Karlsruhe Institute of Technology

Forbidden Patterns in Mixed Linear Layouts

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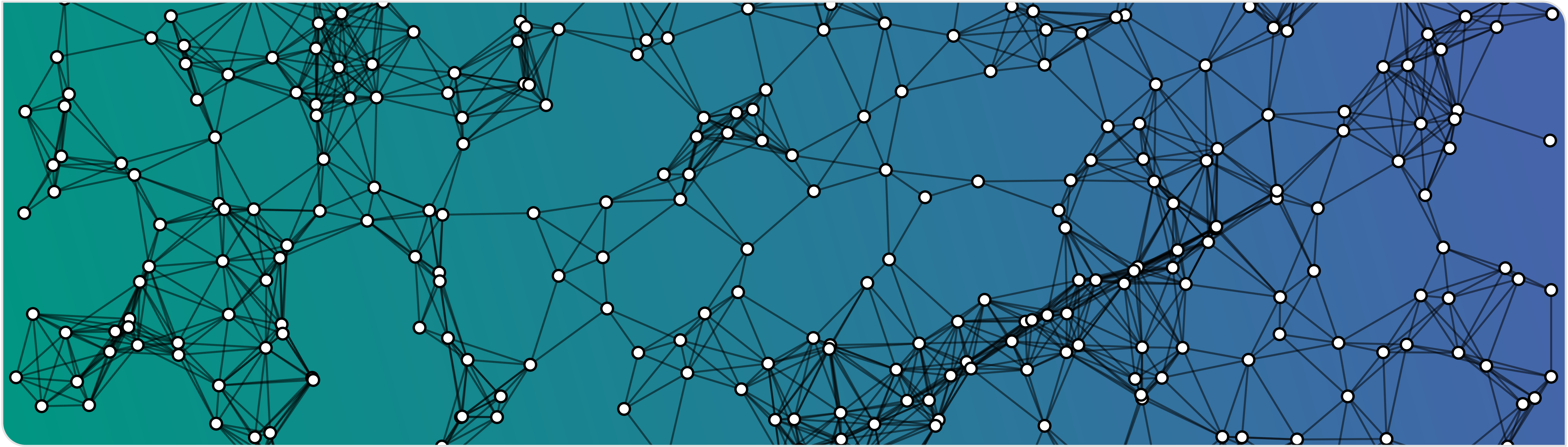
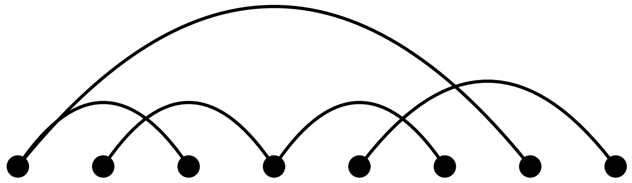
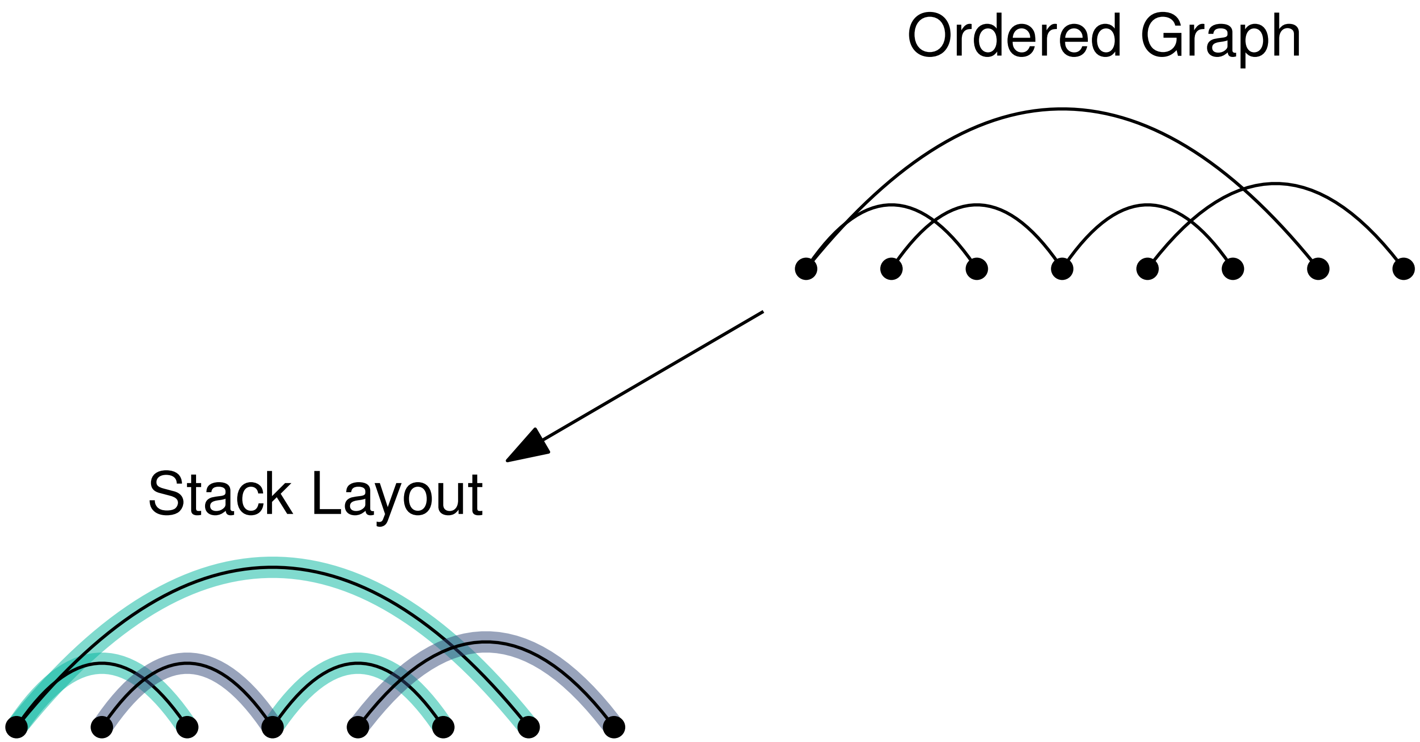


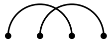
Linear Layouts

Ordered Graph

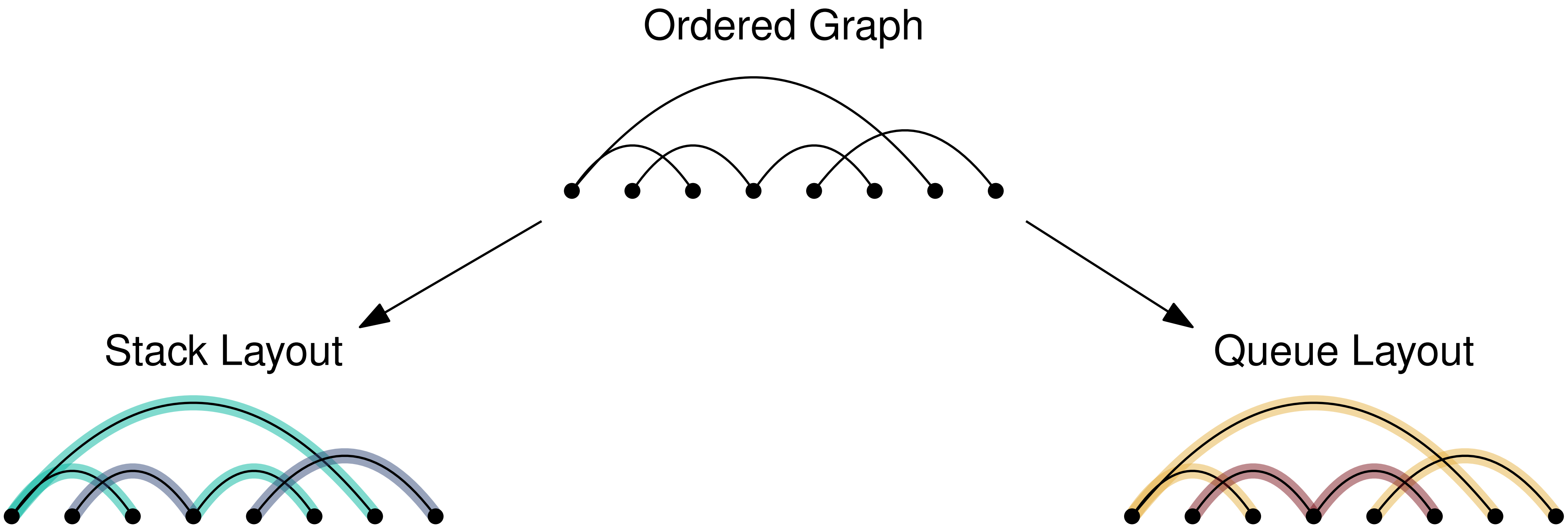


Linear Layouts

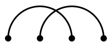


- partition into stacks
  - no  within a stack
- stack number = min. #stacks

Linear Layouts



Stack Layout:

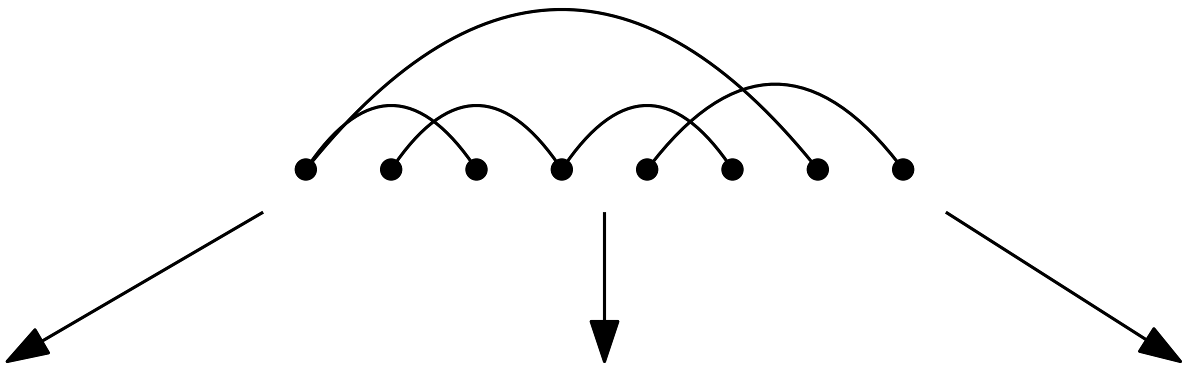
- partition into stacks
  - no  within a stack
- stack number = min. #stacks

Queue Layout:

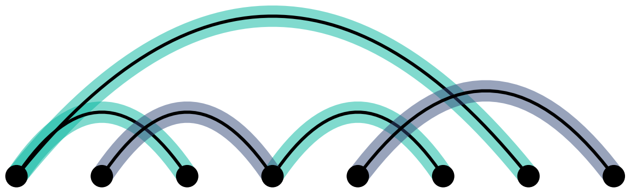
- partition into queues
  - no  within a queue
- queue number = min. #queues

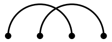
Linear Layouts

Ordered Graph

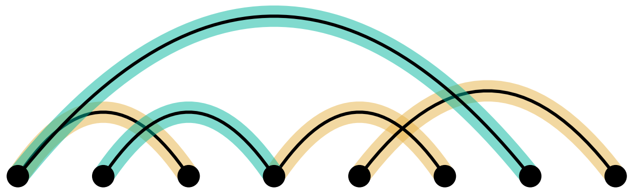


Stack Layout



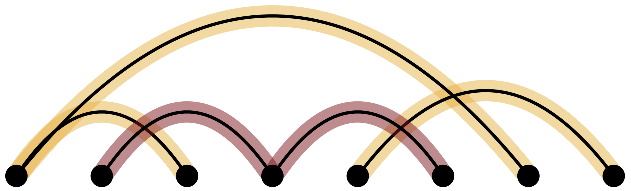
- partition into stacks
  - no  within a stack
- stack number = min. #stacks

