Generalized Inner Product Estimation with Limited Quantum Communication

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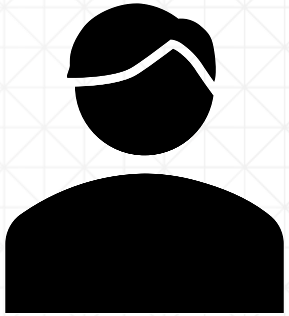
Problem Setup

k copies of
quantum state
$\psi \in \mathbb{C}^d$

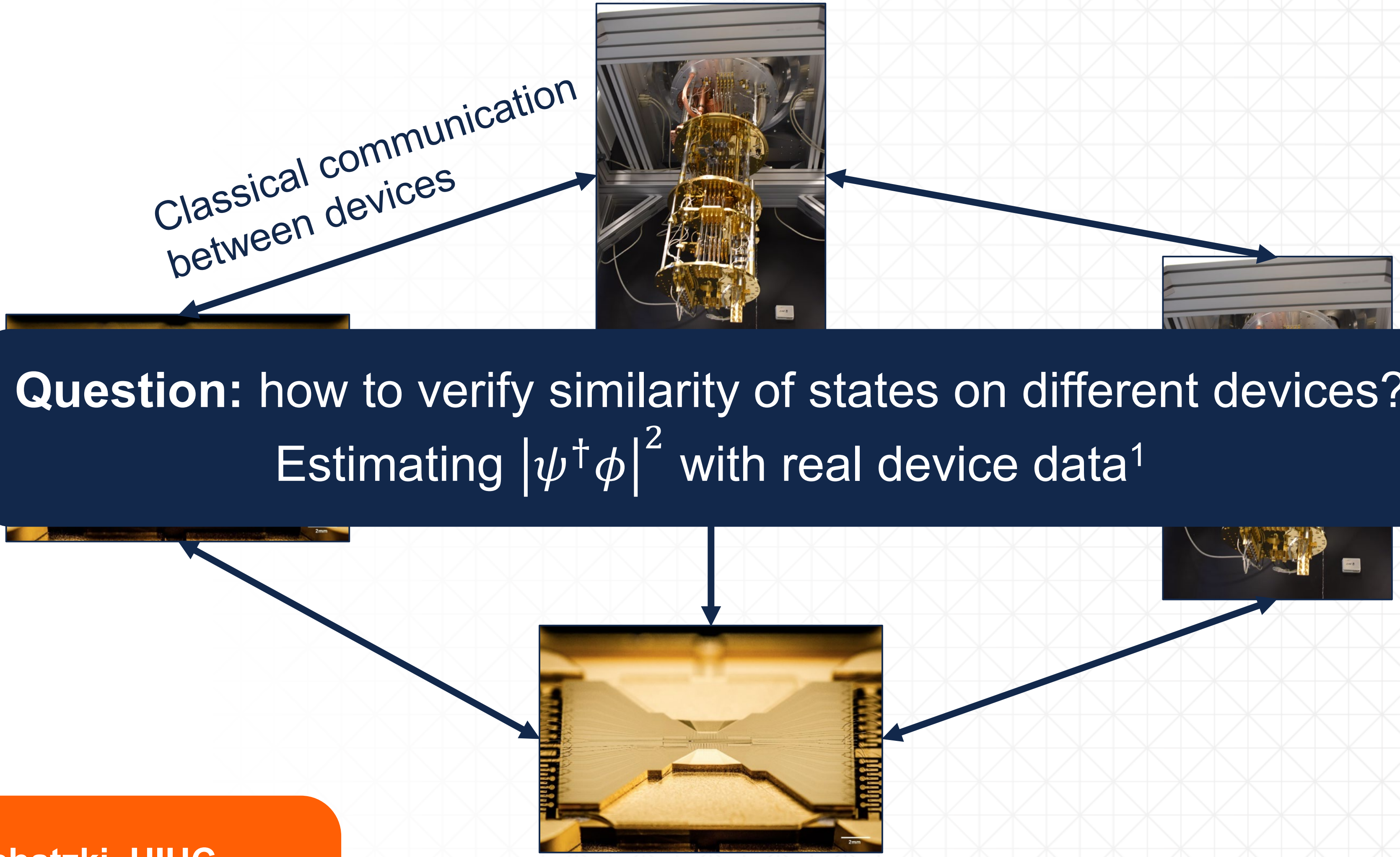
ψ and ϕ unknown



k copies of
quantum state
$\phi \in \mathbb{C}^d$

What can they learn about the relationship between their states with limited communication?

