Commutative $\mathbb{N}$-rational series of polynomial growth

and friends made along the way

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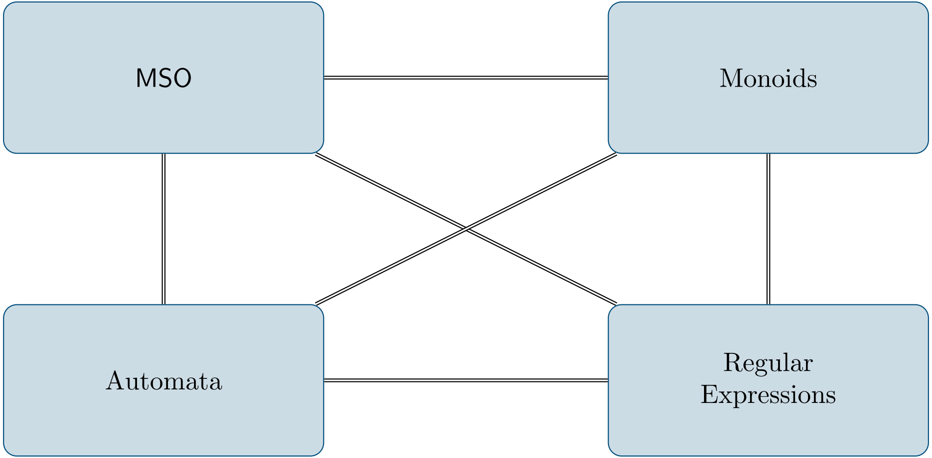
Automata Theory

Seminal results from [Büc60; Tra66; Kle56; Elg61]



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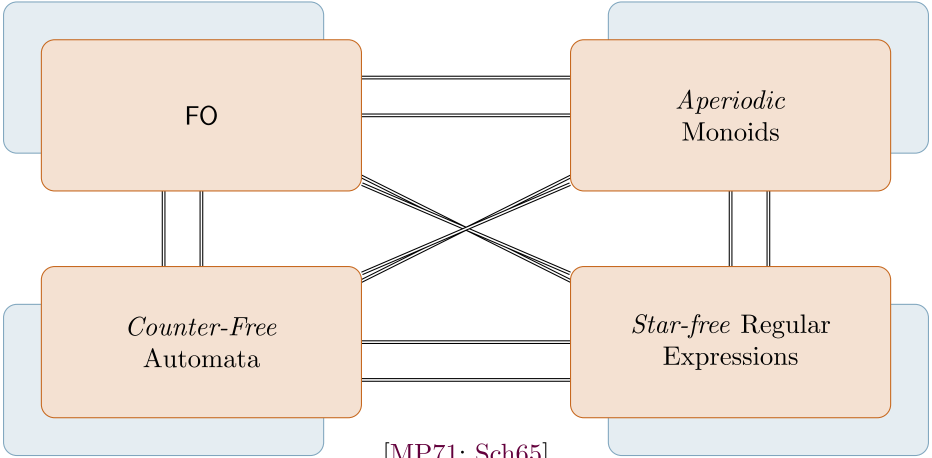

[MP71; Sch65]

Beyond Automata

Extending these results...

$$\Sigma^* \to \mathbb{B}$$

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Quantitative

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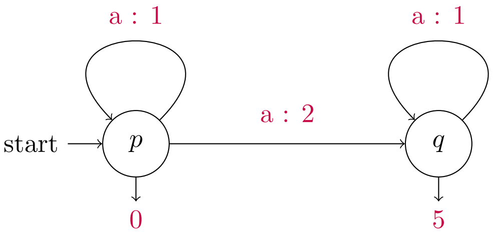
$$\Sigma^* \to (\mathbb{A}, +, \times)$$

$$\Sigma^* \to \Gamma^*$$

$$(\mathbb{N}, +, \times) \simeq \{1\}^*$$

$\mathbb{N}$-weighted automaton



$$f : \{a\}^* \to \mathbb{N}$$
$$f(w) \mapsto 10(|w| - 1)$$

$\mathbb{N}$-polyregular function

$$\varphi(x_1, \ldots, x_n) \in \mathsf{MSO}$$

$$\#\varphi \colon \Sigma^* \to \mathbb{N}$$

$$f = \sum \underbrace{\lambda_i}_{\in \mathbb{N}} \times \#\varphi_i$$

Unary Polyregular Functions

An easier setting, and a subclass of weighted automata! [Sch62]

