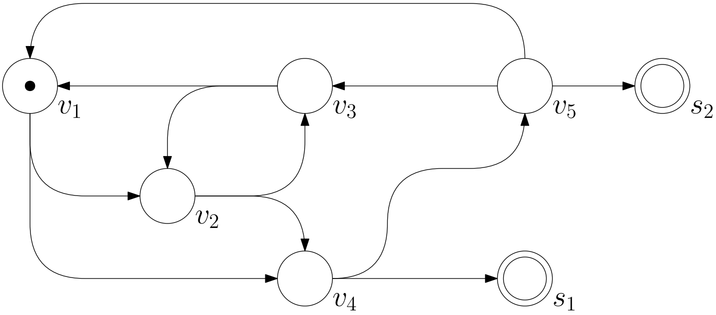
A Quasi-Polynomial Time Algorithm for Multi-Arrival on Tree-Like Multigraphs

Ebrahim Ghorbani, Jonah Leander Hoff, Matthias Mnich

STACS 2025

Arrival

- Arrival: which sink does the particle end up on?
- Each vertex: routes along outgoing arcs in fixed cyclic order



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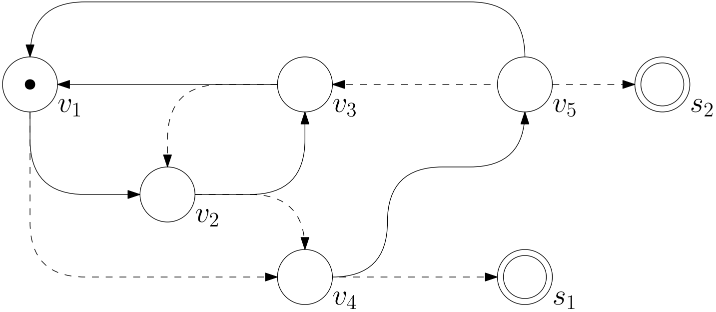

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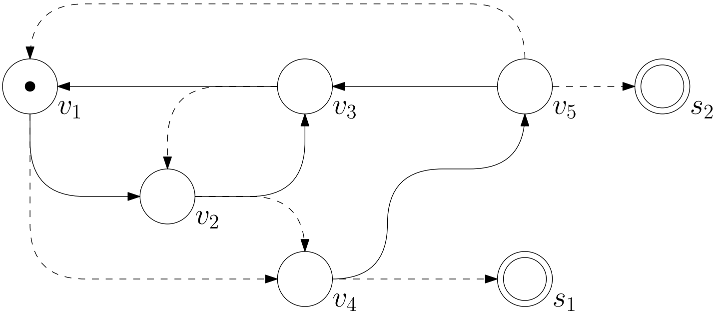

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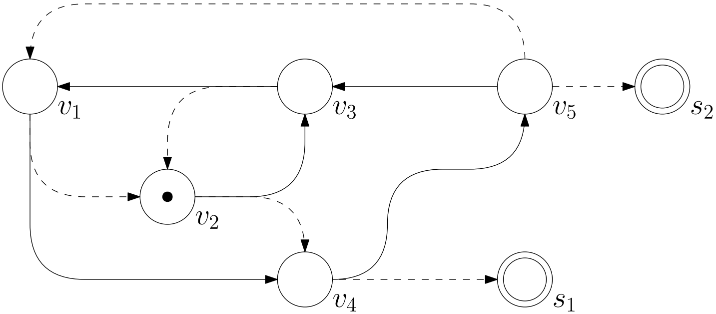