Motivation: Cross Platform Verification



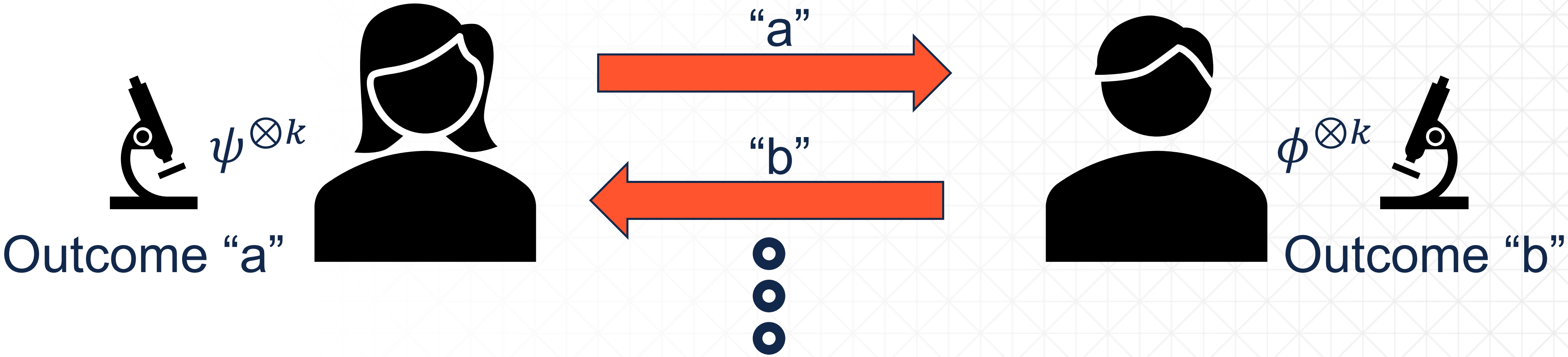
Question: how to verify similarity of states on different devices?

Estimating $|\psi^\dagger \phi|^2$ with real device data[1]

Images: Forbes & AFRL

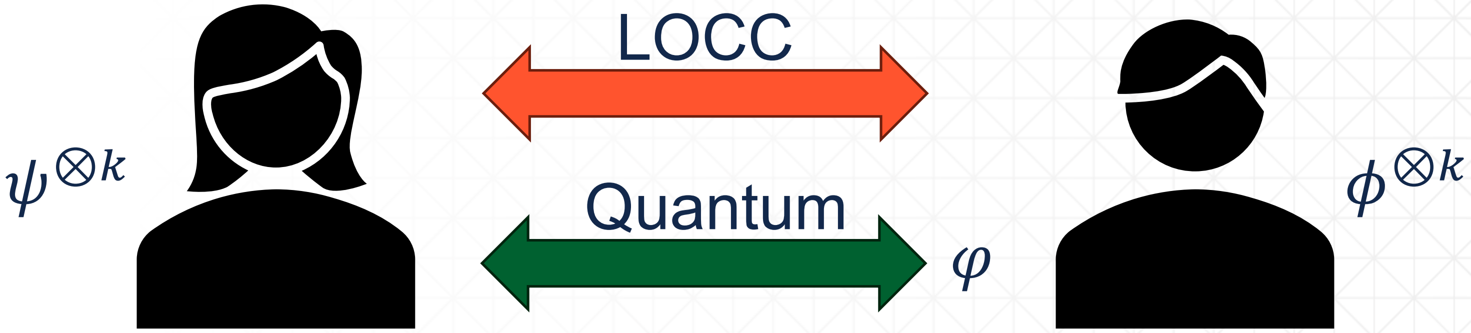
[1]Elben et al., Cross-Platform Verification of Intermediate Scale Quantum Devices.