ℕ-polyregular function

Unary Polyregular Functions

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$w \mapsto |w|^3 + 5|w|$

$w \mapsto |w|_a \times |w|_b$

ℕ-polyregular function

$\sum_{i=1}^{n} \alpha_i \# \varphi_i$

Unary Polyregular Functions

An easier setting, and a subclass of weighted automata! [Sch62]

Polynomial Growth $\wedge$ $\mathbb{N}$-weighted automaton $\quad w \mapsto 2^{|w|}|w|$

$\Updownarrow$

$\mathbb{N}$-polyregular function $\quad \sum_{i=1}^{n} \alpha_i \# \varphi_i$

$w \mapsto |w|^3 + 5|w|$

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Unary Polyregular Functions

An easier setting, and a subclass of weighted automata! [Sch62]

$\exists k \in \mathbb{N}. f(w) = \mathcal{O}(|w|^k)$ $\bigwedge$ $\mathbb{N}$-weighted automaton $w \mapsto 2^{|w|}|w|$

$\Updownarrow$

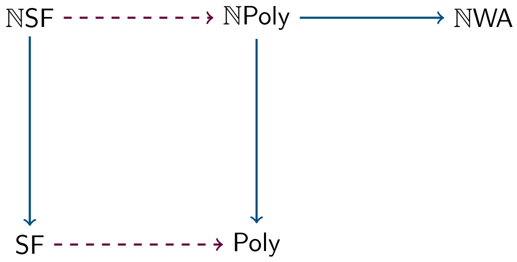
$w \mapsto |w|^3 + 5|w|$

$w \mapsto |w|_a \times |w|_b$

$\mathbb{N}$-polyregular function $\sum_{i=1}^{n} \alpha_i \# \varphi_i$

All protagonists

With open problems from [Dou23; Kar77]



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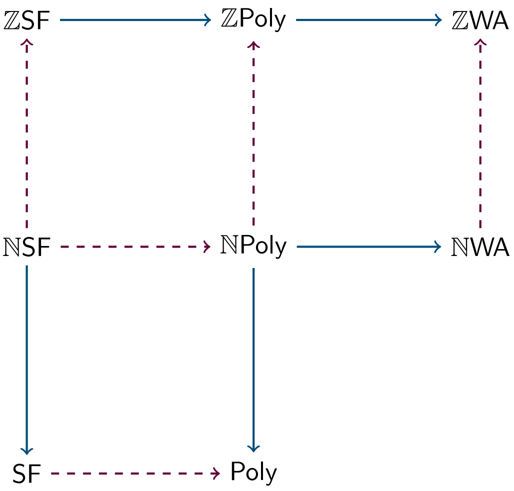

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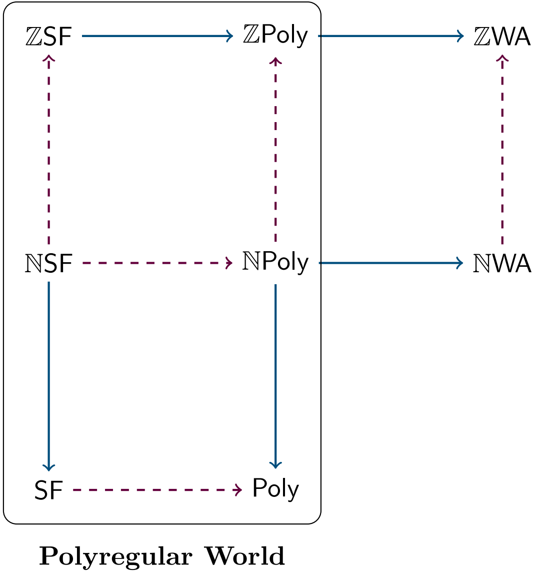