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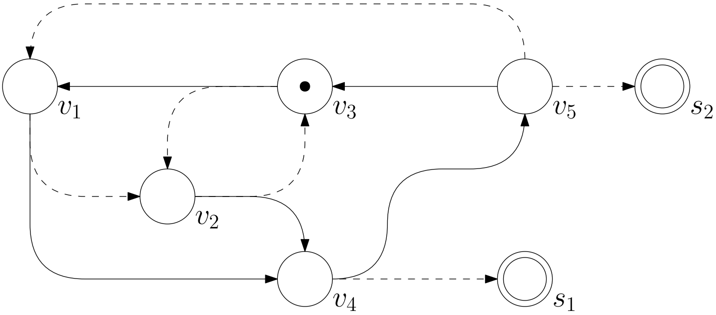

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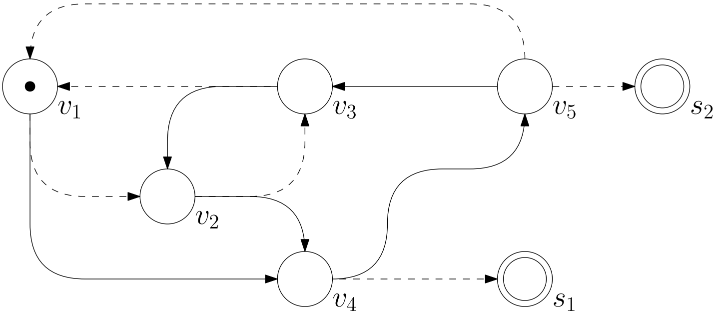

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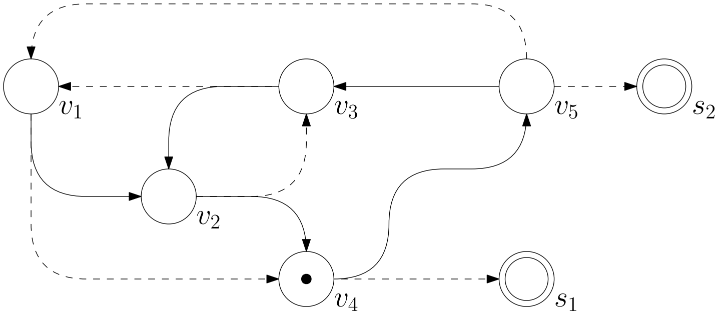

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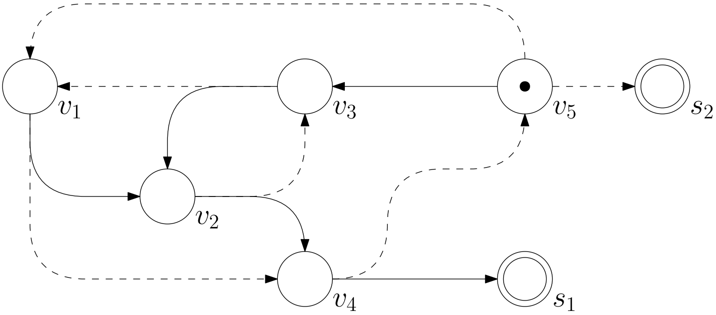

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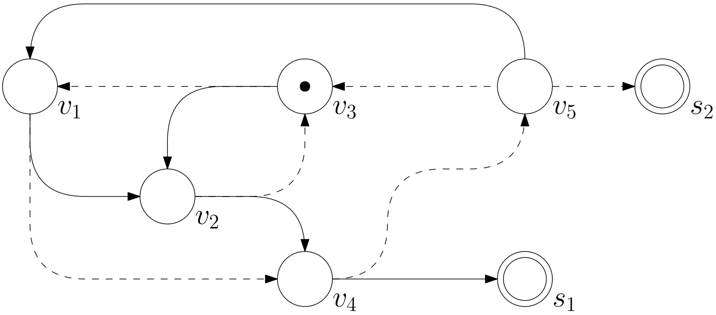