Distributed Protocol



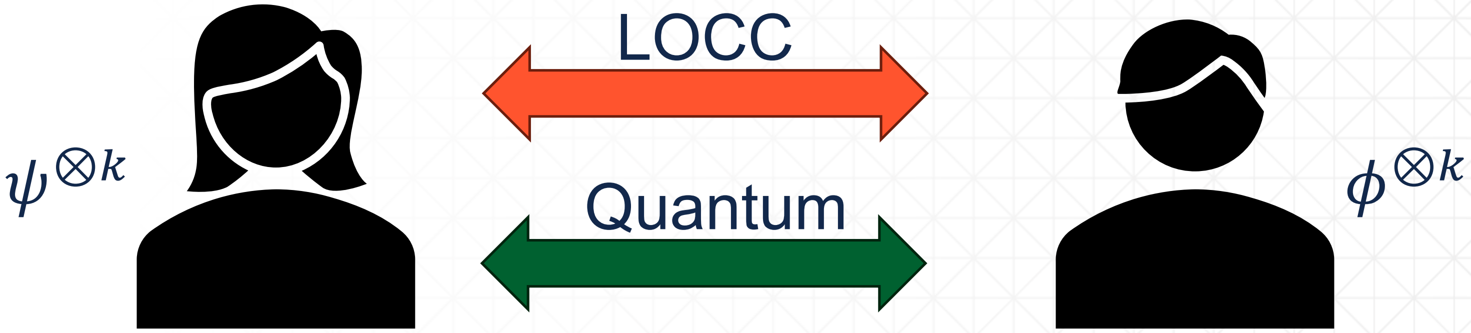
Local operations and
classical communication
(LOCC)

Distributed Protocol



"Small" amounts of
quantum communication

Task: (Generalized) Inner Product Estimation



Goal: estimate $\left|\psi^{\dagger} M \phi\right|^{2}$ for Hermitian M

Minimize sample complexity **k**

Motivation

- $M = \mathbb{I}$:
  - Similarity between pure states
  - Hilbert Schmidt distance $\|\rho - \sigma\|_2$
- M a projector:
  - Overlap in a subspace
- More general metrics on $\mathbb{C}^d$

Central question: how does the allowed communication change the sample complexity of the task?

Constrained Measurements

- Classical communication:
  - ALL'23: $\Theta\left(max\left\{\frac{\sqrt{d}}{\varepsilon}, \frac{1}{\varepsilon^2}\right\}\right)$ to estimate $|\psi^\dagger \phi|$
  - **AS'25:** $\mathcal{O}\left(max\left\{\frac{\|M_\varepsilon\|_2}{\varepsilon}, \frac{1}{\varepsilon^2}\right\}\right)$, $\Omega\left(max\left\{\frac{\|M_\varepsilon\|_2}{\sqrt{\varepsilon}}, \frac{1}{\varepsilon^2}\right\}\right)$ to estimate $\left|\psi^\dagger M \phi\right|^2$
- Limited quantum communication (can send q-dimensional states):
  - **AS'25:** $\Omega\left(\sqrt{\frac{d}{q}}\right)$
  - **AS'25:** protocol using $\Theta(1)$ q-dimensional messages and $\mathcal{O}\left(\sqrt{\frac{d}{q}}\right)$ copies

Generalized Inner Product With Classical Communication

Goal: estimate $|\psi^\dagger M\phi|^2$ with just classical communication

No Constraints

- Allowed unlimited classical and quantum communication
- $\Theta\left(\frac{1}{\varepsilon^2}\right)$ samples to estimate $\left|\psi^\dagger M\phi\right|^2$ to accuracy ε