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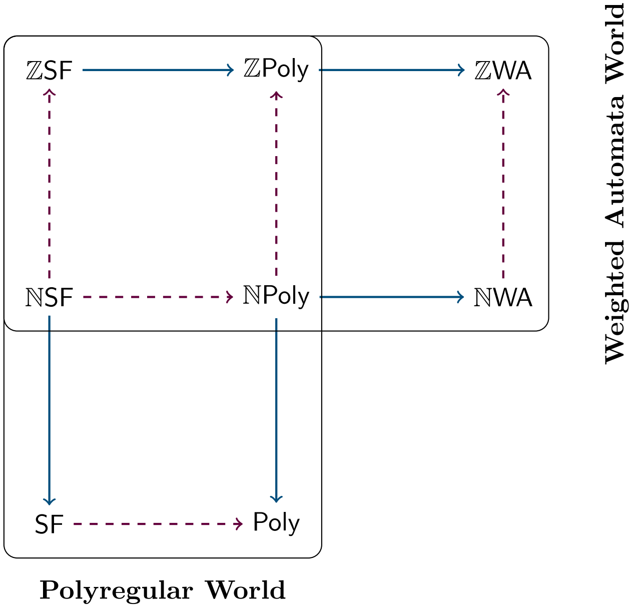

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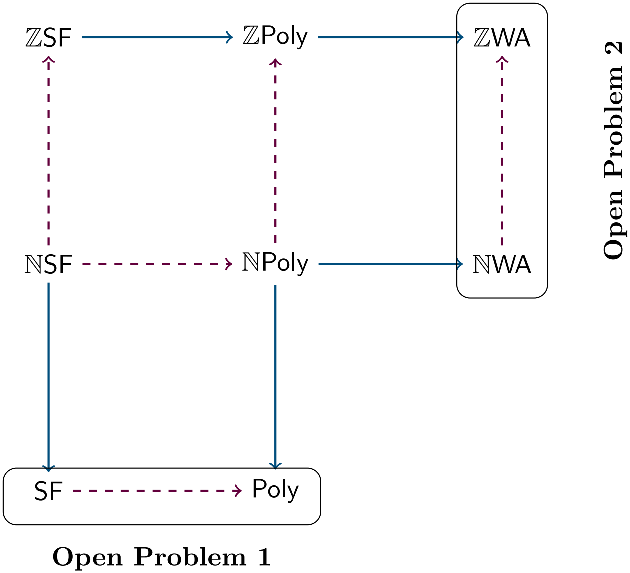

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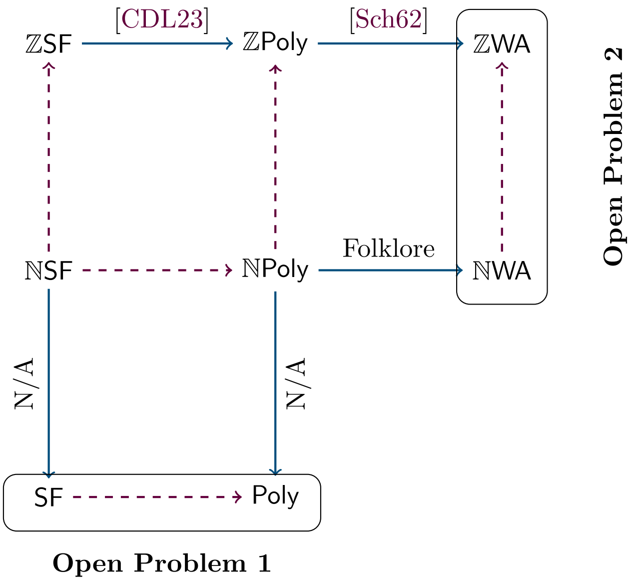
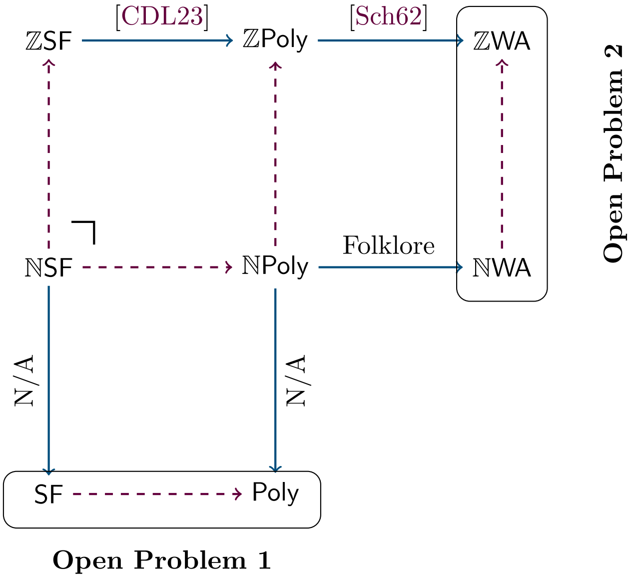


All protagonists

With open problems from [Dou23; Kar77]

$\mathbb{Z}\mathsf{SF} \cap \mathbb{N}\mathsf{Poly} = \mathbb{N}\mathsf{SF}$?