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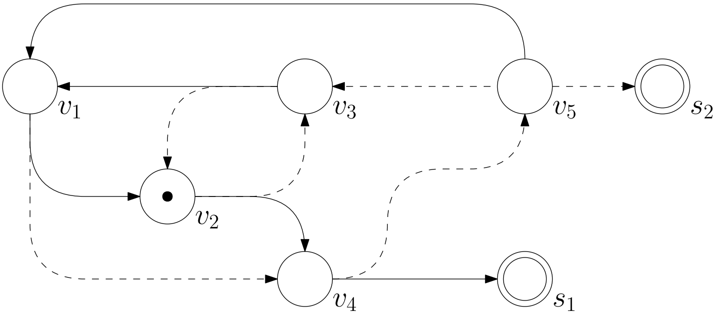

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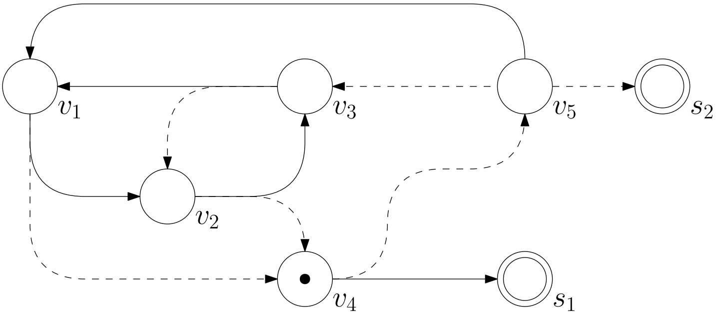

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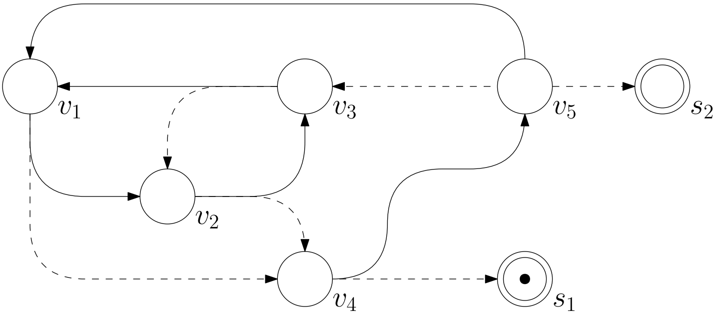

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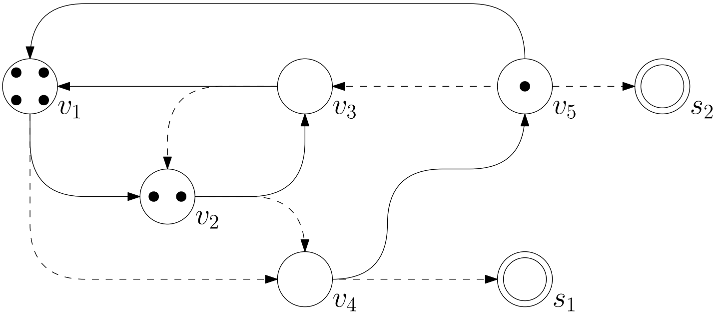

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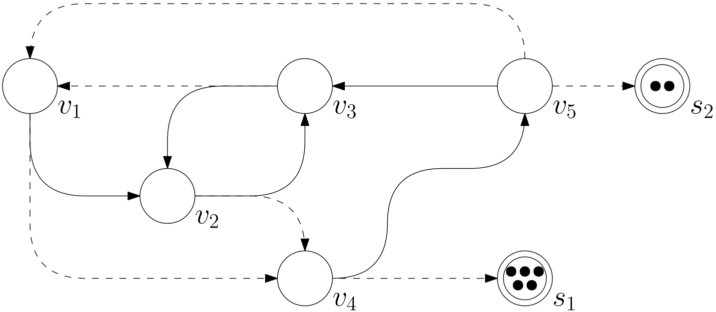
- Multi-Arrival: how many particles end up on each sink?