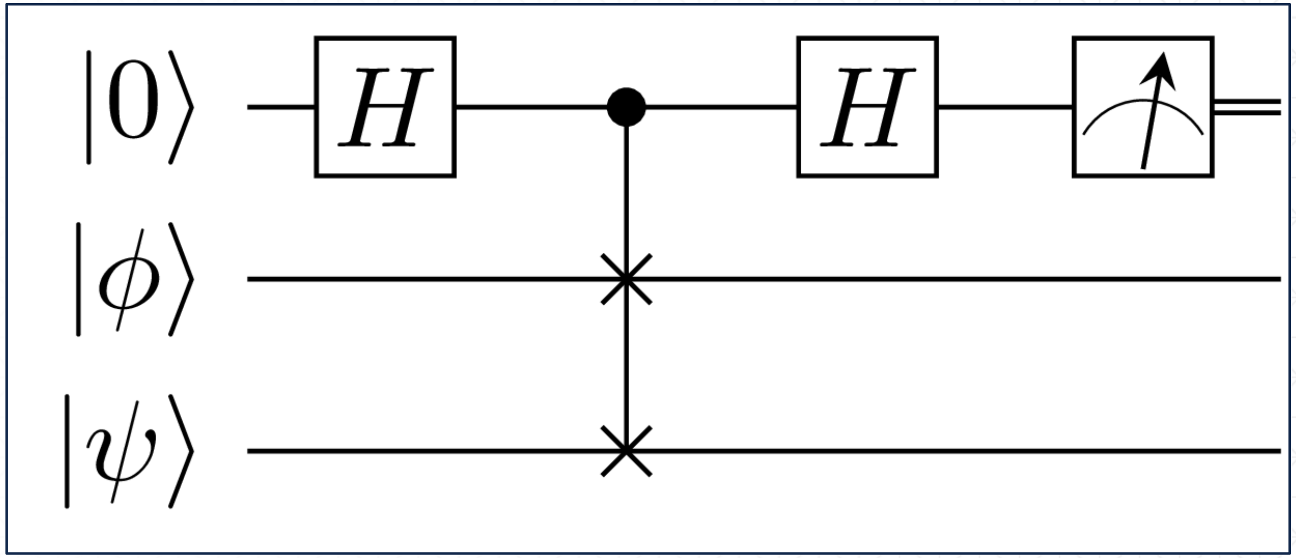


Image: Wikipedia

Generalized Inner Product

- **Goal**: estimate $f = \left|\psi^\dagger M \phi\right|^2$ to error ε
- **Idea**: controlled by largest eigenvalues
- Define M_ε as M with all eigenvalues of norm less than $\varepsilon/2$ replaced with 0
- Estimate $f_\varepsilon = \left|\psi^\dagger M_\varepsilon \phi\right|^2$

$$M = \begin{matrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & \varepsilon/3 \end{matrix}$$

X

Protocol with Classical Communication

Input: An operator M. Alice gets $|\psi\rangle^{\otimes k}$, Bob gets copies of $|\phi\rangle^{\otimes k}$.
Output: An ε-approximation of $\langle\psi|M|\phi\rangle|^2$.

Projector onto support of M_ε (has dimension d_ε)

$$\vee^n V = \{|\phi\rangle \in V^{\otimes n} \mid \pi \cdot |\phi\rangle = |\phi\rangle, \forall \pi \in S_n\}$$

Standard POVM is $\{\dim \vee^n V \ |\psi\rangle\langle\psi|^{\otimes n} d\psi\}$

Yields a state $|\psi\rangle \in V$

Analysis of Protocol

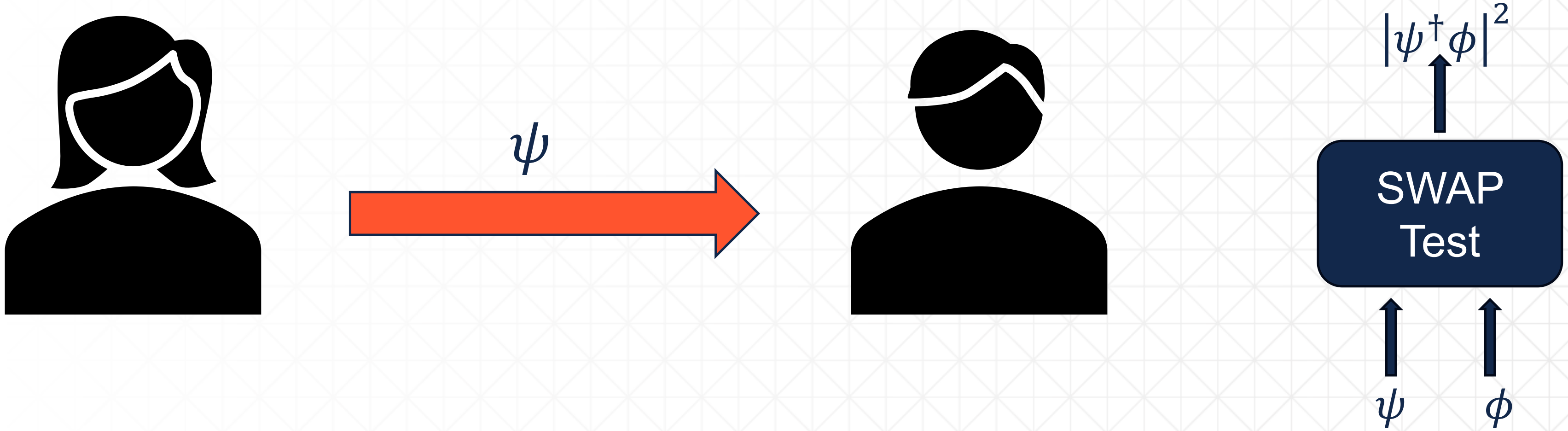
- Bias is $\leq \frac{\varepsilon}{2} + \mathcal{O}\left(\frac{1}{k}\right)$
- $\mathrm{Var}(w) = \mathcal{O}\left(\frac{1}{k} + \frac{\|M_\varepsilon\|_2^2}{k^2} + \frac{\|M_\varepsilon\|_2^4}{k^4}\right)$
- $k = \Omega\left(\max\left\{\frac{1}{\varepsilon^2}, \frac{\|M_\varepsilon\|_2}{\varepsilon}\right\}\right)$ suffices

