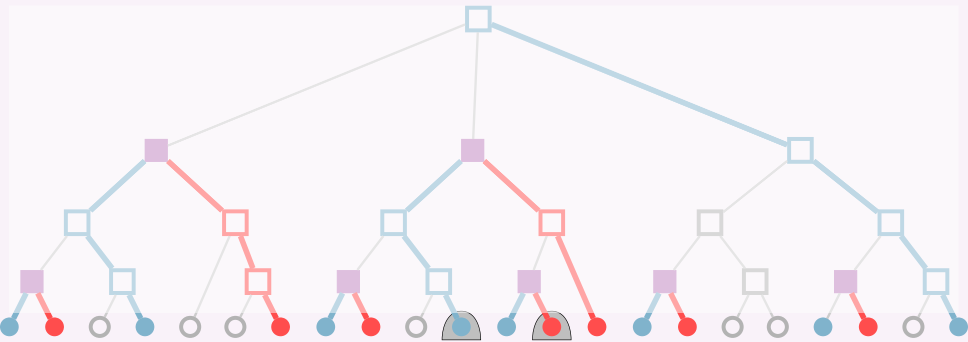

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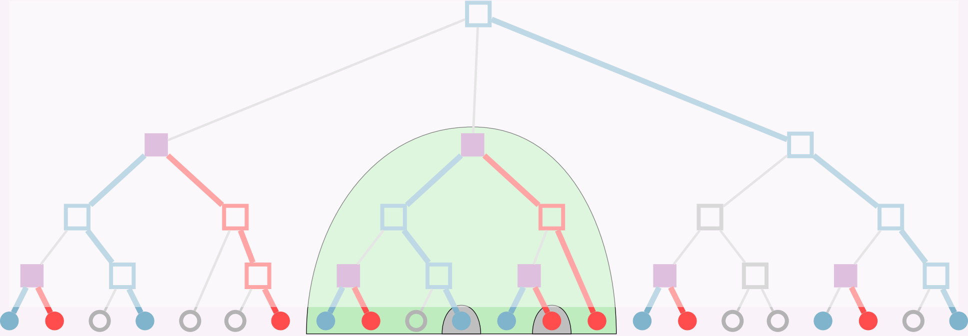

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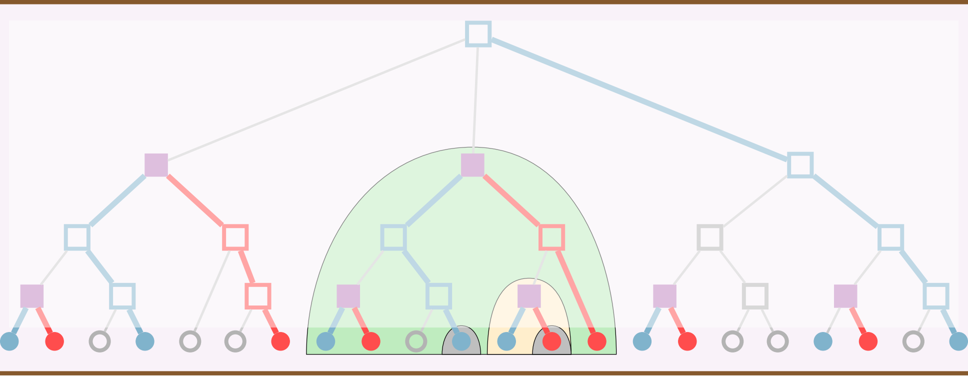

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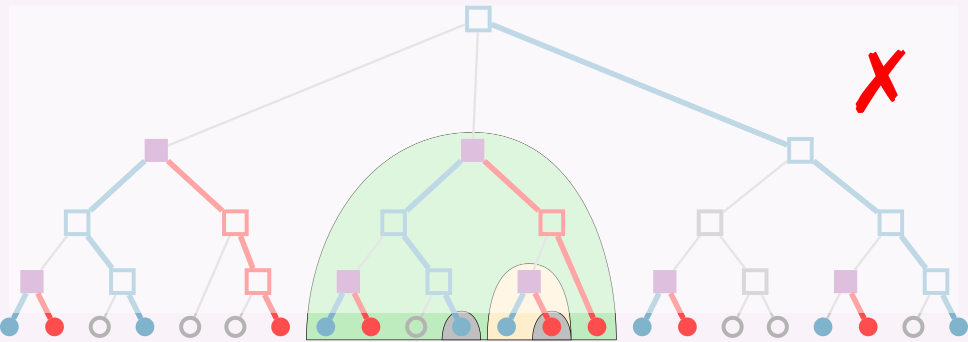

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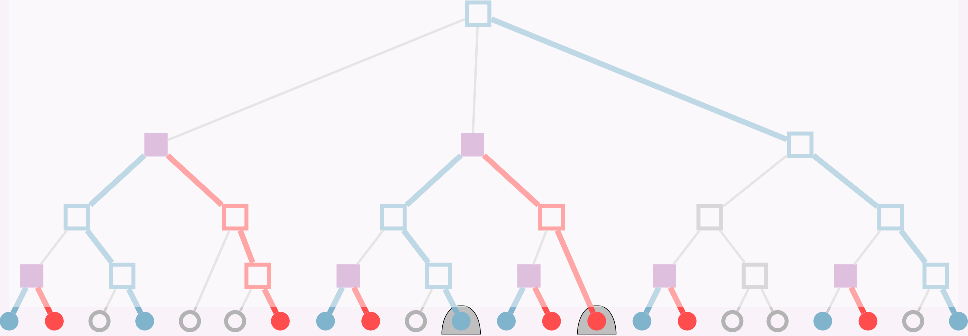