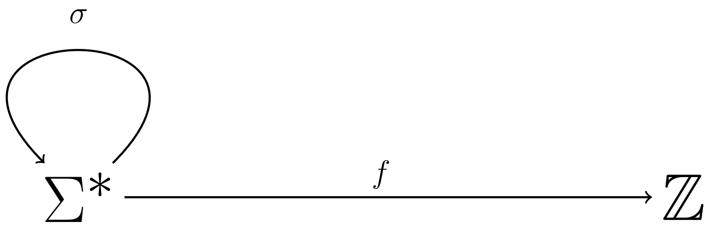


Simplify further: commutative input

Commutative Input?

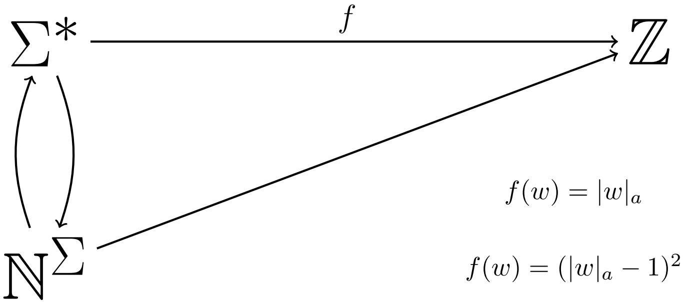
$$\Sigma^* \xrightarrow{\quad f \quad} \mathbb{Z}$$

Commutative Input?



$$f(abab) = f(aabb)$$

Commutative Input?



$$f(w) = |w|_a$$

$$f(w) = (|w|_a - 1)^2$$

$$f(w) = (|w|_a - |w|_b)^2$$

In particular: polynomials in $\mathbb{N}[\vec{X}]$ (resp. $\mathbb{Z}[\vec{X}]$)

All protagonists

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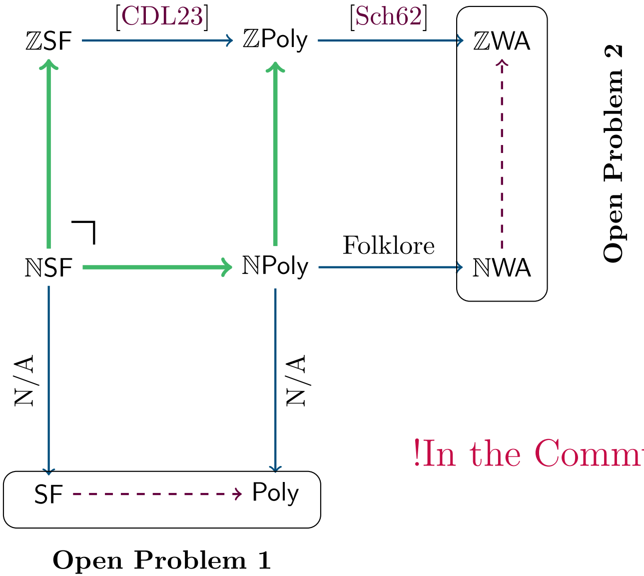

!In the Commutative Setting!

The case of Polynomials

A previous statement

of Karhumäki [Kar77].

$$P = P_{\max} + P_{\text{rest}}$$

$P_{\max} \in \mathbb{N}[\vec{X}]$ $\forall \vec{x} \in \mathbb{N}^k, P(\vec{x}) \geq 0$

$$\Updownarrow$$

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Idea: $P(X_1 + K, \ldots, X_n + K) \in \mathbb{N}[\vec{X}]$ for some large enough K and *hardcode* small values.

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Counterexample: $P = Z(X+Y)^2 + 2(X-Y)^2$.

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$\forall \vec{x} \in \mathbb{N}^k, P(\vec{x}) \geq 0$ UNDECIDABLE!