Lower Bound

- M_ε is “similar” to $\mathbb{I}$ on its support
- Reduce estimating inner product to estimating $\left|\psi^\dagger M_\varepsilon \phi\right|^2$

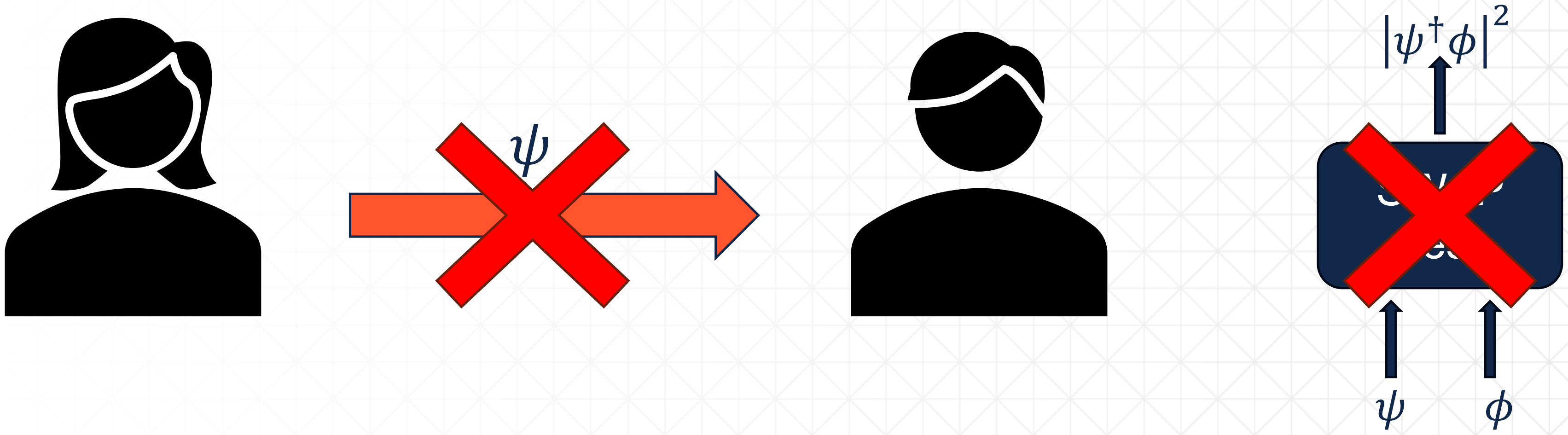
Inner Product Estimation with Entanglement

$\Theta(1)$ samples



What if Alice and Bob have *quantum* communication?