Mixed Linear Layout



- partition into pages (stacks or queues)
- mixed page number = min. #pages

Queue Layout



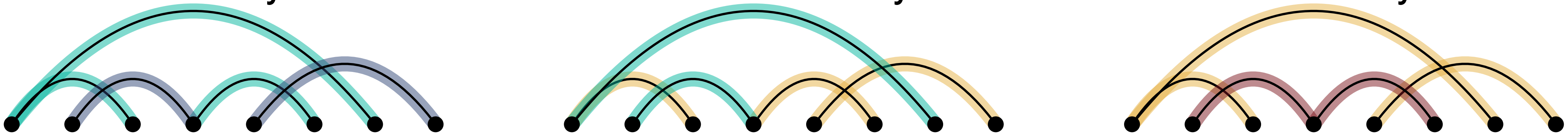
- partition into queues
  - no  within a queue
- queue number = min. #queues

Linear Layouts

Stack Layout

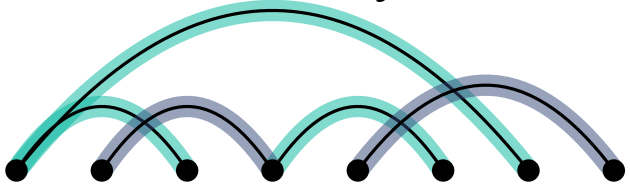
Mixed Linear Layout

Queue Layout

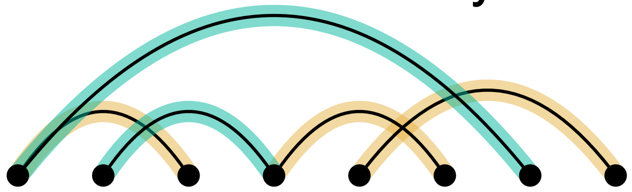


Linear Layouts

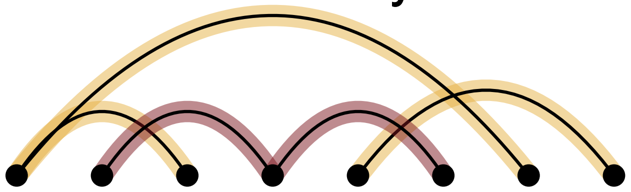
Stack Layout



Mixed Linear Layout

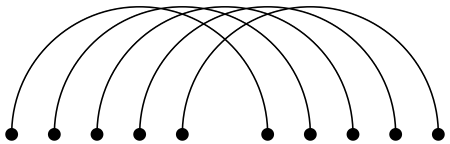


Queue Layout



Characterization via twists:

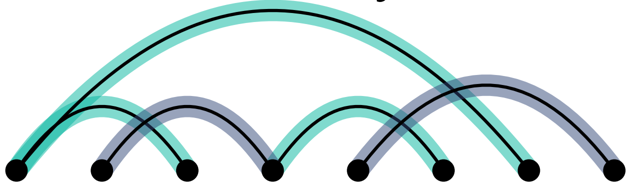
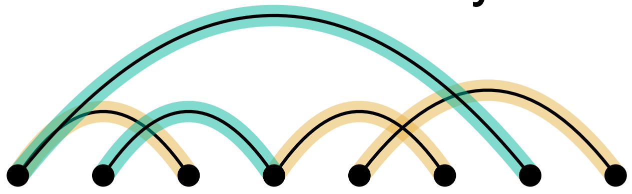
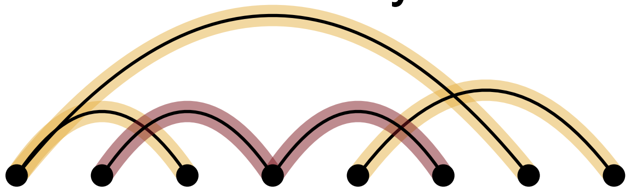
- 5-twist:



- twist number $k =$ size of the largest twist
- $k \leq$ stack number $\leq \mathcal{O}(k \log k)$

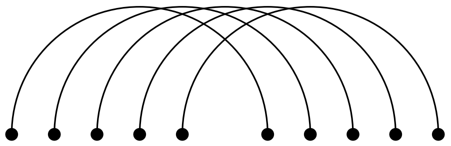
[Davies, 2022]

Linear Layouts

Stack Layout	Mixed Linear Layout	Queue Layout
		

Characterization via twists:

- 5-twist:

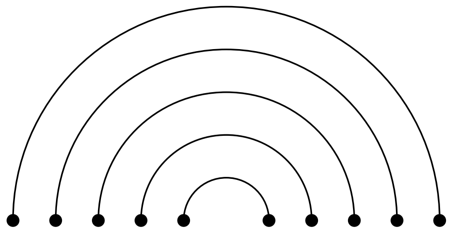


- twist number $k =$ size of the largest twist
- $k \leq$ stack number $\leq \mathcal{O}(k \log k)$

[Davies, 2022]

Characterization via rainbows:

- 5-rainbow:

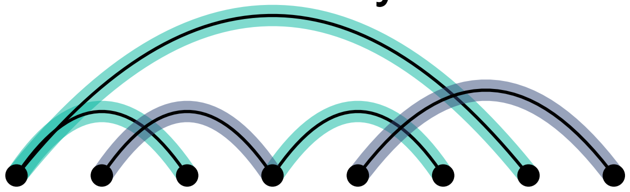


- rainbow number $k =$ size of the largest rainbow
- $k \leq$ queue number $\leq k$

[Heath, Rosenberg, 1992]

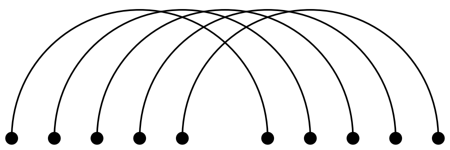
Linear Layouts

Stack Layout



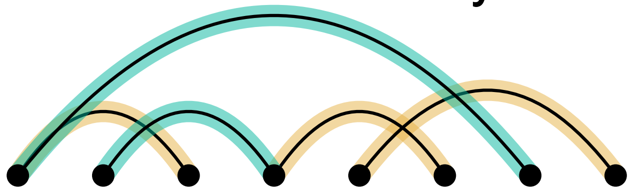
Characterization via twists:

- 5-twist:



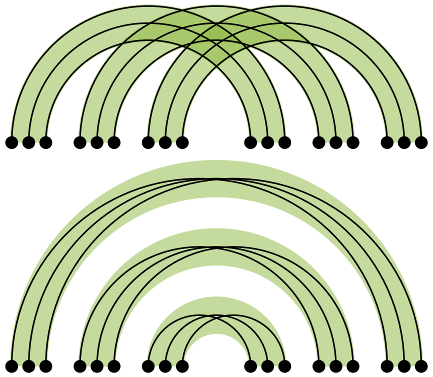
- twist number k = size of the largest twist
- $k \leq$ stack number $\leq \mathcal{O}(k \log k)$ [Davies, 2022]

Mixed Linear Layout



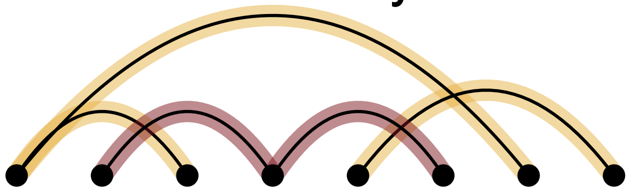
Characterization via thick patterns?

- 3-thick patterns (size 3×3):



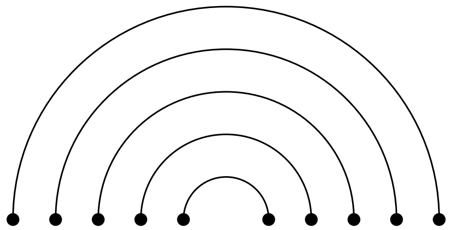
- $k \times k$ = size of the largest thick pattern
- $k \leq$ mixed page number $\overset{?}{\leq} f(k)$

Queue Layout



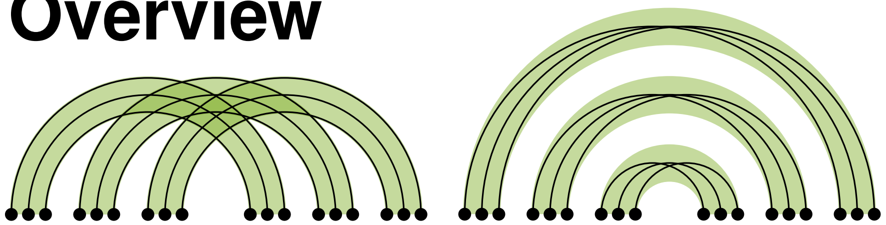
Characterization via rainbows:

- 5-rainbow:

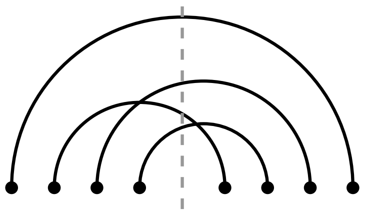
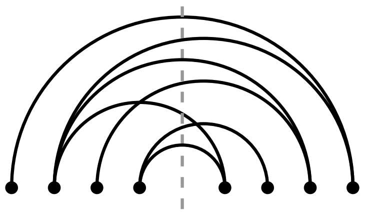
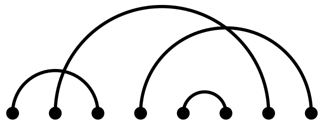
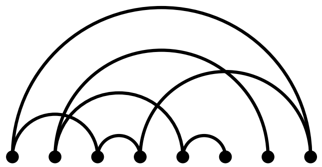


- rainbow number k = size of the largest rainbow
- $k \leq$ queue number $\leq k$ [Heath, Rosenberg, 1992]

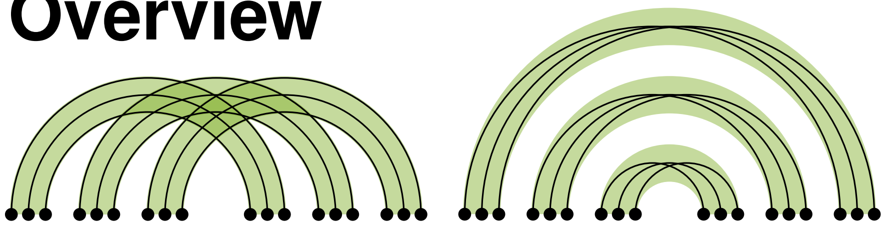
Overview



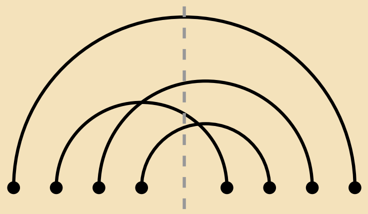
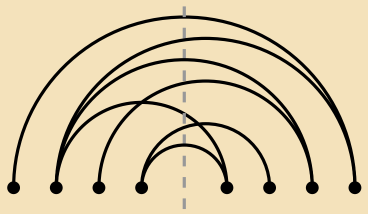
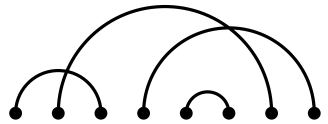
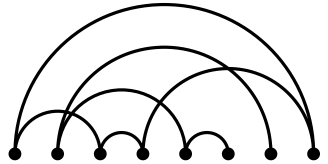
$k \times k =$ size of the largest thick pattern
$\Rightarrow k \leq$ mixed page number $\leq f(k)$?