Idea: $P(X_1 + K, \ldots, X_n + K) \in \mathbb{N}[\vec{X}]$ for some large enough K and *hardcode* small values.

$\Updownarrow$

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Binomial Coefficients

Binomial Monomial:
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positive linear combinations of
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Strongly $\mathbb{N}$ Binomial Polynomials:
partial applications of P are Natural Binomial Polynomials

Back to $\mathbb{Z}$

Let $P \in \mathbb{Q}[\vec{X}]$

Classical result from Pólya [Pól15]

$$P(\mathbb{Z}^k) \subseteq \mathbb{Z} \implies P(\mathbb{N}^k) \subseteq \mathbb{Z} \implies P \in \mathbb{Z}\mathsf{BP}$$

$$P \in \mathbb{Z}\mathsf{BP} \Downarrow P \in \mathbb{Z}\mathsf{SF}$$

Binomials are in $\mathbb{Z}\mathsf{SF}$

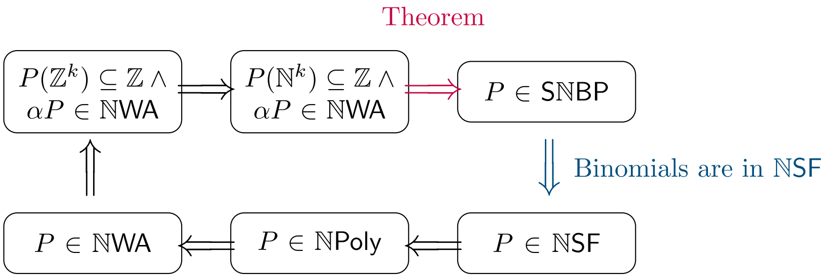
$$P \in \mathbb{Z}\mathsf{WA} \impliedby P \in \mathbb{Z}\mathsf{Poly} \impliedby P \in \mathbb{Z}\mathsf{SF}$$

$$P \in \mathbb{Z}\mathsf{WA} \Uparrow P(\mathbb{Z}^k) \subseteq \mathbb{Z}$$

$$\binom{|w|}{\ell} = \#\left(x_1 < x_2 < \cdots < x_l\right)(w)$$

And now for $\mathbb{N}$

Let $P \in \mathbb{Q}[\vec{X}], \alpha \in \mathbb{N}$, s.t. $\alpha P \in \mathbb{Z}[\vec{X}]$



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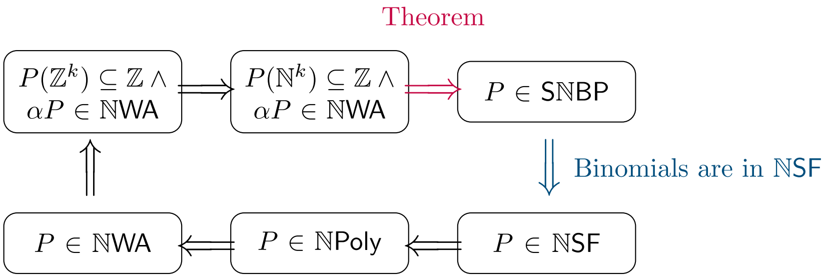


In the case of at most two indeterminates, it coincides with [Kar77]