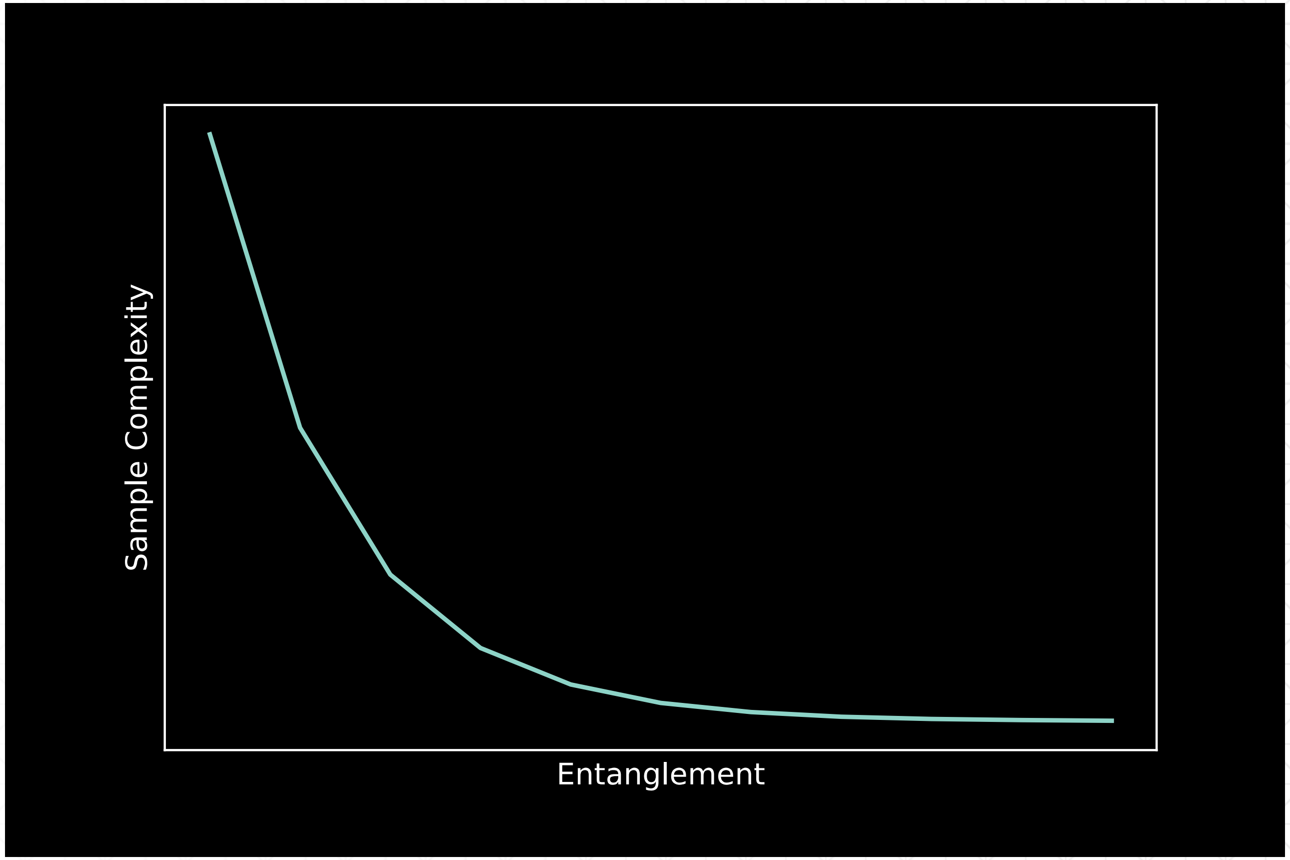
Limited Entanglement



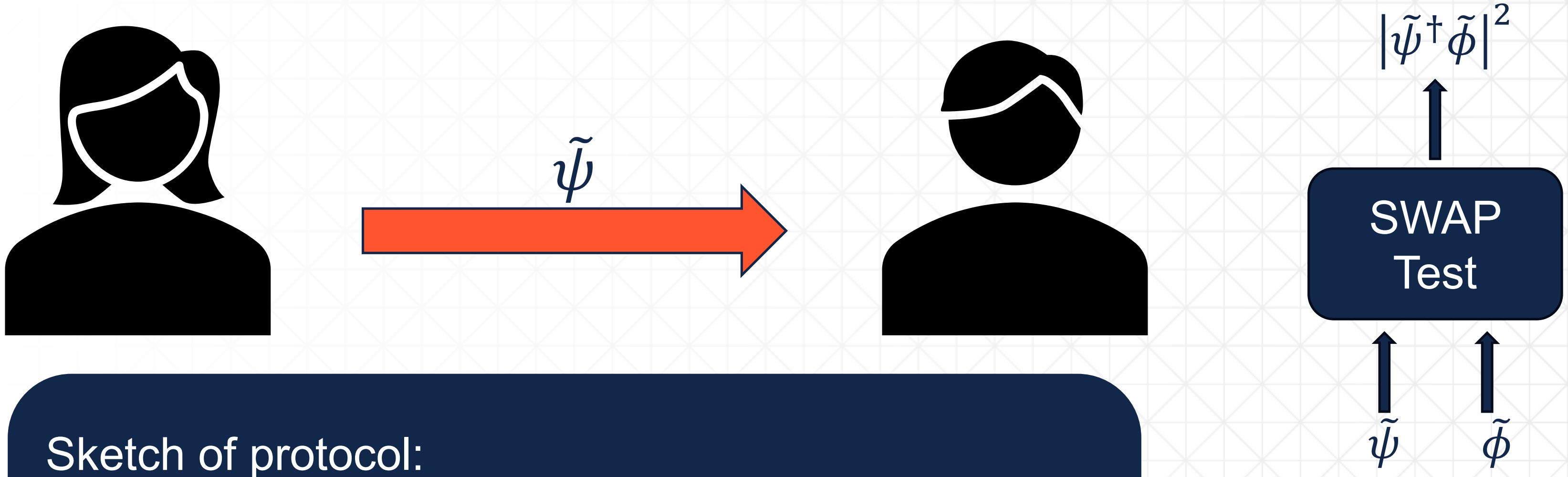
What if Alice and Bob have *very little quantum* communication?