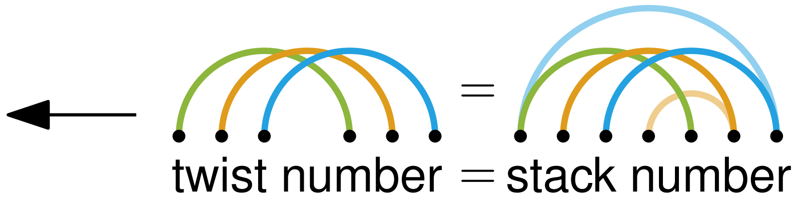
	Matching	Non-matching
Separated		
Non-separated		

Overview

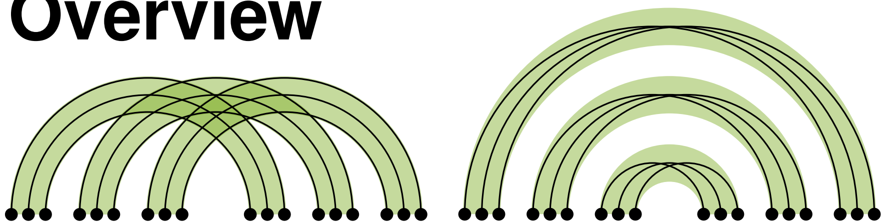


$k \times k =$ size of the largest thick pattern
$\Rightarrow k \leq$ mixed page number $\leq f(k)$?

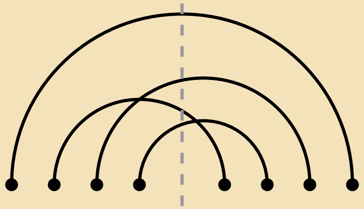
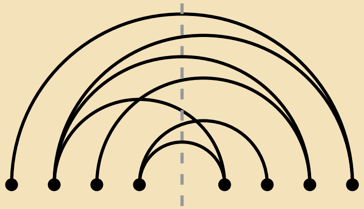
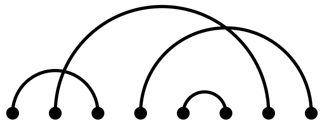
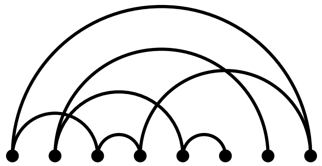
	Matching	Non-matching
Separated		
Non-separated		

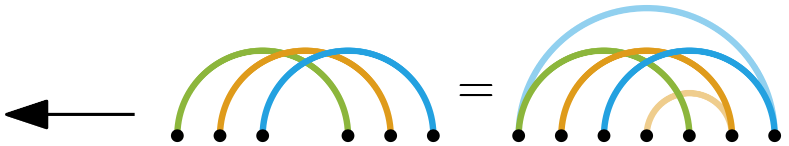


Overview



$k \times k$ = size of the largest thick pattern
$\Rightarrow k \leq$ mixed page number $\leq f(k)$?

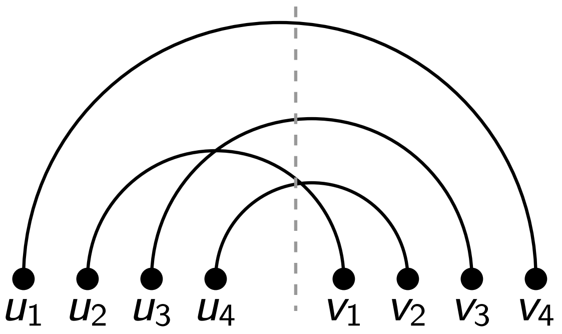
	Matching	Non-matching
Separated		
Non-separated		

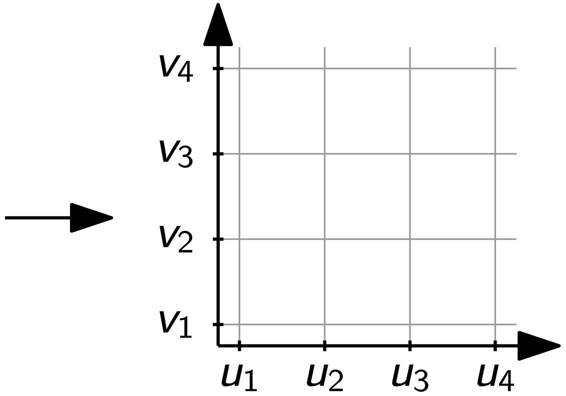


twist number = stack number

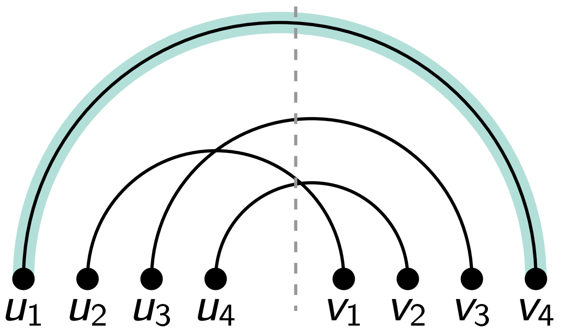
Plan:
- separated matchings
- non-separated matchings
- non-matchings

