Faster Algorithms on Linear Delta-Matroids

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Introduction to Delta-Matroids

Delta-matroids are an extension of matroids introduced by Bouchet in the 1980s.
A delta-matroid is defined by a pair $(V, \mathcal{F})$:

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For any sets $A, B \in \mathcal{F}$ and any element $x \in A \Delta B$, there exists an element $y \in A \Delta B$ such that:

$$A \Delta \{x, y\} \in \mathcal{F},$$

where Δ denotes the symmetric difference, i.e., $A \Delta X = (A \setminus X) \cup (X \setminus A)$.

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The bases of a matroid are defined using the symmetric exchange axiom along with the condition that all $F \in \mathcal{F}$ have the same cardinality.

Examples of Delta-Matroids

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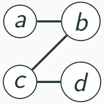

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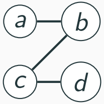

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It is not possible to define something analogous to matroid truncation.
Example: $\mathcal{F} = \{\emptyset, \{a, b\}, \{c, d\}\}$ does not satisfy the symmetric exchange axiom.

Linear Representation of Delta-Matroids

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Linear Representation of a Delta-Matroid:

- Consider a skew-symmetric matrix A with rows and columns labeled by a set V.
- Define $\mathcal{F} = \{F \subseteq V \mid A[F] \text{ is nonsingular}\}$, where $A[F]$ denotes the submatrix of A restricted to rows and columns indexed by F.

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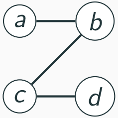
Example: The **Tutte matrix** provides a linear rep. of matching delta-matroids.



$$\begin{array}{c c} & \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{pmatrix} 0 & x_{ab} & 0 & 0 \\ -x_{ab} & 0 & x_{bc} & 0 \\ 0 & -x_{bc} & 0 & x_{cd} \\ 0 & 0 & -x_{cd} & 0 \end{pmatrix} \end{array}$$

$$\mathcal{F} = \{\emptyset, \{a,b\}, \{b,c\}, \{c,d\}, \{a,b,c,d\}\}$$

Caveat. $\emptyset$ is always feasible in $\mathbf{D}(A)$.

Linear Representation Using Twist

The definition of the twist operation for a subset $S \subseteq V$ of a delta-matroid is:

$$D = (V, \mathcal{F}) \xrightarrow{\text{twist by } S} D \Delta S = (V, \{F \Delta S \mid F \in \mathcal{F}\}).$$

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Example: When $\mathcal{F} = \{\{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}\}$, a representation is given by $\mathbf{D}(A)\Delta\{a, c\}$, where

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \text{ since } \mathcal{F}(\mathbf{D}(A)) = \{\emptyset, \{a, b\}, \{c, d\}, \{a, b, c, d\}\}$$

Linear Representation Using Contraction

The definition of the contraction operation for a subset $T \subseteq V$ of a delta-matroid is:

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Example: When $\mathcal{F} = \{\{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}\}$, a representation is given by $\mathbf{D}(A)/\{e, f\}$, where

$$A = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \\ f \end{array} \begin{array}{c} \begin{array}{cccccc} a & b & c & d & e & f \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 \end{pmatrix} \end{array}$$

Equivalence between twist representation and contraction representation 1

Suppose that a twist representaiton $\mathbf{D}(A)\Delta S$ is given for $A \in \mathbb{F}^{V \times V}$.
Then, a contraction reprsentation is give by $\mathbf{D}(A^*)/T$, where A^* is indexed by $V \cup T$:

$$A^* = \begin{array}{c} \\ T \\ S \\ V \setminus S \end{array} \begin{array}{c} \begin{array}{ccc} T & S & V \setminus S \end{array} \\ \begin{pmatrix} A[S] & -I & -A[S, V \setminus S] \\ I & O & O \\ -A[V \setminus S, S] & O & A[V \setminus S] \end{pmatrix} \end{array}$$

Theorem: A twist representation can be converted to a contraction representation in $O(n^2)$ time.

Equivalence between twist representation and contraction representation 2

Suppose that a contraction representaiton $\mathbf{D}(A)/T$ is given for $A \in \mathbb{F}^{(V \cup T)\times(V \cup T)}$.
Then, a twist reprsentation is given by $\mathbf{D}(A^*)\Delta F$, where F is a fixed feasible set:

$$A^* = \begin{array}{c} \\ F \\ \overline{F} \end{array} \begin{array}{c} \begin{array}{cc} F & \overline{F} \end{array} \\ \begin{pmatrix} B^{-1}[F] & B^{-1}[F, T \cup F]A[T \cup F, \overline{F}] \\ A[\overline{F}, T \cup F]B^{-1}[F, T \cup F] & A[\overline{F}] + A[\overline{F}, T \cup F]B^{-1}A[T \cup T, \overline{F}] \end{pmatrix} \end{array},$$

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Theorem: A contraction representation can be converted to a twist representation in $O(n^{\omega})$ time.

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Problem: Given $D_i = (V, \mathcal{F}_i)$, $i = 1, 2$, determine if there exists an $F \in \mathcal{F}_1 \cap \mathcal{F}_2$.

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- D_1: A delta-matroid consisting of the bases of a matroid.
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Based on the contraction representation, an $O(n^\omega)$ time algorithm can be derived.

Given skew-symmetric matrices A_i labeled by $V \cup T_i$, let $D_i = \mathbf{D}(A_i)/T_i$. Define:

$$A_{12} = \begin{array}{c} \\ V_1 \cup T_1 \\ V_2 \cup T_2 \end{array} \begin{array}{c} \begin{array}{cc} V_1 \cup T_1 & V_2 \cup T_2 \end{array} \\ \begin{pmatrix} A_1 & O \\ O & A_2 \end{pmatrix} \end{array} \text{ and } B' = \begin{array}{c} \\ V_1 \\ T_1 \\ V_2 \\ T_2 \end{array} \begin{array}{c} \begin{array}{cccc} V_1 & T_1 & V_2 & T_2 \end{array} \\ \begin{pmatrix} O & O & B & O \\ O & O & O & O \\ -B & O & O & O \\ O & O & O & O \end{pmatrix} \end{array}$$

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Define $A = A_{12} + B'$, then: $\mathrm{Pf}\, A = \sum_{V' \subseteq V} \pm z^{n-|V'|} \cdot \mathrm{Pf}\, A_1[V_1' \cup T_1] \cdot \mathrm{Pf}\, A_2[V_2' \cup T_2] \prod_{v \in V \setminus V'} y_v$.

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$\Rightarrow$ The minimum exponent of z corresponds to the size of the maximum common feasible set.

Theorem: Maximum common feasible set can be found in randomized polynomial time.

Concluding remarks

Other results:

- The delta-sum of linear delta-matroids is also a linear delta-matroid.
- Transformation from projected linear delta-matroids to elementary ones.
- Decision problem $\rightarrow$ search problem (takes $O(n)$ time overhead for maximization).

Open question: Since the proposed method is randomized, is derandomization possible?

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Remark: An FPT paper on delta-matroids on arXiv

FPT algorithms over linear delta-matroids with applications

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Thank you