Computability of extender sets in multidimensional subshifts

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Subshifts

Definition

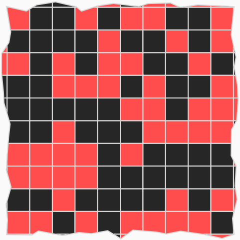
Alphabet: finite set of symbols

$$\mathcal{A} = \{ ■, ■ \}$$

A set $\mathcal{F}$ of finite forbidden patterns:

$$\mathcal{F} = \emptyset$$

Configuration: $\mathcal{A}$-colouring of $\mathbb{Z}^d$ avoiding $\mathcal{F}$



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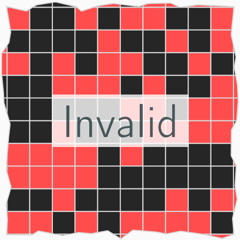

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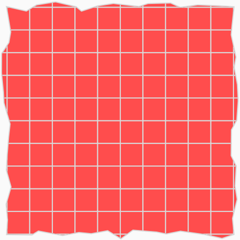

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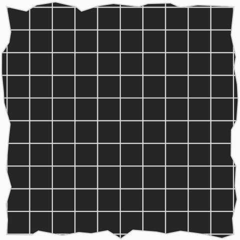

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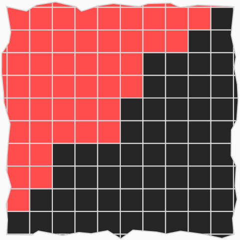

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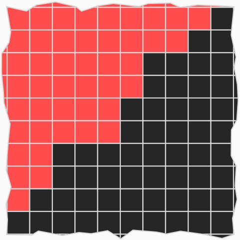

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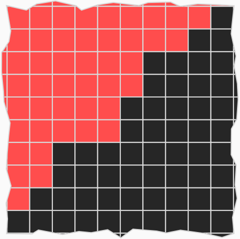

For $\mathcal{F}$ ...

- Finite: **Subshift of Finite Type**
- Enumerable: **effective**

Sofic subshifts: general definition

Sofic subshifts on $\mathbb{Z}^d$: letter-by-letter projections of SFTs (less formally, SFT "up to some construction lines").

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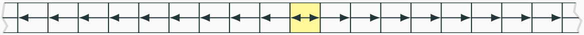

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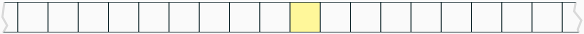

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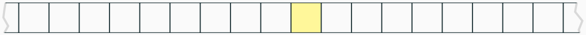

$\mathbb{Z}$-sofic = *regular language* as forbidden patterns

Distinguishing subshifts classes

Question

Given an effective $\mathbb{Z}^d$-subshift X, determine whether it is sofic.

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Not an easy problem: multidimensional sofic subshifts can be very complicated.

Lifts: link effective and sofic subshifts

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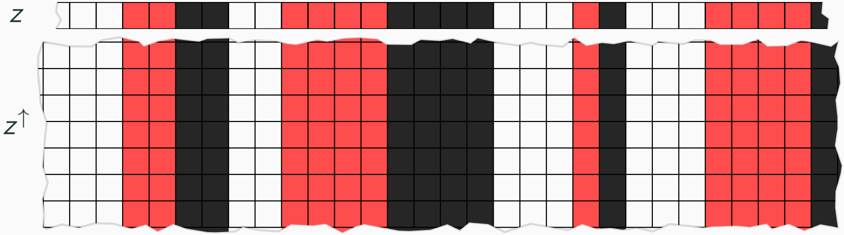

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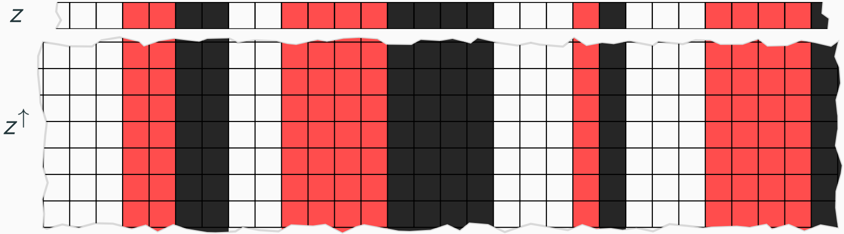

Theorem — DRS, 2012, Aubrun and Sablik 2013

Let Z an effective $\mathbb{Z}$-subshift. Then, $Z^{\uparrow}$ is a sofic subshift on $\mathbb{Z}^2$.