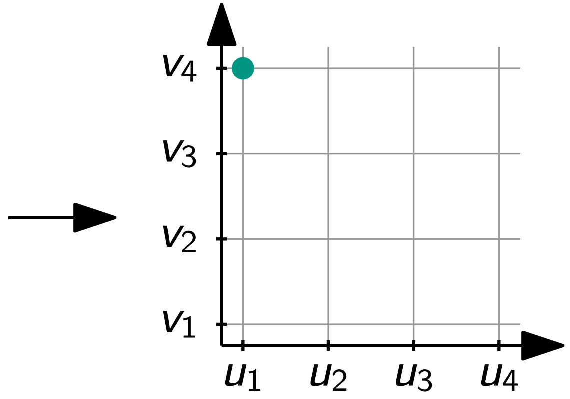
Separated Matchings



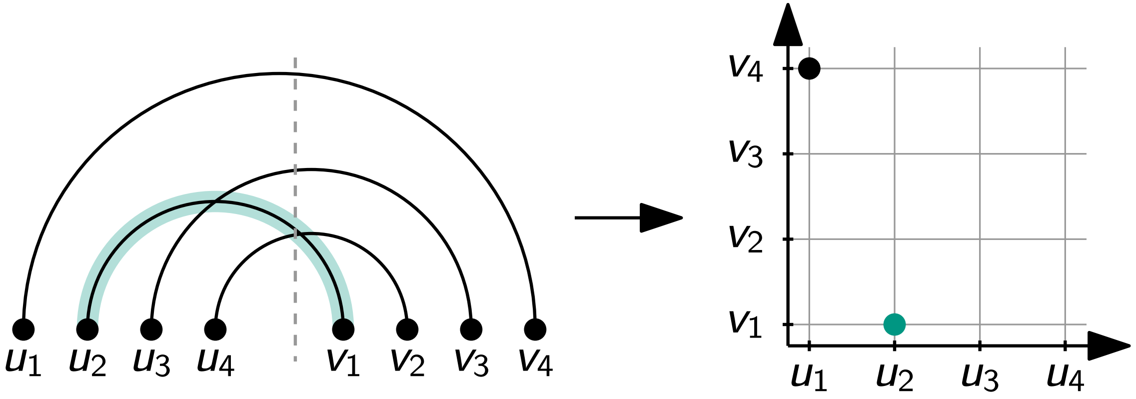


Separated Matchings

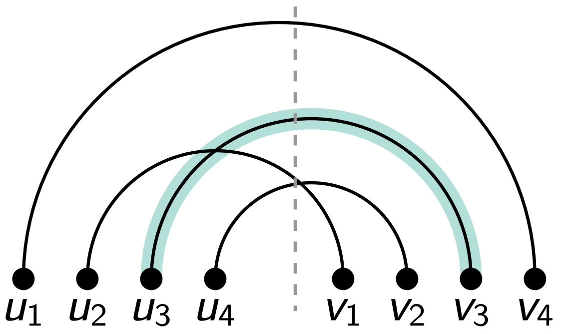


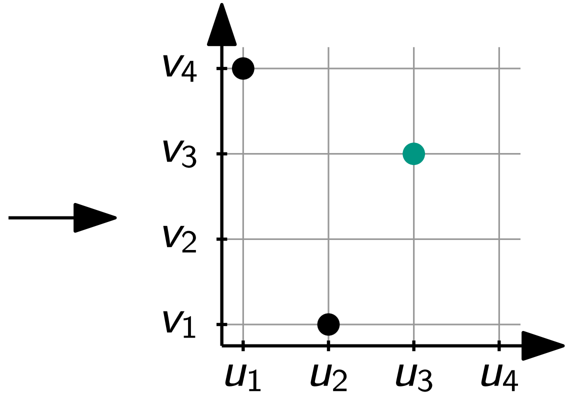


Separated Matchings

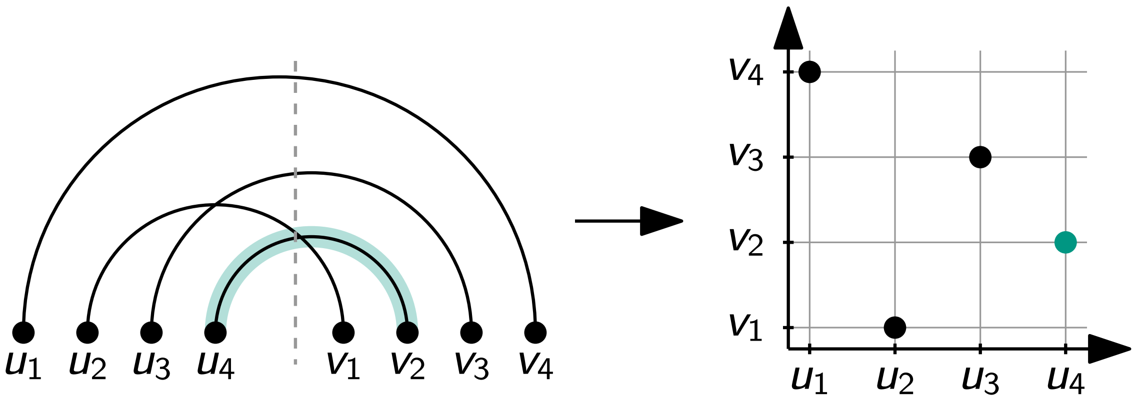


Separated Matchings

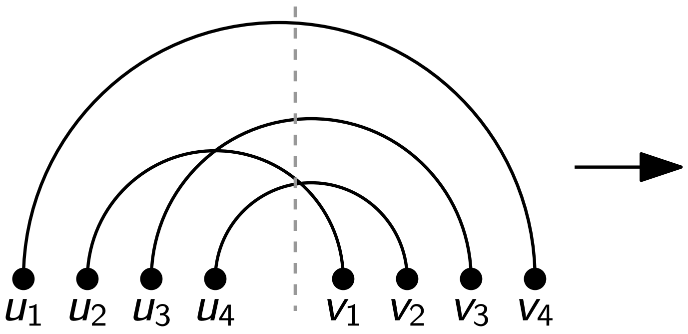


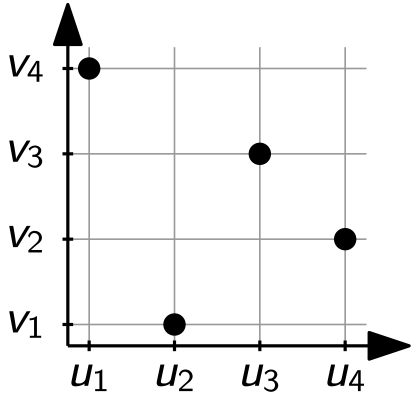


Separated Matchings

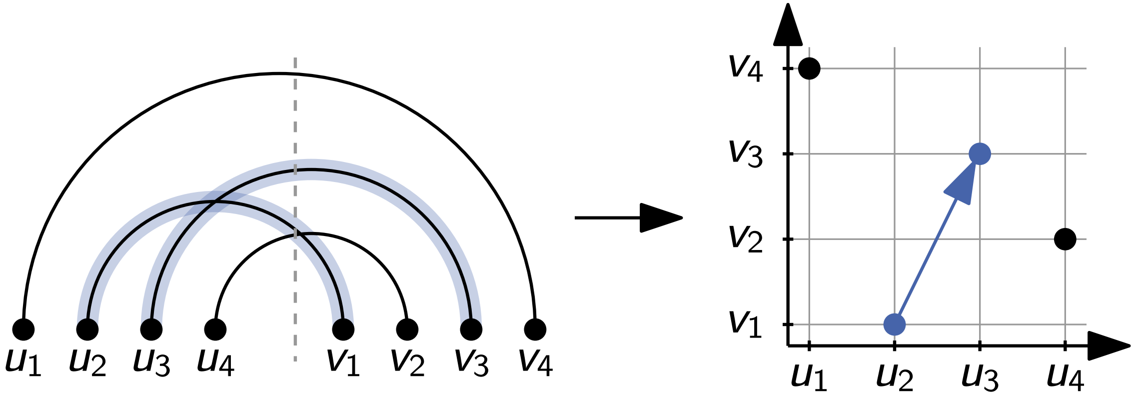


Separated Matchings



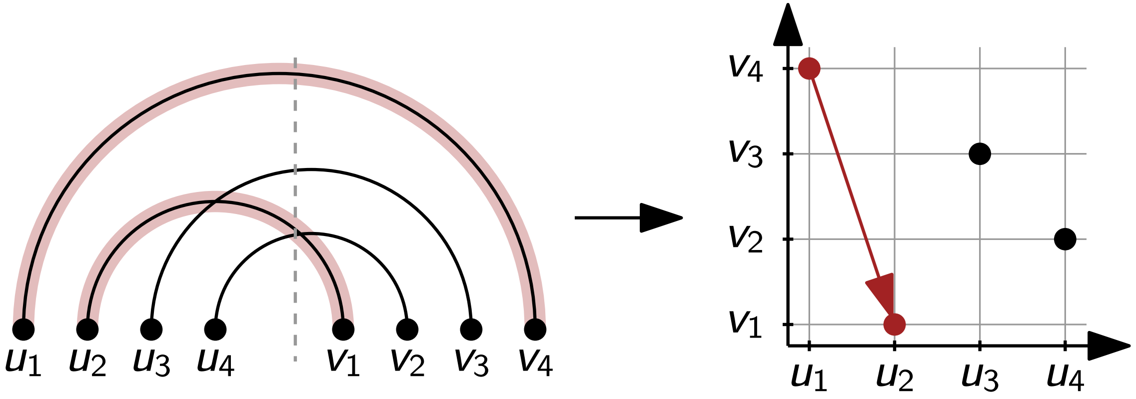


Separated Matchings



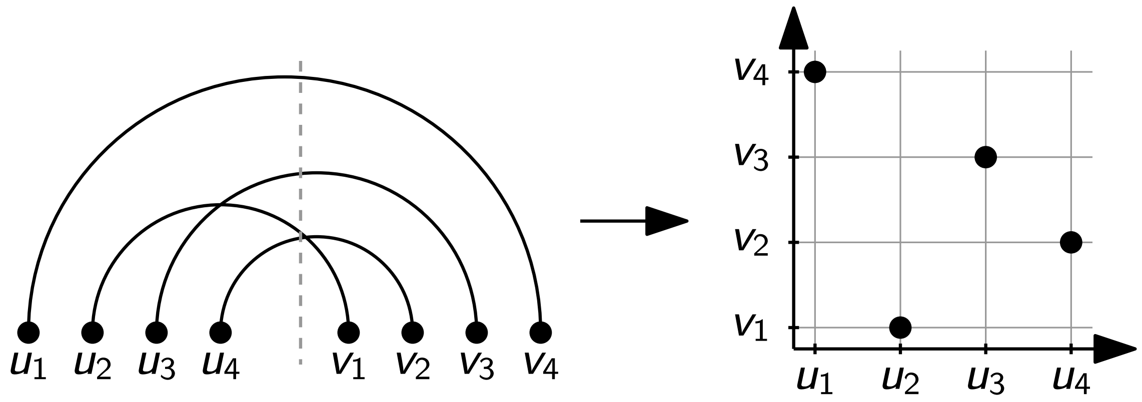
- e_1, e_2 cross $\Leftrightarrow$ $e_1 \nearrow e_2$ or $e_2 \nearrow e_1$

Separated Matchings



- e_1, e_2 cross $\Leftrightarrow$ $e_1 \nearrow e_2$ or $e_2 \nearrow e_1$
- e_1, e_2 nest $\Leftrightarrow$ $e_1 \searrow e_2$ or $e_2 \searrow e_1$

Separated Matchings



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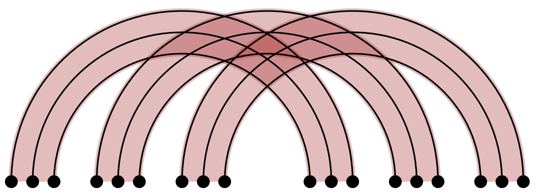
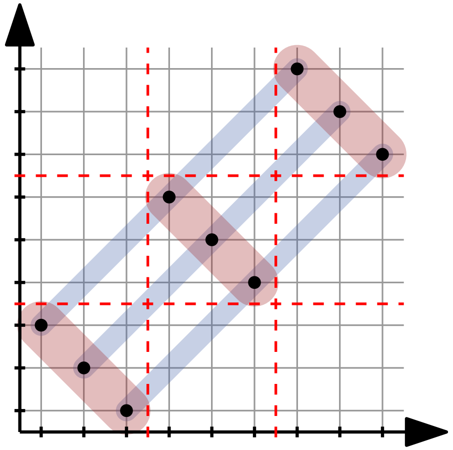
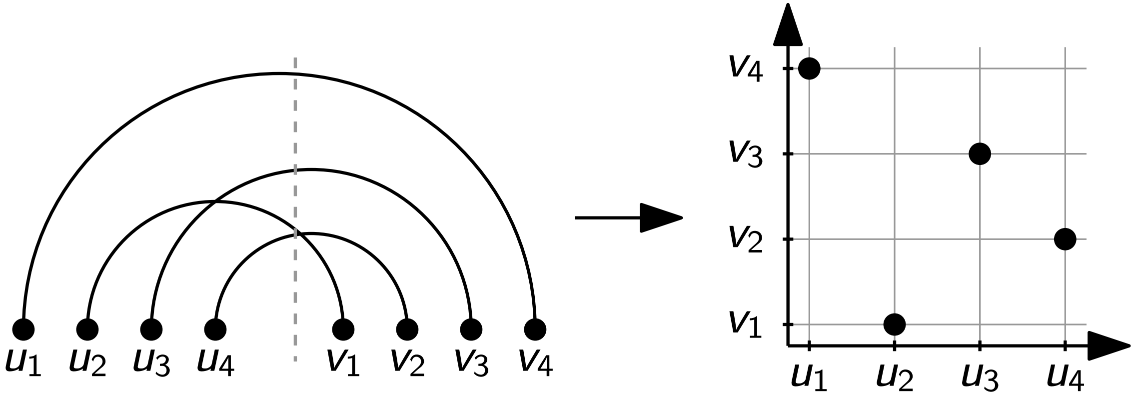
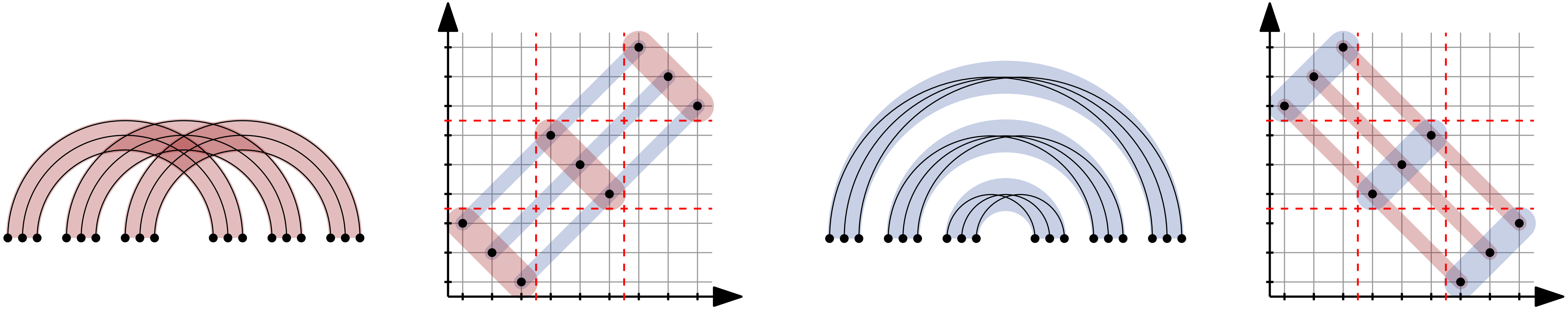




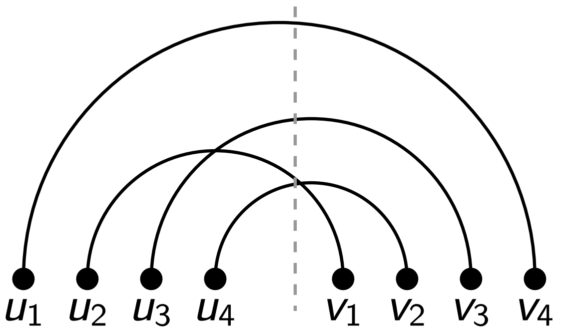
Separated Matchings

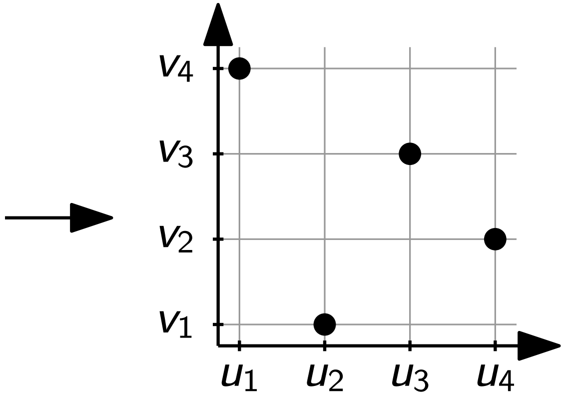


- e_1, e_2 cross $\Leftrightarrow$ $e_1 \nearrow e_2$ or $e_2 \nearrow e_1$
- e_1, e_2 nest $\Leftrightarrow$ $e_1 \searrow e_2$ or $e_2 \searrow e_1$