Routing Multiple Particles



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Result

- Multi-Arrival on tree-like multigraphs in time $\mathcal{O}^*(\log^\kappa m)$
- $\kappa \approx$ measure of balanced nested branchings
  - Path-Like: $\kappa = 1$
  - Tree-Like: $\kappa \leq \log n_b$, where n_b number of branching vertices

Prior Work

Type	Complexity	Graph	Reference
Arrival	$2^{\mathcal{O}(\sqrt{n}\log n)}$	Any	[1]
Arrival	$\mathcal{O}^*(1)$	Tree-like	[2]
Multi-Arrival	$\mathcal{O}^*(1)$	Uniform path-like	[3]
Multi-Arrival	$\mathcal{O}^*(1)$	Path-like	This paper
Multi-Arrival	$\mathcal{O}^*(\log^{\kappa} m)$	Tree-like	This paper

- If Multi-Arrival $\mathcal{O}^*(1)$ on $G[V \setminus X]$ with $|X|$ bounded
  - Then Arrival $\mathcal{O}^*(1)$ on $G[V]$ **[Gärtner, Haslebacher, and Hoang 2021]**

[1] **[Gärtner, Haslebacher, and Hoang 2021]**
[2] **[Auger, Coucheney, and Duhaze 2022]**
[3] **[Auger, Coucheney, Duhazé, et al. 2023]**

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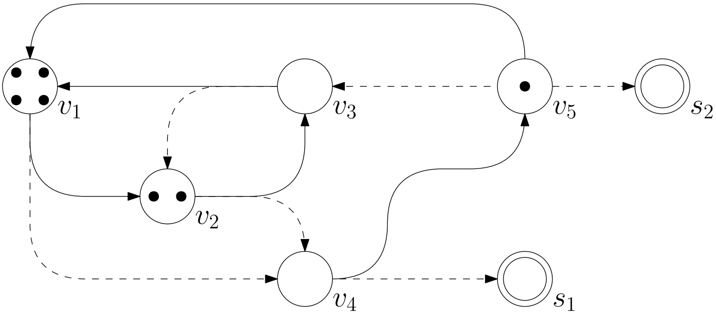
Legal Routing Vectors

- Routing (vector) $r \in \mathbb{Z}^V$: how often each vertex is routed
- Legal routing: corresponds to some legal routing sequence
- Maximal legal routing: legal routing with all particles on sinks



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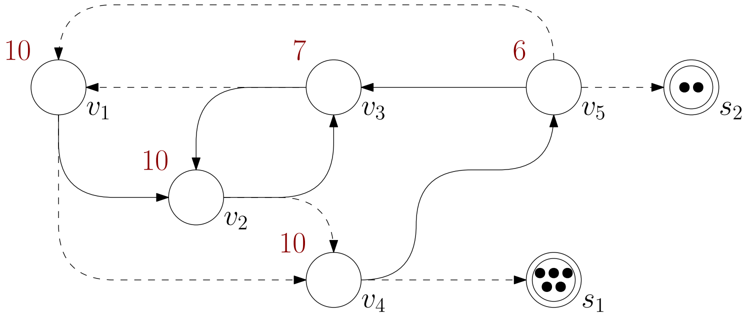
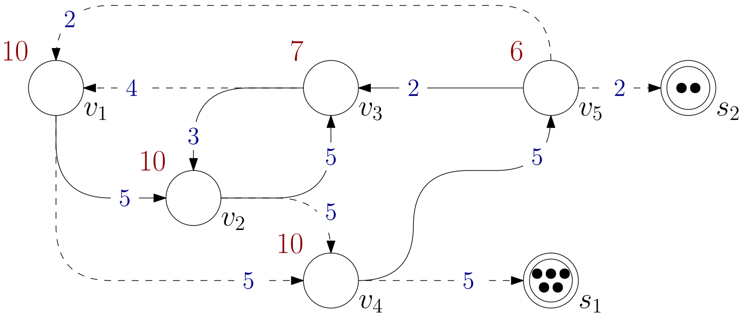


