Efficiently computing the minimum rank of a matrix in a monoid of zero-one matrices

joint work with Stefan Kiefer

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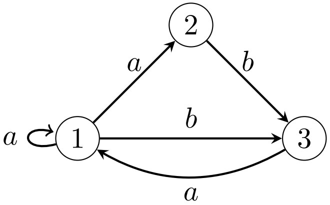

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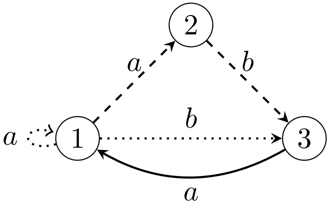
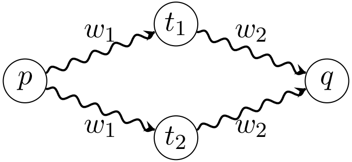


Diamonds



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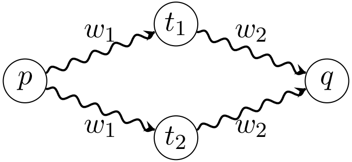

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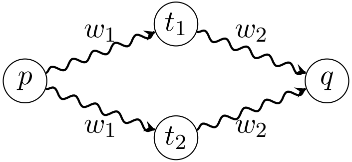

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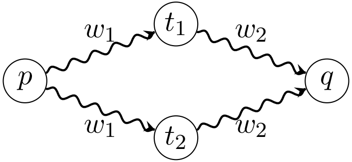

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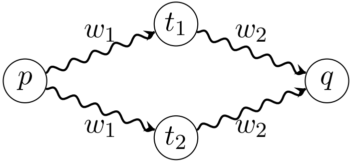

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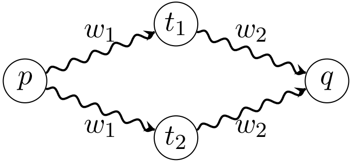

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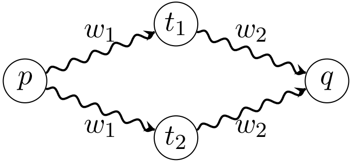

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- Indeed, it's enough to check if $\bar{A}x = x$ has a non-zero solution.

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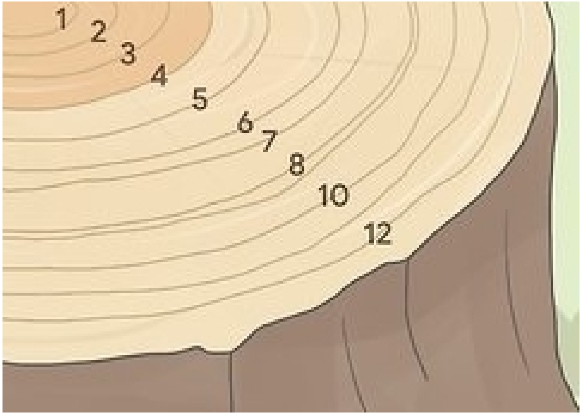
- Similarly, the rank conjecture generalises it to products of length $(n-r)^2$ in the case of minimum rank $r \geq 1$.
- Minimum rank of a matrix is computable in polynomial time (Carpi, TCS 1988 + Kiefer, Mascle, STACS 2019).

Why are these problems nice?



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- If the minimum rank is positive, can we still compute it by linear algebra (in NC^2)?– Wasn't known even for DFAs.

Yes

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Minimum rank in NC^2

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Given a zero-one matrix monoid, one can compute the minimum rank of a matrix in it in NC^2.

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- By using this result, we define nicely behaving weights of states in the unambiguous automaton.
- Some sort of Kraft–McMillan (in)equality?

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- Given two matrices generating a zero-one matrix monoid, can we multiply them faster than in $\mathcal{O}(n^\omega)$?
- If $\mathcal{M}$ contains the zero matrix, Kiefer and Mascle showed that this matrix can be represented as a product of at most n^5 generators. The best known lower bound is $n(n+1)/2$. Can we do anything about that?

Thank you!