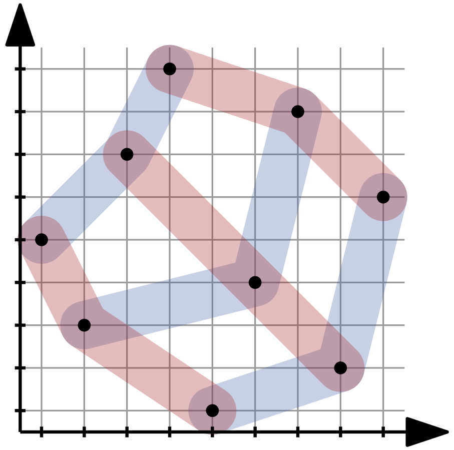
Separated Matchings

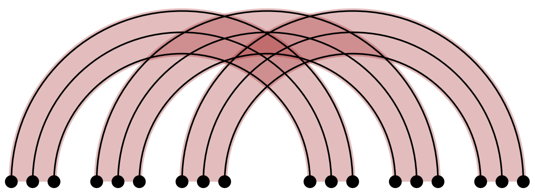


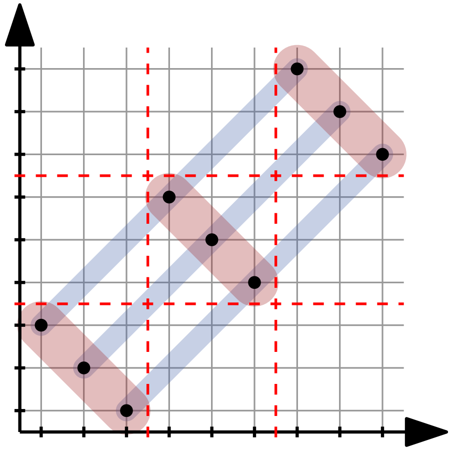


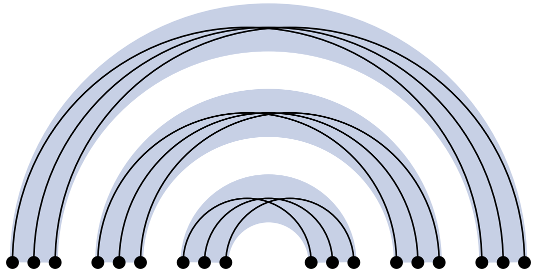
- e_1, e_2 cross $\Leftrightarrow$ $e_1 \nearrow e_2$ or $e_2 \nearrow e_1$
- e_1, e_2 nest $\Leftrightarrow$ $e_1 \searrow e_2$ or $e_2 \searrow e_1$

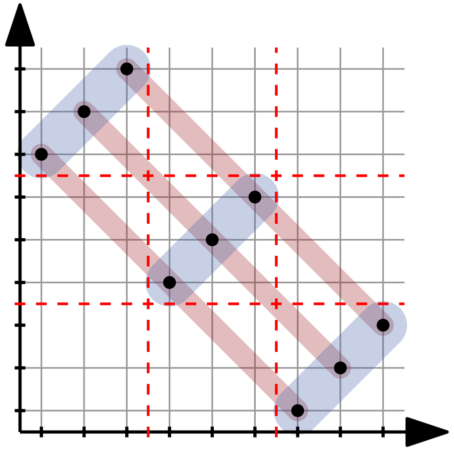
3-⊗-pattern (size 3×3):



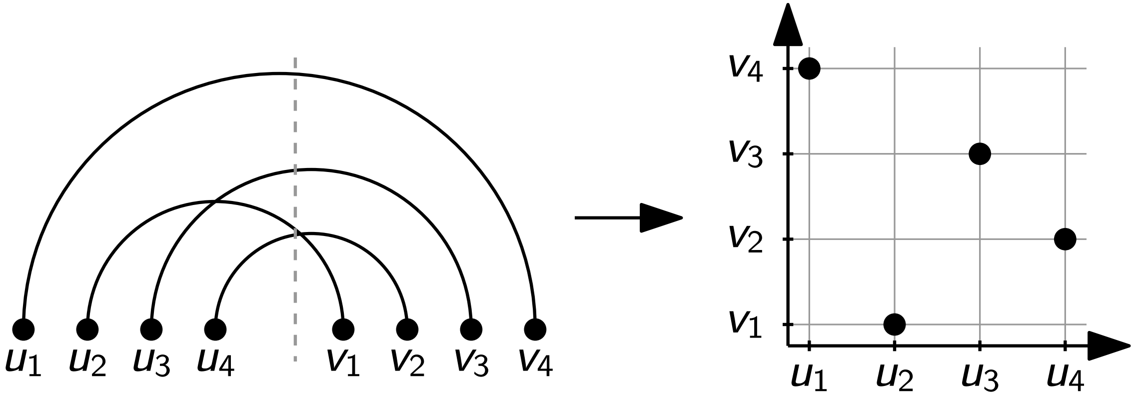






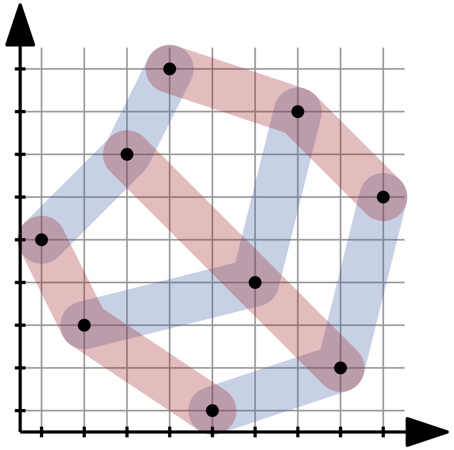


Separated Matchings



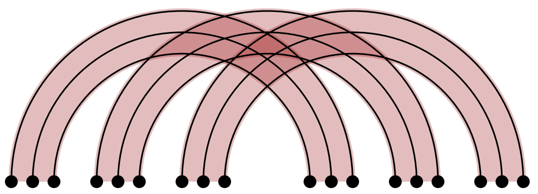
- e_1, e_2 cross $\Leftrightarrow$ $e_1 \nearrow e_2$ or $e_2 \nearrow e_1$
- e_1, e_2 nest $\Leftrightarrow$ $e_1 \searrow e_2$ or $e_2 \searrow e_1$

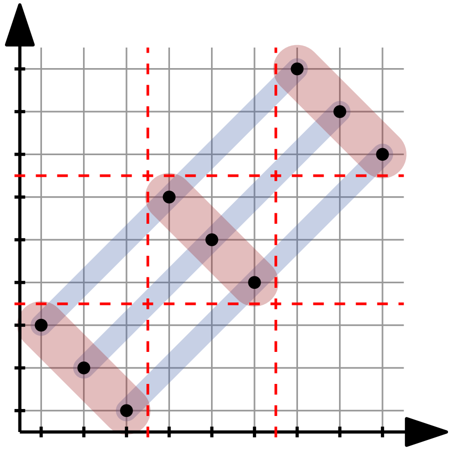
3-$\diamondplus$-pattern (size 3×3):

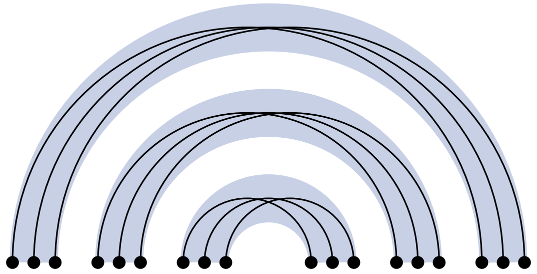


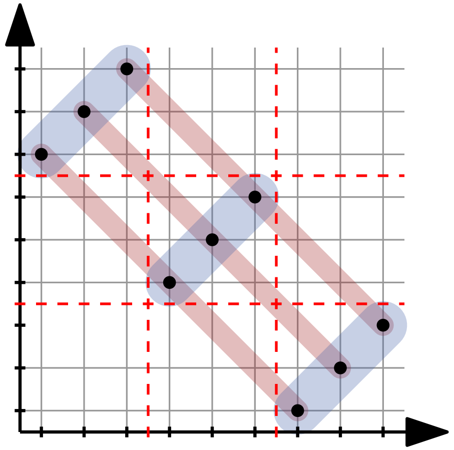
New goal:

- large $\diamondplus$-pattern $\Leftrightarrow$ large thick pattern
- $k \times k =$ size of the largest $\diamondplus$-pattern $\Rightarrow$ mixed page number $\leq f(k)$

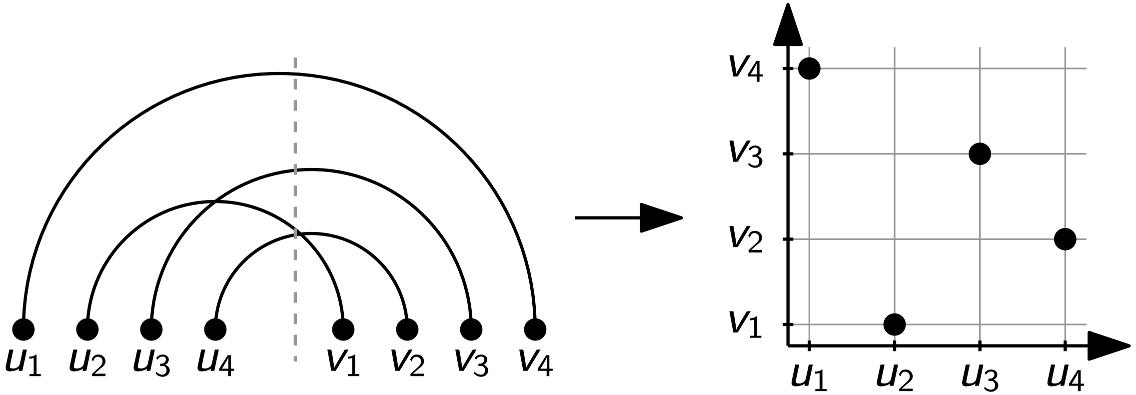








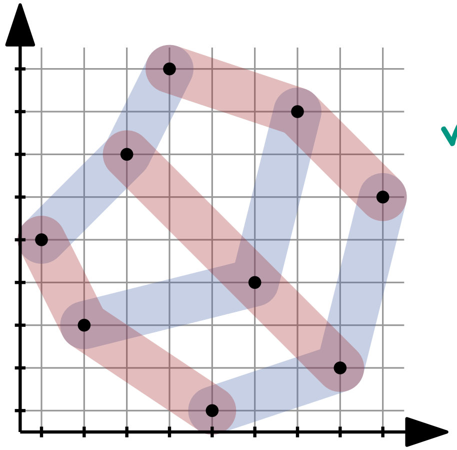
Separated Matchings



- e_1, e_2 cross $\Leftrightarrow$ $e_1 \nearrow e_2$ or $e_2 \nearrow e_1$
- e_1, e_2 nest $\Leftrightarrow$ $e_1 \searrow e_2$ or $e_2 \searrow e_1$

- Every k-thick pattern is a k-◈-pattern.
- Every k^7-◈-pattern contains a k-thick pattern.