Compensated Routing Vectors

- Compensated routing: each vertex has out-flow less or equal in-flow
- Lower-bound flow into sinks
- Switching flows **[Dohrau et al. 2017]**



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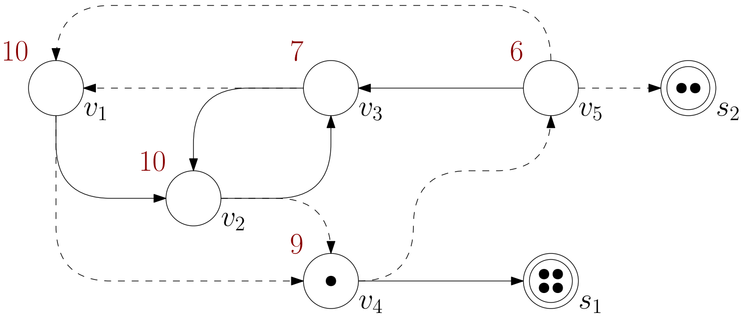

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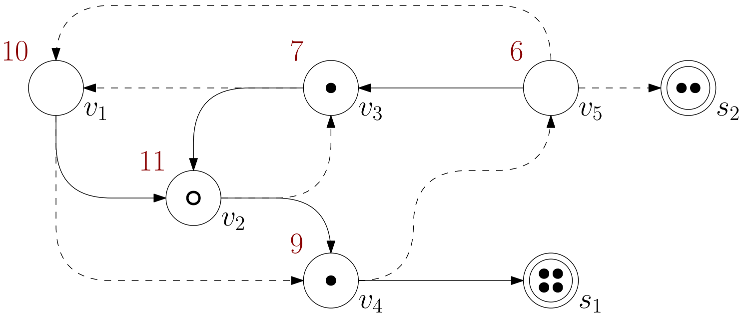

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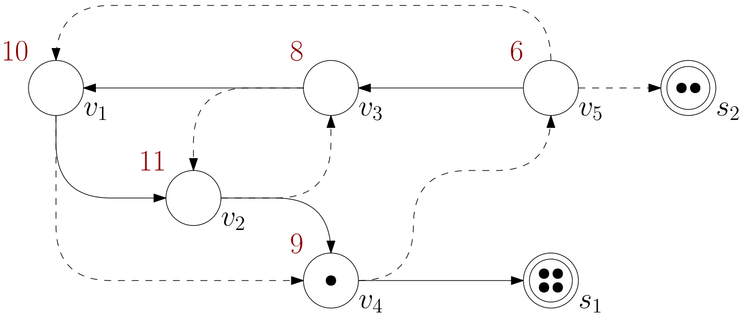
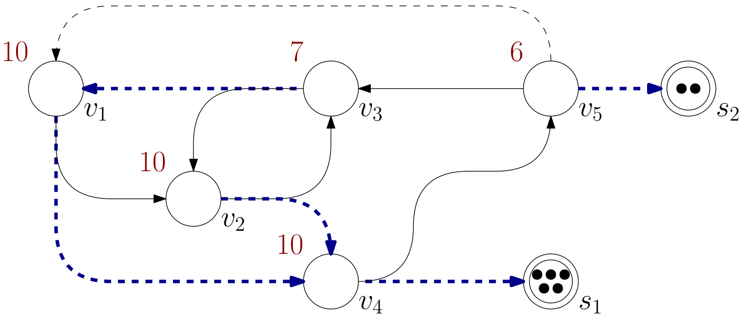


Directed Routing Vectors

- Idea: maximize flow into fixed sink *s*
- *s*-directed routing: last particles sent were towards *s*