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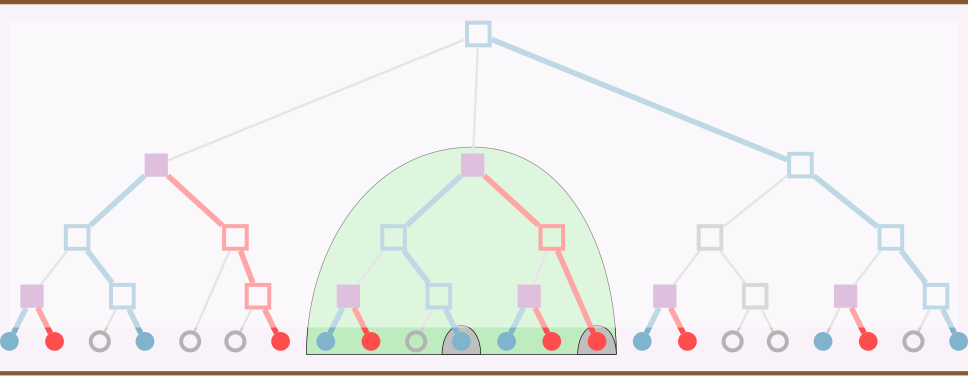

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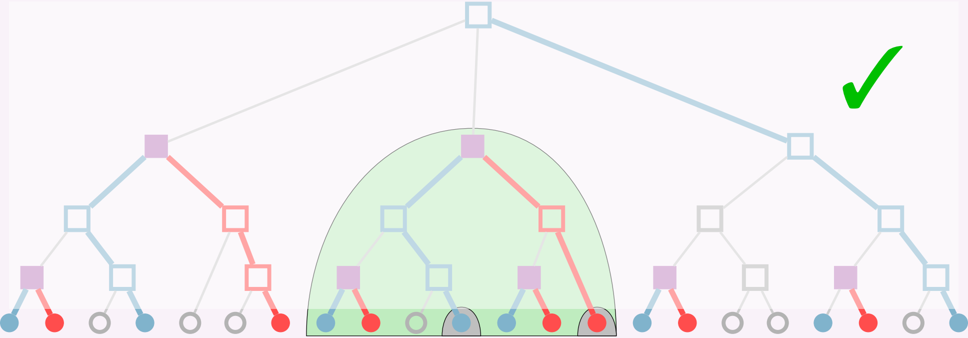

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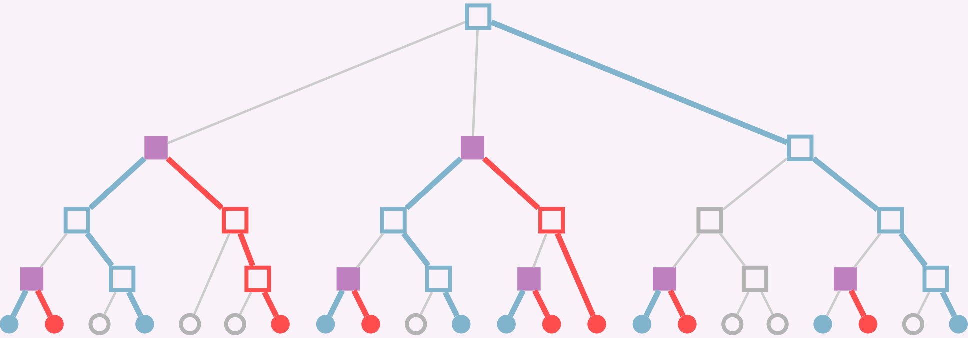

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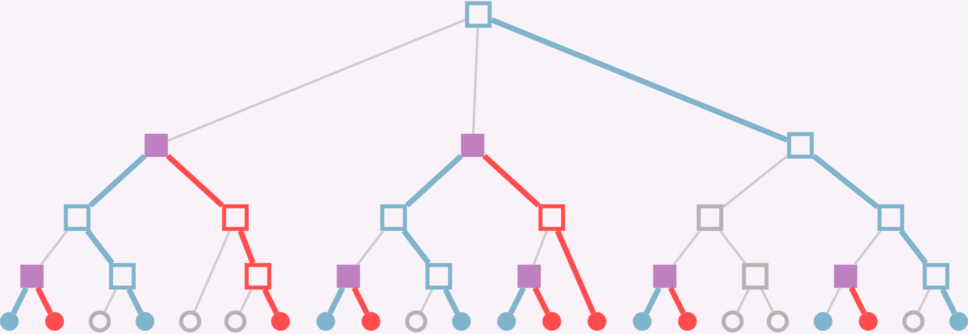

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Conclusion

Theorem

There are CMSO-transductions from set systems:

- to laminar trees (if laminar);

and from (both directed and undirected) graphs:

- to modular decompositions;
  - to cotrees (if cograph);
- to split decompositions;
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Conclusion

Theorem

There are CMSO-transductions from set systems:

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Open directions:

- which decompositions of which structures?
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Thank you for your attention.