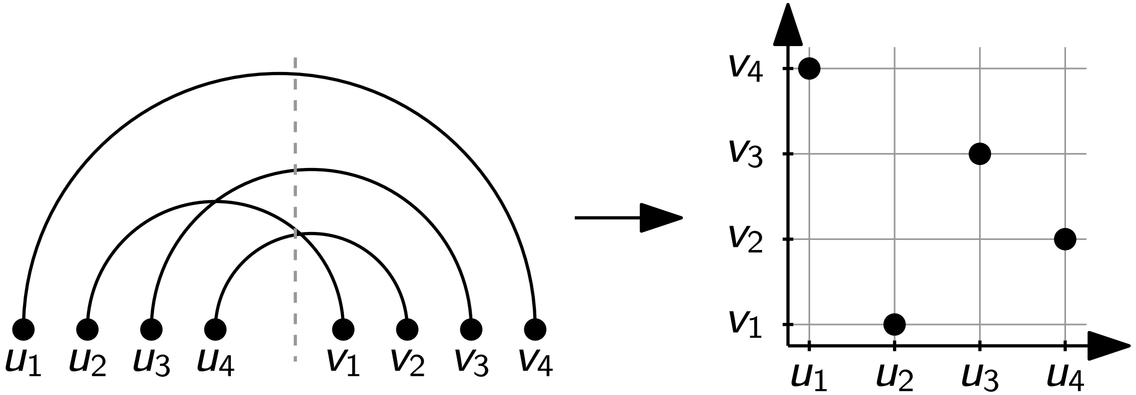
3-◈-pattern (size 3×3):



New goal:

- ✓ large ◈-pattern $\Leftrightarrow$ large thick pattern
- $k \times k$ = size of the largest ◈-pattern $\Rightarrow$ mixed page number $\leq f(k)$

Separated Matchings

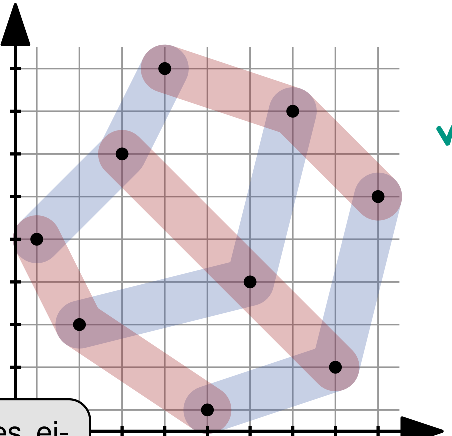


- e_1, e_2 cross $\Leftrightarrow$ $e_1 \nearrow e_2$ or $e_2 \nearrow e_1$ $\Leftrightarrow: e_1 \prec e_2$ or $e_2 \prec e_1$
- e_1, e_2 nest $\Leftrightarrow$ $e_1 \searrow e_2$ or $e_2 \searrow e_1$ $\Leftrightarrow: e_1 || e_2$

Each pair of edges either crosses or nests.

- Every k-thick pattern is a k-⨳-pattern.
- Every k^7-⨳-pattern contains a k-thick pattern.

3-⨳-pattern (size 3×3):



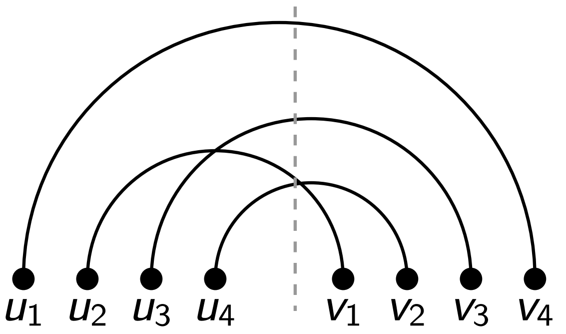
New goal:

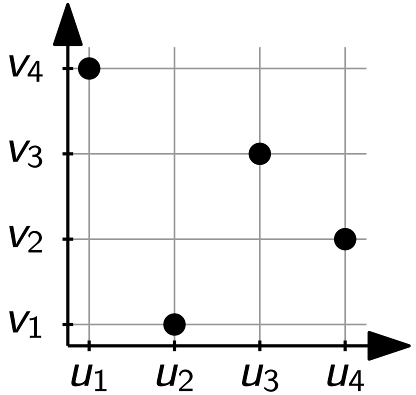
- ✓ large ⨳-pattern $\Leftrightarrow$ large thick pattern
- $k \times k$ = size of the largest ⨳-pattern $\Rightarrow$ mixed page number $\leq f(k)$

$\Rightarrow$ **P**artially **o**rdered **set** (Poset)

- partition into stacks and queues $\Leftrightarrow$ chain-/antichain-decomposition

Separated Matchings



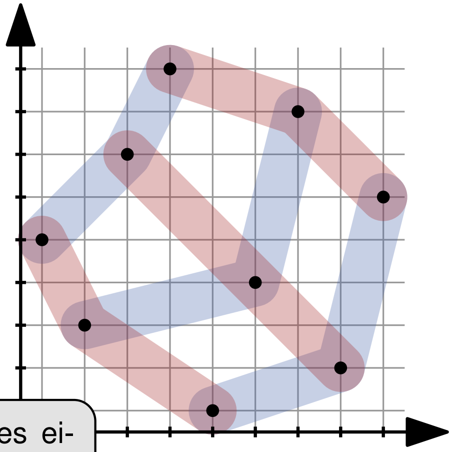


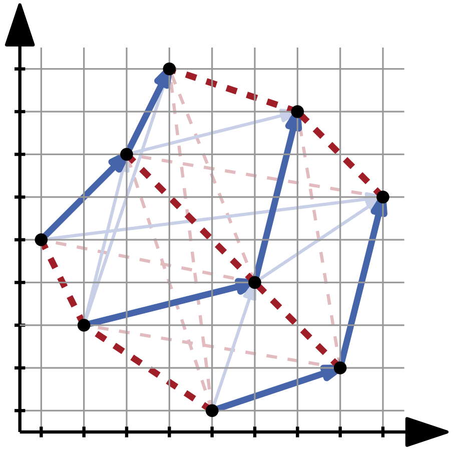
- e_1, e_2 cross ⇔ $e_1 \nearrow e_2$ or $e_2 \nearrow e_1$ ⇔: $e_1 \prec e_2$ or $e_2 \prec e_1$
- e_1, e_2 nest ⇔ $e_1 \searrow e_2$ or $e_2 \searrow e_1$ ⇔: $e_1 || e_2$

Each pair of edges either crosses or nests.

- Every k-thick pattern is a k-◈-pattern.
- Every k^7-◈-pattern contains a k-thick pattern.

3-◈-pattern (size 3 × 3):

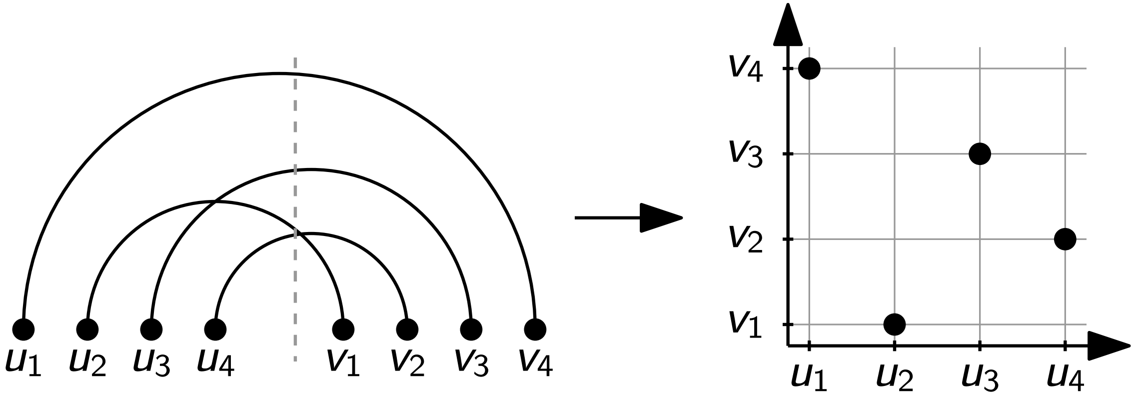




⇒ **P**artially **o**rdered **set** (Poset)

- partition into stacks and queues
  ⇔ chain-/antichain-decomposition

Separated Matchings

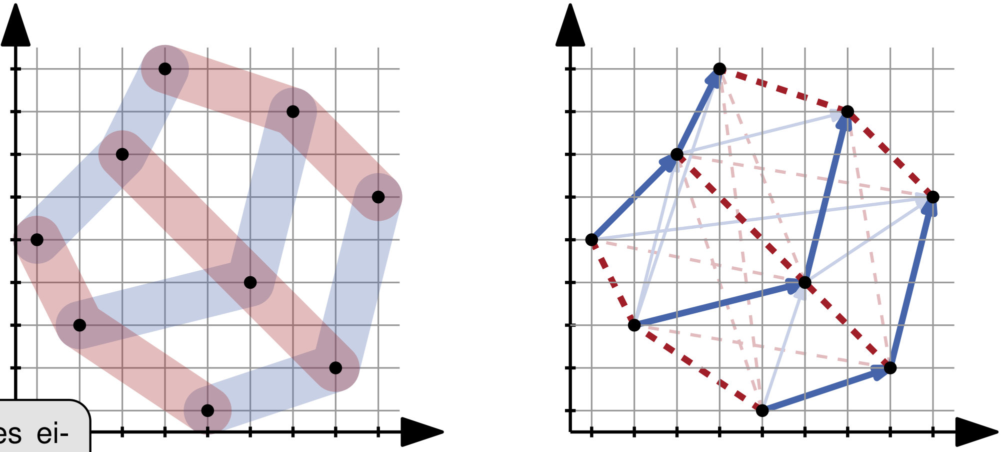


- e_1, e_2 cross $\Leftrightarrow$ $e_1 \nearrow e_2$ or $e_2 \nearrow e_1$ $\Leftrightarrow$: $e_1 \prec e_2$ or $e_2 \prec e_1$
- e_1, e_2 nest $\Leftrightarrow$ $e_1 \searrow e_2$ or $e_2 \searrow e_1$ $\Leftrightarrow$: $e_1 || e_2$

Each pair of edges either crosses or nests.

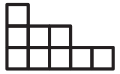
- Every k-thick pattern is a k-$\diamondplus$-pattern.
- Every k^7-$\diamondplus$-pattern contains a k-thick pattern.