Pattern Extension

Goal: characterize patterns equivalent for the "exchangeability" relation

Given a subshift X, $w \sim w' \iff$ if we can exchange w and w' in any valid pattern of X (in particular, w, w' have the same domain).

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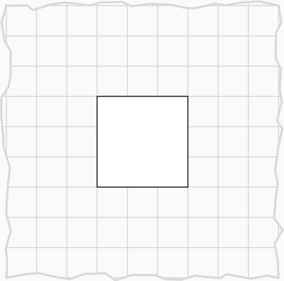
We write $E_X(n)$ for the number of **equivalence classes** of patterns of domain $\{0, \ldots, n-1\}^d$

Example: X = sunny-side-up, configurations on {🟨, ⬜} with ≤ 1 🟨.



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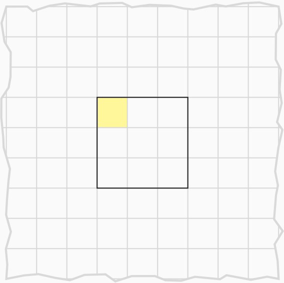

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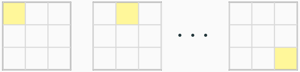

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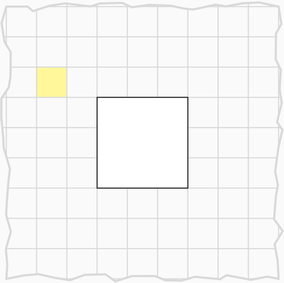

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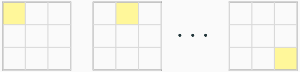
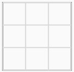


Second equivalence class:



$E_X(n) = 2$ for all n.

The SFT case, the $E_X(n)$ map

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Theorem — Ormes, Pavlov [OP16]

For $X \subseteq \mathcal{A}^{\mathbb{Z}^d}$ any subshift, if there exists $n > 0$ such that $E_X(n) \leq n$ then X is sofic.
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Question

What are the possible asymptotic behaviours of E_X, for various subshifts classes ?

$(E_X(n))_{n\in\mathbb{N}}$ = too fine-grained, but what seems to matter is its growth rate:

Definition : Extender entropy — French, Pavlov [FP19]

Define the **extender entropy** of a $\mathbb{Z}^d$ subshift X as

$$h_E(X) = \lim_{n\to+\infty} \frac{\log E_X(n)}{n^d} = \inf_{n\to+\infty} \frac{\log E_X(n)}{n^d}$$

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$$E_X(n) \sim |\mathcal{A}|^{h_E(x)n^d}$$

Hence, $0 \leq h_E(X) \leq \log \mathcal{A}$, and is always 0 for an SFT X.

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Idea: $\alpha \in \mathbb{R}$ is computable if we can approximate it, $|\alpha - r_n| < 2^{-n}$ with $(r_n)_{n\in\mathbb{N}}$ computable.
Removing the computability hypothesis on (r_n) gives non-computable real numbers:

$$\Pi_n = \inf_{k_1} \sup_{k_2} \inf_{k_3} \dots r_{k_1,\dots,k_n} \subset \mathbb{R}$$

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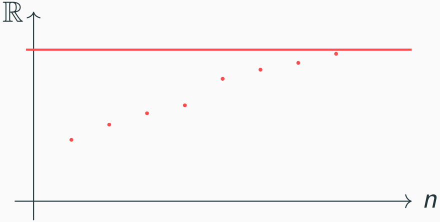

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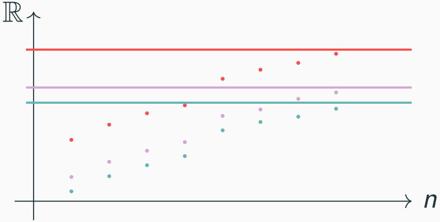

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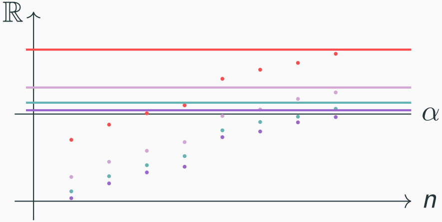

Everything is non-computable

Theorem — Callard, P., Vanier [CPV25]

The extender entropies of the effective subshifts of $\mathbb{Z}$ are exactly the non-negative Π_3 real numbers.