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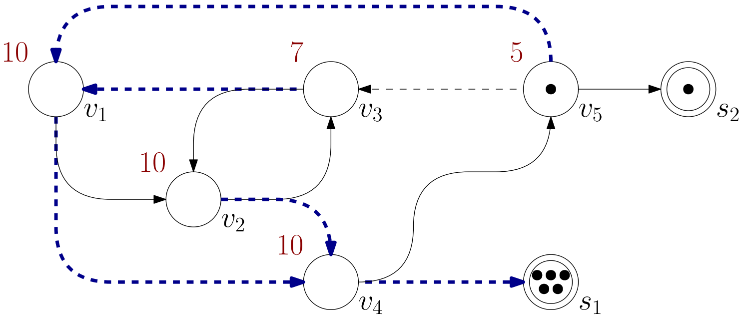


Problem

Original problem:

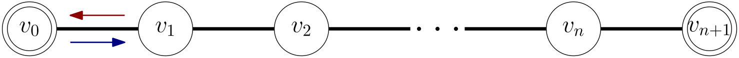
$$\begin{array}{ll} \text{maximize} & \text{flow of } \hat{r} \text{ into sinks} \\ \text{subject to} & \hat{r} \in \mathbb{Z}^V \text{ is compensated.} \end{array}$$

Directed problem for sink s:

$$\begin{array}{ll} \text{maximize} & \text{flow of } r_s \text{ into } s \\ \text{subject to} & r_s \in \mathbb{Z}^V \text{ is } s\text{-directed and compensated.} \end{array}$$

Path-Like Case

1. Guess flow $v_0 \leftarrow v_1$
2. Iteratively construct r
   - Since s-directed: flow $v_{i-1} \leftarrow v_i$ gives $r(v_i)$
   - Since compensated: $r(v_i) \leq \sigma_i + (v_{i-1} \rightarrow v_i) + (v_i \leftarrow v_{i+1})$
3. If missing flow $v_n \leftarrow v_{n+1}$ is zero: guess is achievable
4. Otherwise: guess is too large



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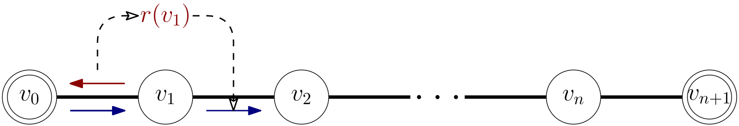

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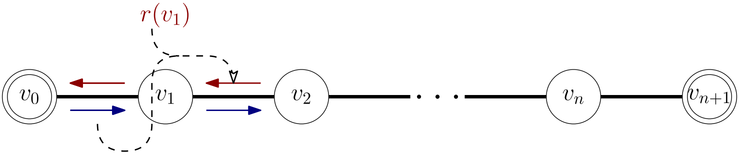

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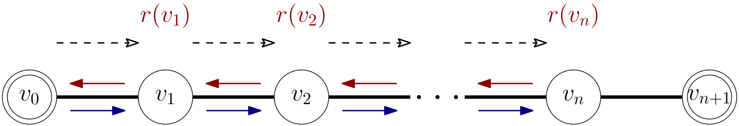

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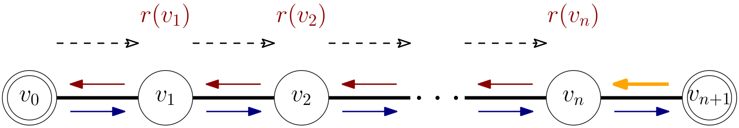
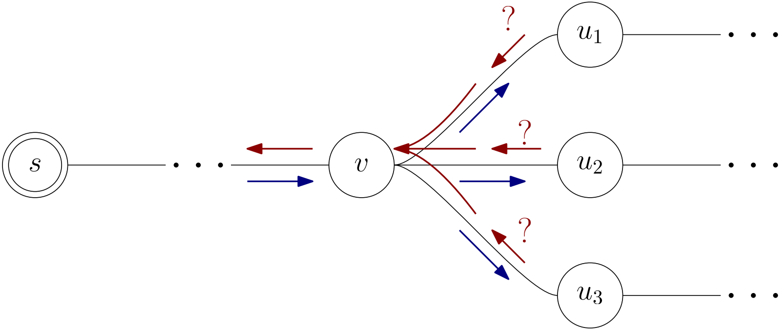