Tying the knot: commutative input

Polynomial Pumping

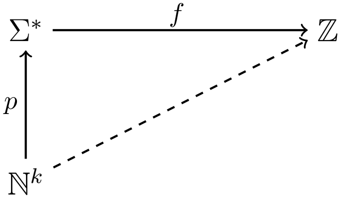
Also known as: forcing commutativity

$$\Sigma^* \xrightarrow{\quad f \quad} \mathbb{Z}$$

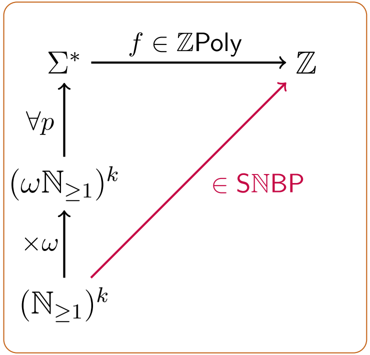
Polynomial Pumping

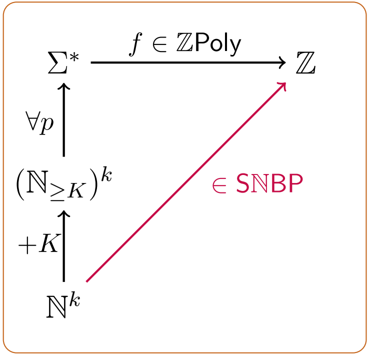
Also known as: forcing commutativity

$p(x_1, \ldots, x_k) = v_0 \prod_{i=1}^{n} (u_i)^{x_i} v_i$

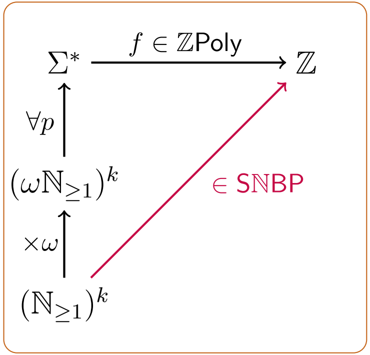


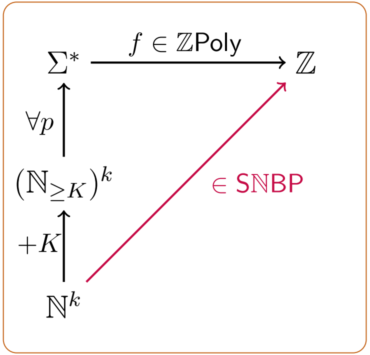
Commutative Input





Commutative Input

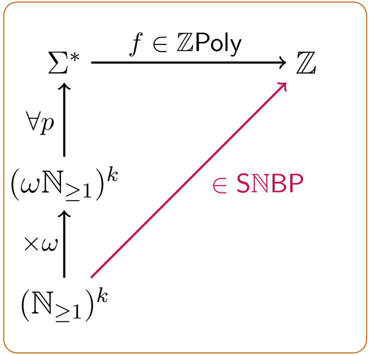


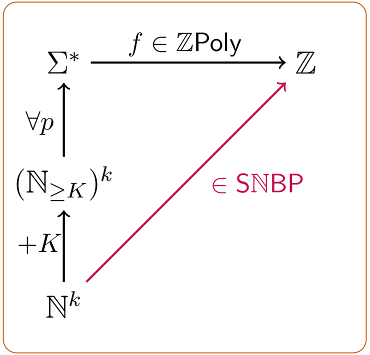


$\exists\omega \Updownarrow$ $\qquad$ $\Updownarrow \exists K$

$f \in \mathbb{N}\mathsf{Poly}$ $\qquad$ $f \in \mathbb{N}\mathsf{SF}$

Commutative Input



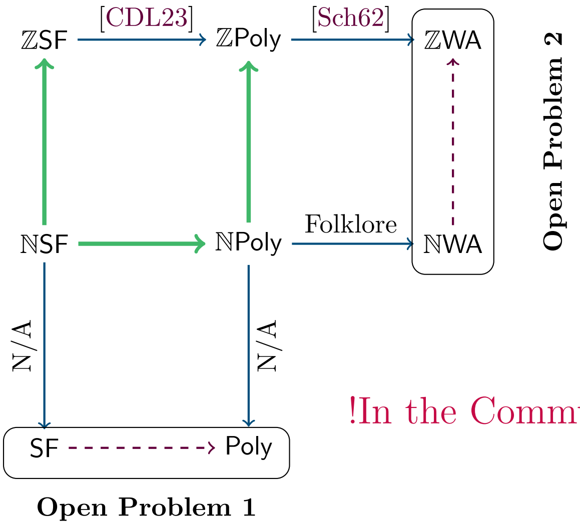


$\exists\omega \Updownarrow$ $\qquad\qquad$ $\Updownarrow \exists K$

$f \in \mathbb{N}\mathsf{Poly}$ $\qquad\qquad$ $f \in \mathbb{N}\mathsf{SF}$

Conjecture: true without commutative assumption

Conclusion



Towards Series: define *aperiodic* ℤWA and ℕWA (but not like [DG05; DG19]).

Spectrum in $\mathbb{R}_{\geq 0}$, aperiodic finite quotients, pumping arguments

Dropping Commutation: attempted a *WQO* approach building a canonical model [Lop24].

Using aperiodic approximations?

!In the Commutative Setting!

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