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The extender entropies of the sofic subshifts of $\mathbb{Z}^d$, $d \geq 2$ are exactly the non-negative Π_3 real numbers.

Easy direction: $h_E(X) \in \Pi_3$

Proof sketch and construction

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$$+$$

$$\#\text{Patterns of size } n \approx E_X(n)$$

$\rightarrow$ few equivalent (="exchangeable") patterns.

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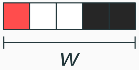

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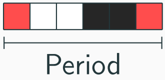

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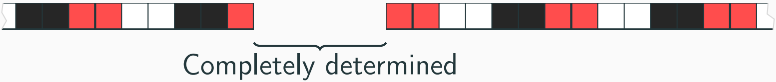

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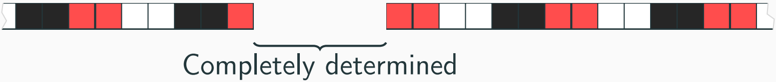

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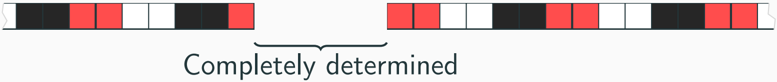

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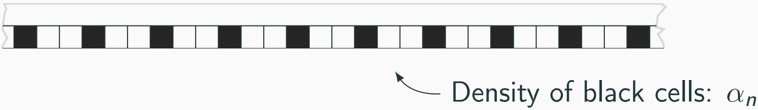

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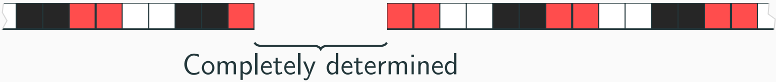

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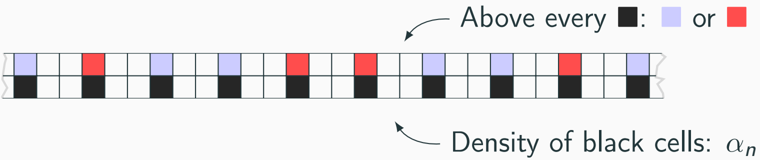

A proportion α_n of the cells "matter for extensibility".

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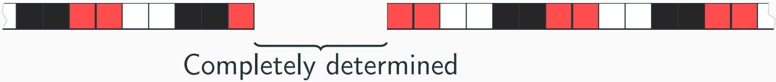

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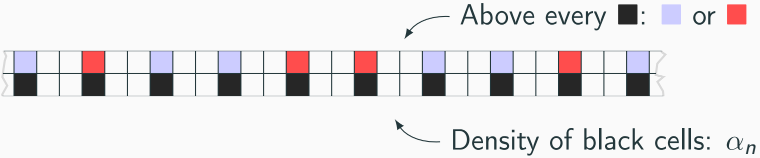

A proportion α_n of the cells "matter for extensibility".

+ Lots of hidden details (α_n non-computable but X must be effective, adapting it to sofic subshifts ...)

Summary

Another isomorphism invariant in subshifts:

- Fully characterized by computability theory
- For which sofic subshifts are “as expressive” as the effective subshifts
- Which is very different in dimensions $d = 1$ and $d \geq 2$ in the case of sofic subshifts

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More results:

	$\mathbb{Z}$	$\mathbb{Z}^{d \geq 2}$
SFT	$\{0\}$	
Sofic	$\{0\}$	Π_3
Effective	Π_3	
Computable	Π_2	
Effective and minimal	Π_1	
Effective and 1-mixing/block-gluing	Π_3	