3-$\diamondplus$-pattern (size 3×3):

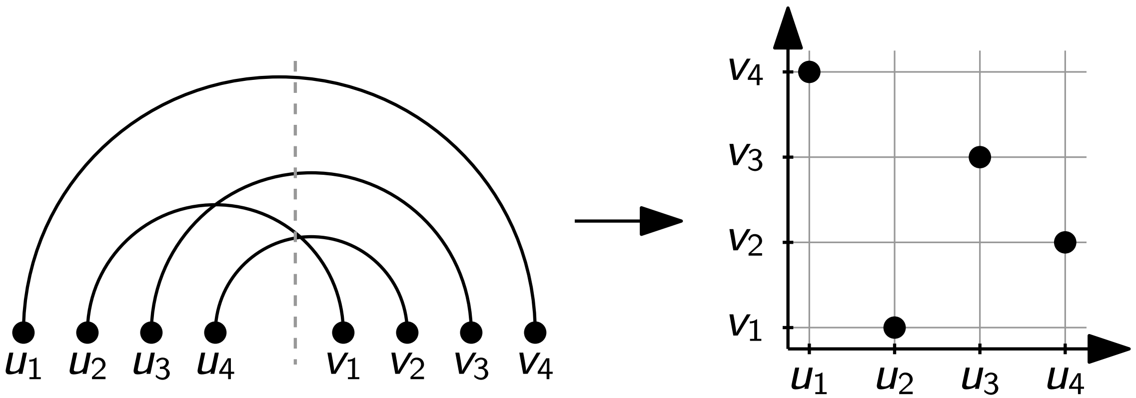


$\Rightarrow$ **P**artially **o**rdered **set** (Poset)

- partition into stacks and queues
  $\Leftrightarrow$ chain-/antichain-decomposition
- Ferrer's diagrams [Greene, 1976]
  - count the number of chains and antichains needed for a decomposition



Separated Matchings



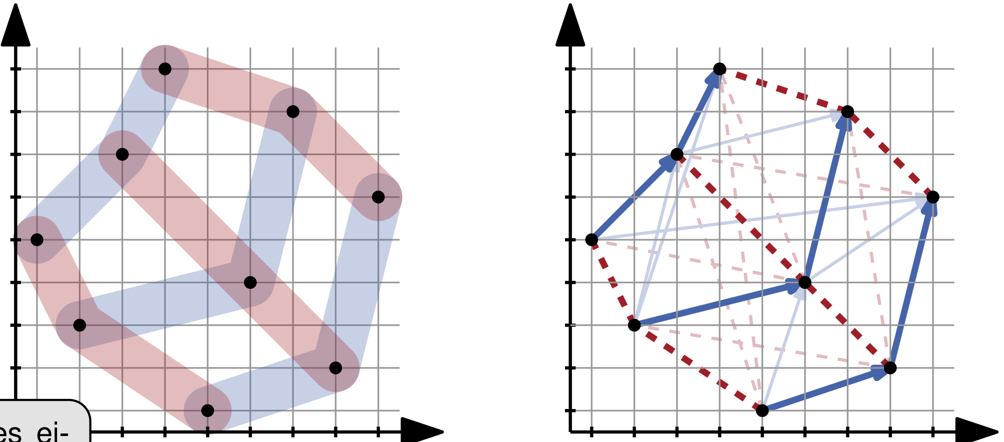
- e_1, e_2 cross $\Leftrightarrow$ $e_1 \nearrow e_2$ or $e_2 \nearrow e_1$ $\Leftrightarrow$: $e_1 \prec e_2$ or $e_2 \prec e_1$
- e_1, e_2 nest $\Leftrightarrow$ $e_1 \searrow e_2$ or $e_2 \searrow e_1$ $\Leftrightarrow$: $e_1 || e_2$

Each pair of edges either crosses or nests.

- Every k-thick pattern is a k-$\otimes$-pattern.
- Every k^7-$\otimes$-pattern contains a k-thick pattern.

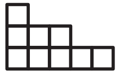
$k \times k =$ size of the largest $\otimes$-pattern
$\Rightarrow k \leq$ mixed page number $\leq 2k$

3-$\otimes$-pattern (size 3×3):

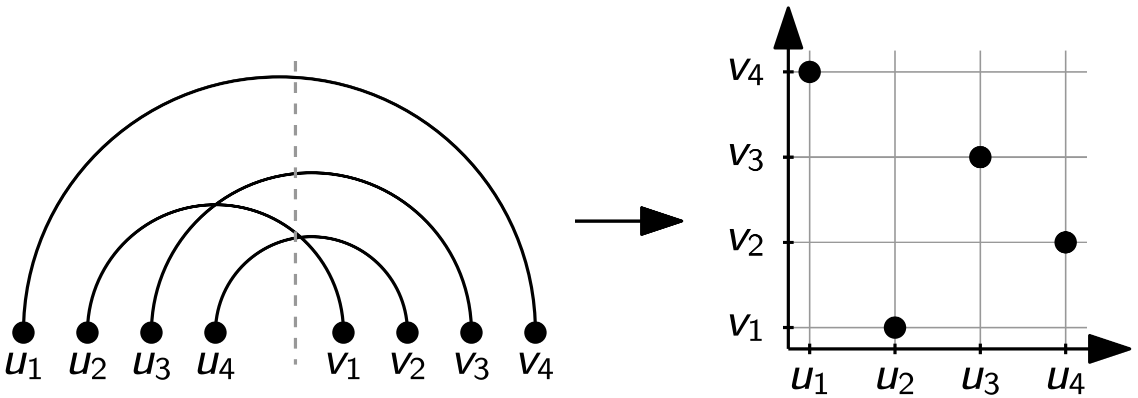


$\Rightarrow$ **P**artially **o**rdered **set** (Poset)

- partition into stacks and queues
  $\Leftrightarrow$ chain-/antichain-decomposition
- Ferrer's diagrams [Greene, 1976]
  - count the number of chains and antichains needed for a decomposition



Separated Matchings



- e_1, e_2 cross $\Leftrightarrow$ $e_1 \nearrow e_2$ or $e_2 \nearrow e_1$ $:\Leftrightarrow$ $e_1 \prec e_2$ or $e_2 \prec e_1$
- e_1, e_2 nest $\Leftrightarrow$ $e_1 \searrow e_2$ or $e_2 \searrow e_1$ $:\Leftrightarrow$ $e_1 || e_2$

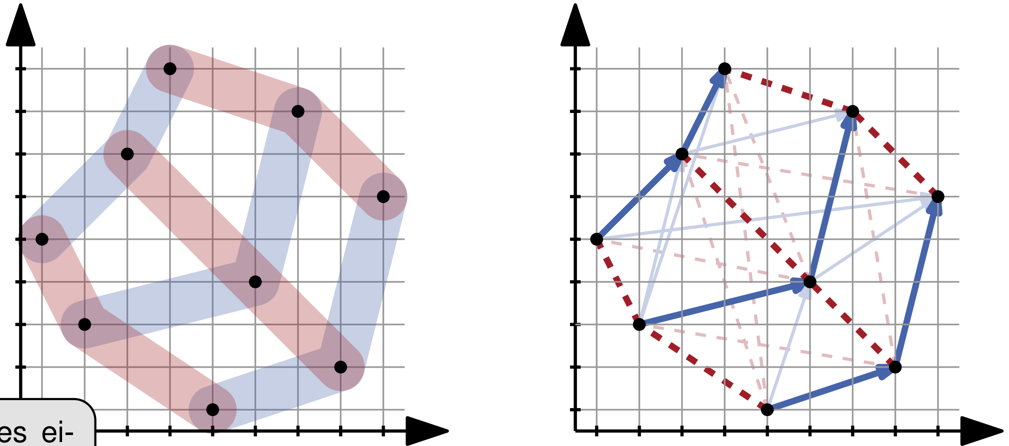
Each pair of edges either crosses or nests.

- Every k-thick pattern is a k-⨳-pattern.
- Every k^7-⨳-pattern contains a k-thick pattern.

$k \times k$ = size of the largest ⨳-pattern
$\Rightarrow k \leq$ mixed page number $\leq 2k$

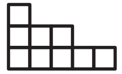
$k \times k$ = size of the largest thick pattern
$\Rightarrow k \leq$ mixed page number $\leq 2k^7$

3-⨳-pattern (size 3×3):

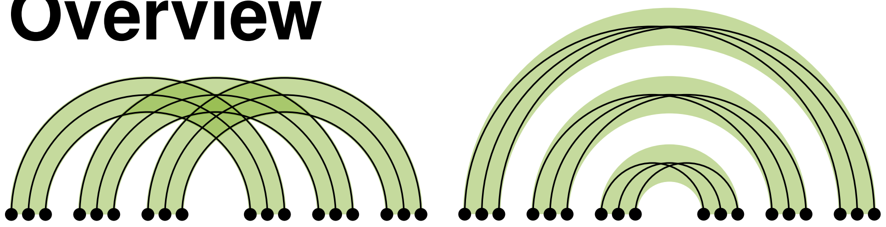


$\Rightarrow$ **P**artially **o**rdered **set** (Poset)

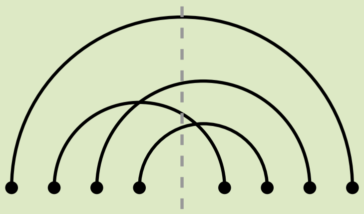
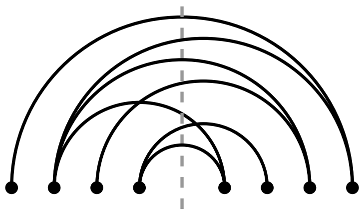
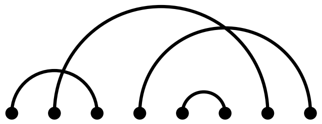
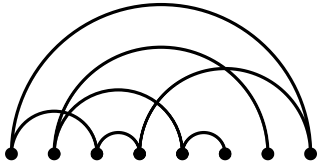
- partition into stacks and queues
  $\Leftrightarrow$ chain-/antichain-decomposition
- Ferrer's diagrams [Greene, 1976]
  - count the number of chains and antichains needed for a decomposition



Overview



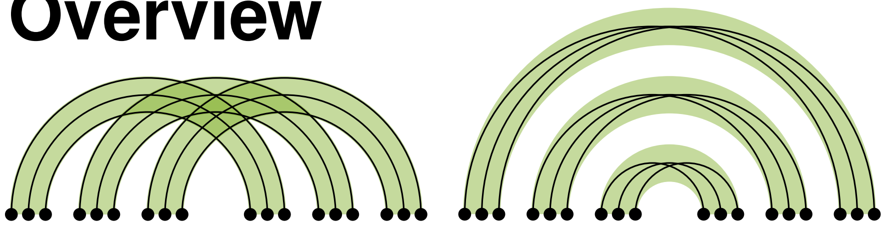
$k \times k =$ size of the largest thick pattern
$\Rightarrow k \leq$ mixed page number $\leq f(k)$?

	Matching	Non-matching
Separated		
Non-separated		

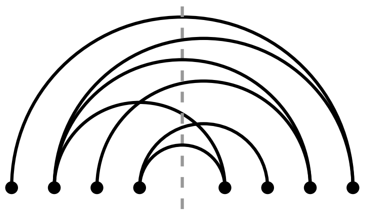
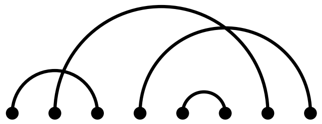
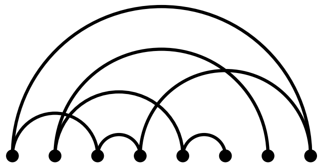
Plan:

- separated matchings
- non-separated matchings
- non-matchings

Overview

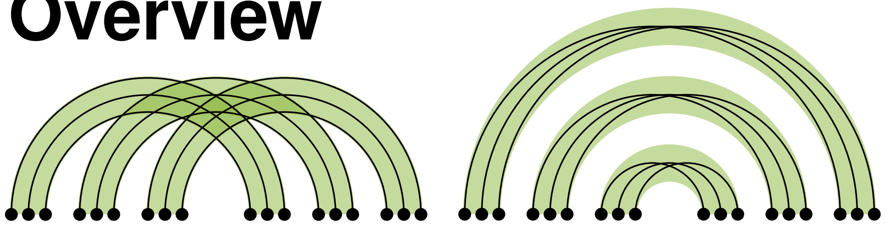


$k \times k =$ size of the largest thick pattern
$\Rightarrow k \leq$ mixed page number $\leq f(k)$?