Tree-Like Extension

- Can only infer $v \leftarrow u_1 + u_2 + \ldots$
- Each $v \leftarrow u_i$ upper-bound depends on $v \rightarrow u_i$
- Solve by recursion with fixed $v \leftarrow u_j$ inferred



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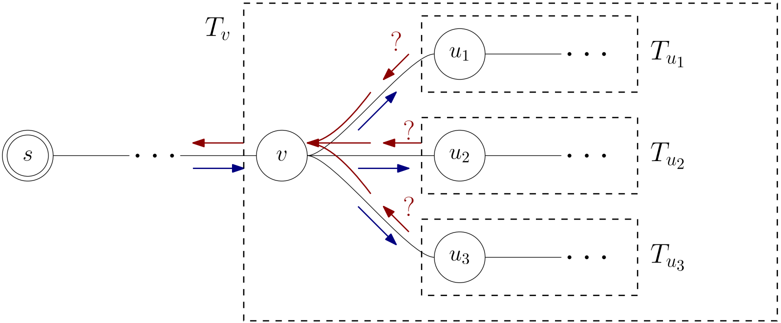

Contracted-Height

- $\kappa = 1 + \max\{\mathrm{ch}(T_r) \mid r \in N^-(S)\}$
- *Contracted height* $\mathrm{ch}(T_r)$: minimal height obtained by contracting per vertex one edge to child

$$\mathrm{ch}(T_v) = \begin{cases} 1 + \mathrm{ch}(T_{u_1}) & \mathrm{ch}(T_{u_1}) = \mathrm{ch}(T_{u_2}) \\ \mathrm{ch}(T_{u_1}) & \mathrm{ch}(T_{u_1}) > \mathrm{ch}(T_{u_2}) \end{cases}$$

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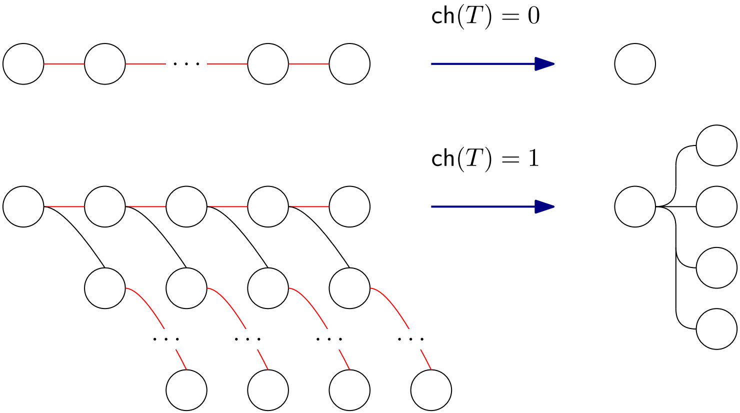
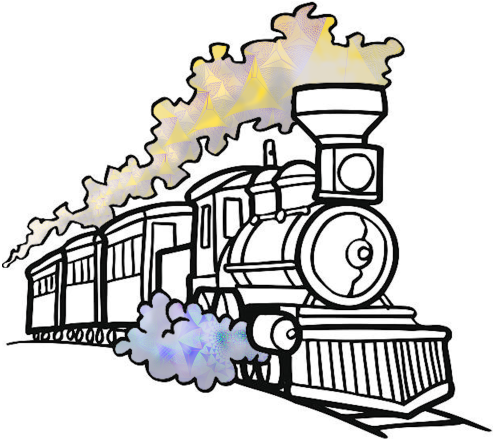


Summary

- Decomposition of maximal legal routing into s-directed compensated routings
- Recursive search to find s-directed compensated routings
  - Linear approximation guides exponential search
  - Recursion is contracted on one sub-tree for each vertex
- Dynamic program to find s-directed compensated routings



Thank you for your attention!

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