Faster Edge Coloring by Partition Sieving

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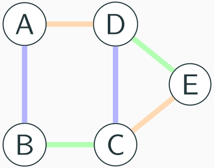
Edge Coloring

Definition: An *edge coloring* of a graph $G = (V, E)$ is an assignment of colors to E so that no two adjacent edges share the same color.



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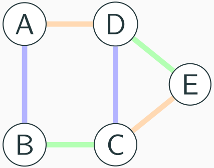

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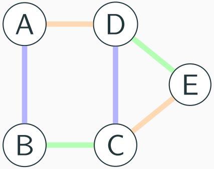

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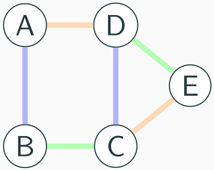

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NP-hardness: Determining the chromatic index is NP-hard even for $\Delta = 3$.

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Implications for Edge Coloring:

- Since vertex coloring can be solved in $O^*(2^n)$ time, the reduction implies that edge coloring can be solved in $O^*(2^m)$ time (but requires exponential space).

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- Since vertex coloring can be solved in $O^*(2^n)$ time, the reduction implies that edge coloring can be solved in $O^*(2^m)$ time (but requires exponential space).
- For bounded-degree graphs, a refined subset convolution technique yields an $O^*(2^{(1-\varepsilon)m})$-time algorithm, where $\varepsilon = 1/2^{\Theta(\Delta)}$.

Motivation: Space

Bjorklund et al. [JCSS 2017] developed randomized polynomial-space solutions for edge coloring:

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Our contribution:

Edge coloring can be solved in randomized $O^*(2^{m-3n/5})$ time and polynomial space.

Outline

Algorithm outline:

- **Step 1. Polynomial construction:**
  Design a polynomial that can be efficiently evaluted.
- **Step 2. Sieving:**
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Talk:

- We first review a simpler $O^*(2^m)$ time algorithm.
- We then discuss our improved algorithm, which refines both steps.

$O^*(2^m)$-time algorithm for edge coloring: Polynomial

Polynomial construction.

Define a variable x_e for each edge $e \in E$; let $X = \{x_e\}_{e \in E}$.

Define a polynomial $P(X)$ over a field $\mathbb{F}$ of characteristic 2:

$$P(X) = \sum_{M_1,\ldots,M_k} \prod_{i=1}^{k} \prod_{e \in M_i} x_e,$$

where $M_1, \ldots, M_k$ are matchings with $|M_1| + \cdots + |M_k| = m$.

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Edge coloring can be reformulated as: Is there a collection of k matchings $M_1, \dots, M_k$ that covers the graph, i.e., $M_1 \cup \cdots \cup M_k = E$?

This is equivalent to checking whether $P(X)$ contains the monomial $\prod_{e \in E} x_e$.

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Polynomial Evaluation.

Given $\mathbf{a} = \{a_e \in \mathbb{F} \mid e \in E\}$, we can evaluate $P(\mathbf{a})$ in polynomial time since $P(X)$ can be expressed as a product of the Pfaffians of the Tutte matrix.

A monomial $x_1^{d_1} x_2^{d_2} \cdots x_n^{d_n}$ is multilinear if each individual degree is at most 1 (i.e., $d_i \leq 1$ for all i).

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For a polynomial $P(X)$ over a field of char. 2, we can determine whether $P(X)$ contains a multilinear monomial of degree ℓ using randomized $O^*(2^\ell)$ evaluations of $P(X)$.

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We test whether $P(X)$ contains a multilinear term of degree m.

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Theorem: Edge coloring can be solved in randomized $O^*(2^m)$ time and poly. space.

Our algorithm: Polynomial

We refine the formulation of $P(X)$ while ensuring efficient evaluation.

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Given $\mathbf{a} = \{a_e \in \mathbb{F} \mid e \in E\}$, we can evaluate $P(\mathbf{a})$ in polynomial time using a generalization of the Cauchy-Binet formula to skew-symmetric matrices (known as the Ishikawa-Wakayama formula).

Our algorithm: Partition Sieving

Let X be a set of variables, and let $P(X)$ be a polynomial.

Let $\mathcal{X} = X_1 \sqcup \cdots \sqcup X_p$ be a partition of X.
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For a polynomial $P(X)$ over a field of char. 2 that is compatible with $(\mathcal{X}, \mathbf{d})$, we can determine whether $P(X)$ contains a multilinear monomial of degree ℓ using randomized $O^*(2^{\ell-p})$ evaluations of $P(X)$.

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This result is based on the *determinantal sieving* framework.

[Eiben, Koana, and Wahlström, SODA 24]

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Final theorem: Edge coloring can be solved in randomized $O^*(2^{m-3n/5})$ time.

Concluding remarks

Additional Results.

- For a d-regular graph, a dominating set of size $(H_{d+1}/(d+1))n$ exists, where H_k is the k-th harmonic number. This implies that edge coloring can be solved in $O^*(2^{m-\frac{H_{d+1}}{d+1}n})$ time, which approaches $O^*(2^{m-n})$ as $d \to \infty$.
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Open Questions.

- Other applications of partition sieving?
- Is there an algorithm for edge coloring running in $O^*(1.9999^m)$ time?
- Can edge coloring be solved in $O^*(2^{m-n})$ time?
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