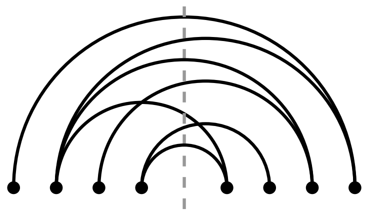
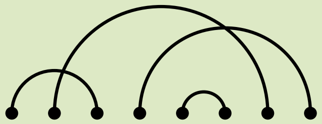
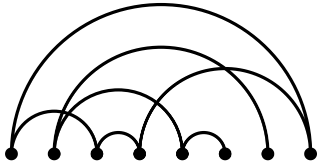
	Matching	Non-matching
Separated	$f(k) \leq 2k^7$	
Non-separated		

Plan:
- separated matchings ✓
- non-separated matchings
- non-matchings

Overview

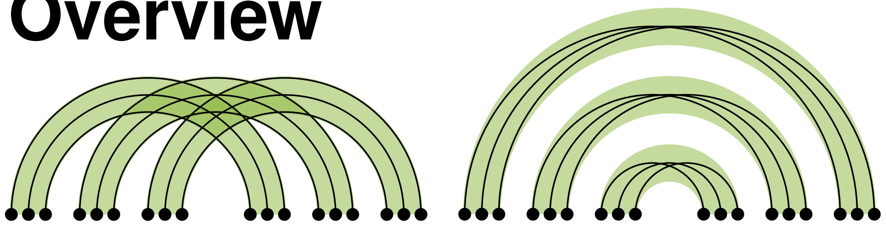


$k \times k =$ size of the largest thick pattern
$\Rightarrow k \leq$ mixed page number $\leq f(k)$?

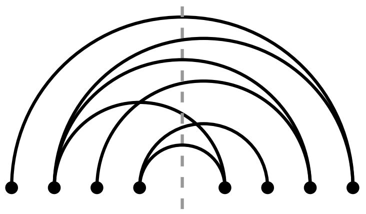
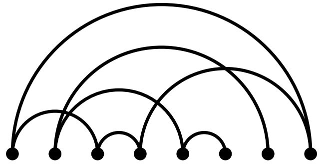
	Matching	Non-matching
Separated	$f(k) \leq 2k^7$	
Non-separated		

Plan:
- separated matchings ✓
- non-separated matchings
- non-matchings

Overview

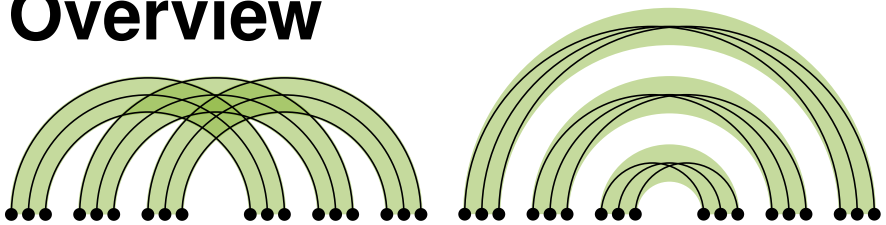


$k \times k =$ size of the largest thick pattern
$\Rightarrow k \leq$ mixed page number $\leq f(k)$?

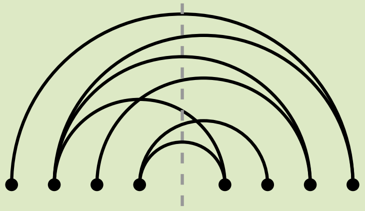
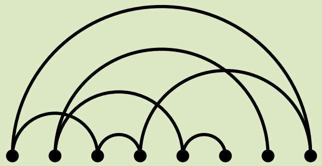
	Matching	Non-matching
Separated	$f(k) \leq 2k^7$ Quotients ↓	
Non-separated	$f(k) \leq k^{\mathcal{O}(k)}$	

Plan:
- separated matchings ✓
- non-separated matchings ✓
- non-matchings

Overview



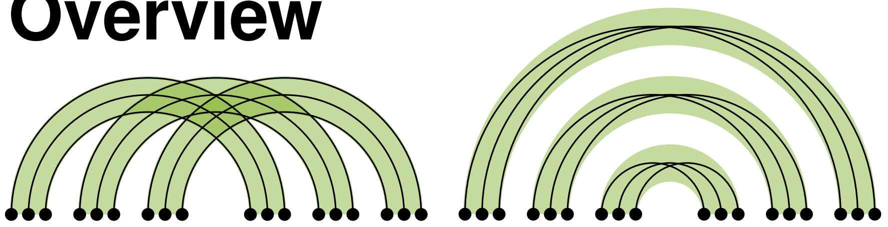
$k \times k =$ size of the largest thick pattern
$\Rightarrow k \leq$ mixed page number $\leq f(k)$?

	Matching	Non-matching
Separated	$f(k) \leq 2k^7$	
Non-separated	$f(k) \leq k^{\mathcal{O}(k)}$	

Quotients

Plan:
- separated matchings ✓
- non-separated matchings ✓
- non-matchings

Overview

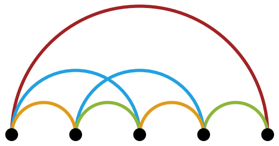


$k \times k =$ size of the largest thick pattern
$\Rightarrow k \leq$ mixed page number $\leq f(k)$?

	Matching	Non-matching
Separated	$f(k) \leq 2k^7$ — Vizing's → ; Quotients ↓	$f(k, \Delta) \leq 2(\Delta + 1)k^7$
Non-separated	$f(k) \leq k^{\mathcal{O}(k)}$ — Vizing's →	$f(k, \Delta) \leq \Delta k^{\mathcal{O}(k)}$

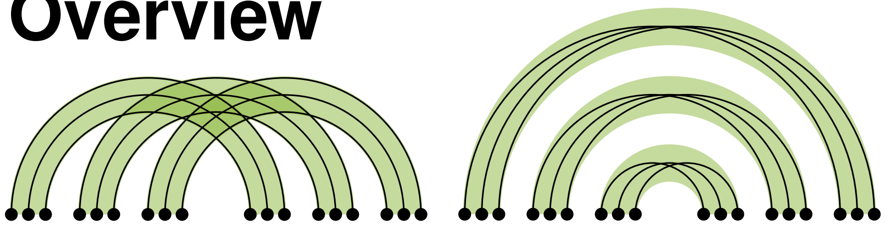
$\Delta =$ max. degree

Vizing's theorem [Vizing, 1965]
Every graph with max. degree Δ has a proper edge coloring with at most $\Delta + 1$ colors.



Plan:
- separated matchings ✓
- non-separated matchings ✓
- non-matchings (✓)

Overview



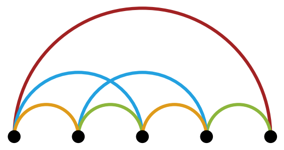
$k \times k =$ size of the largest thick pattern
$\Rightarrow k \leq$ mixed page number $\leq f(k)$?

	Matching	Non-matching
Separated	$f(k) \leq 2k^7$ $f(k) \geq k^2$ Vizing's → Quotients ↓	$f(k, \Delta)$ $\leq 2(\Delta + 1)k^7$
Non-separated	$f(k) \leq k^{\mathcal{O}(k)}$ $f(k) \geq k^2$ Vizing's →	$f(k, \Delta) \leq \Delta k^{\mathcal{O}(k)}$

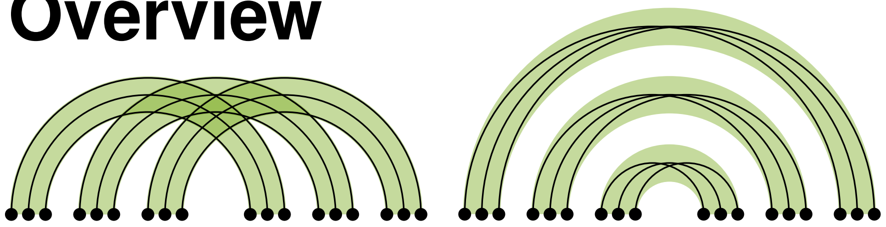
$\Delta =$ max. degree

Plan:
- separated matchings ✓
- non-separated matchings ✓
- non-matchings (✓)

Vizing's theorem [Vizing, 1965]
Every graph with max. degree Δ has a proper edge coloring with at most $\Delta + 1$ colors.



Overview



$k \times k$ = size of the largest thick pattern
$\Rightarrow k \leq$ mixed page number $\leq f(k)$?

some other patterns

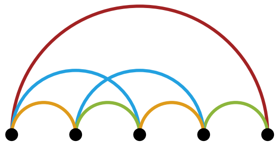
$k \times k$ = size of the largest pattern
$\Rightarrow$ mixed page number $= k$?

	Matching	Non-matching
Separated	$f(k) \leq 2k^7$ $f(k) \geq k^2$ Vizing's → Quotients ↓	$f(k, \Delta)$ $\leq 2(\Delta + 1)k^7$
Non-separated	$f(k) \leq k^{\mathcal{O}(k)}$ $f(k) \geq k^2$ Vizing's →	$f(k, \Delta) \leq \Delta k^{\mathcal{O}(k)}$

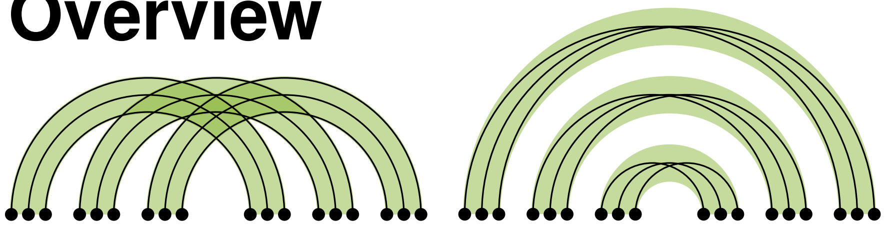
Δ = max. degree

Plan:
- separated matchings ✓
- non-separated matchings ✓
- non-matchings (✓)

Vizing's theorem [Vizing, 1965]
Every graph with max. degree Δ has a proper edge coloring with at most $\Delta + 1$ colors.



Overview



$k \times k$ = size of the largest thick pattern
$\Rightarrow k \leq$ mixed page number $\leq f(k)$?

some other patterns

$k \times k$ = size of the largest pattern
$\Rightarrow$ mixed page number $= k$?

in general, no!

	Matching	Non-matching
Separated	$f(k) \leq 2k^7$ $f(k) \geq k^2$	$f(k, \Delta)$ $\leq 2(\Delta + 1)k^7$
Non-separated	$f(k) \leq k^{\mathcal{O}(k)}$ $f(k) \geq k^2$	$f(k, \Delta) \leq \Delta k^{\mathcal{O}(k)}$

Vizing's (Matching → Non-matching, both rows)

Quotients (Separated → Non-separated, Matching column)

Δ = max. degree

Plan:
- separated matchings ✓
- non-separated matchings ✓
- non-matchings (✓)

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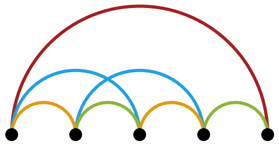

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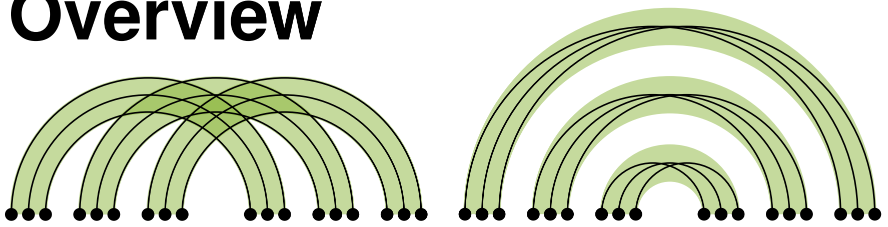

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Vizing's (Separated/Matching → Separated/Non-matching)
Quotients (Separated/Matching → Non-separated/Matching)
Vizing's (Non-separated/Matching → Non-separated/Non-matching)

Δ = max. degree

Open Questions

- matchings: close the gap between upper and lower bounds
- non-matchings: dependency only on k

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