Limited Entanglement (AS'24)

- Allowed a q dimensional quantum message, $k = \Omega\left(\sqrt{\frac{d}{q}}\right)$
- Must communicate $\Omega(n)$ qubits for $\mathrm{poly}(n)$ sample complexity
- With $\Theta(1)$ quantum messages of dimension q, $k = O\left(\sqrt{\frac{d}{q}}\right)$ suffices

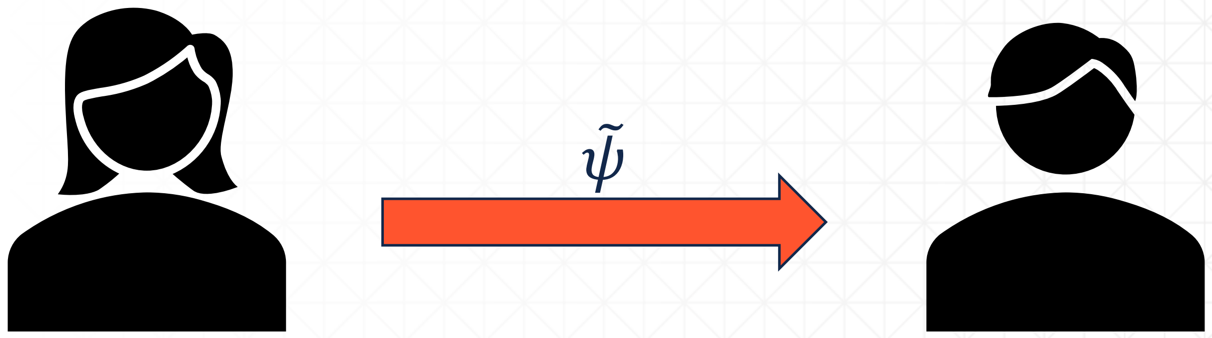


Sketch of Protocol



Sketch of protocol:
- Divide Hilbert space into many random "buckets"
- Each perform projections into these buckets and keep track of collisions
- Teleport and swap test on collisions

Sketch of Protocol

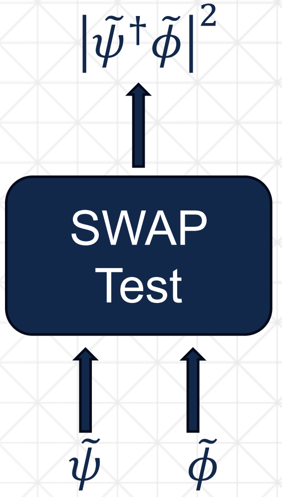


$$\psi \in \mathbb{C}^d \longmapsto \tilde{\psi} \in \mathbb{C}^q$$

$$\tilde{\psi} = \frac{UPU^\dagger \psi}{\|UPU^\dagger \psi\|}$$

$U \in \mathcal{U}(d)$ Haar random

$$\tilde{\phi} = \frac{UPU^\dagger \phi}{\|UPU^\dagger \phi\|}$$



Random projection maintains overlaps with high probability

Conclusion and Open Problems

Summary

Classical Communication

- Estimate $|\psi^\dagger M \phi|^2$
- Sample complexity controlled by $||M_\varepsilon||_2$

Limited Quantum Communication

- Estimate $|\psi^\dagger \phi|^2$
- Entanglement helps, but not much

Open Problems

- Estimate trace distance of mixed states?
  - Without locality constraints, $\mathrm{k} = \Theta(d)$[1]
  - **Question:** does $k = O(d)$ suffice with LOCC?

- Relationship to oblivious quantum state compression
  - Can we compress $\rho^{\otimes k}$ better than ρ

[1]Badescu et al., Quantum State Certification, 2017.

Questions?