How to Play the Accordion

Uniformity and the (Non-)Conservativity of the Linear Approximation of the λ-Calculus

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STACS, Jena, 5 March 2025

Linear Approximation of Functional Programs

program := instruction (program, ..., program)

Linear Approximation of Functional **Programs**

program := instruction

program … program

program := instruction ⇝ **result**

program … program

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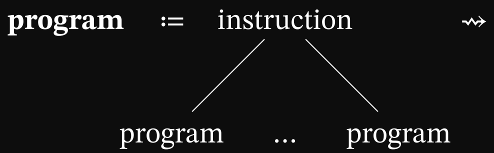
instruction → program … program

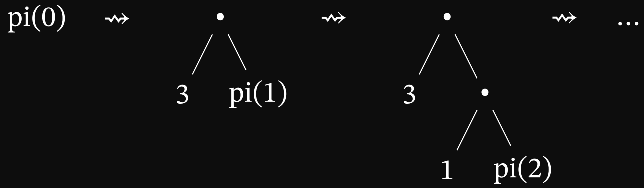
= program that cannot be further executed (*normal form*)



result

= program that cannot be further executed (*normal form*)

But the result might be *infinite* and *infinitely far*:



What we can compute in finite time are *partial results*:

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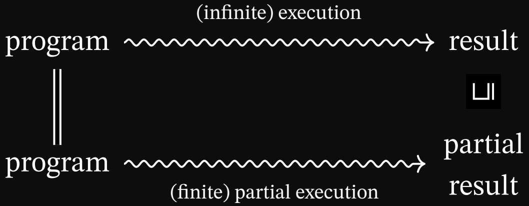
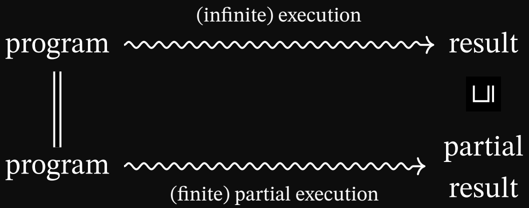
As a summary: [Wadsworth, Hyland, Barendregt, 1970s]



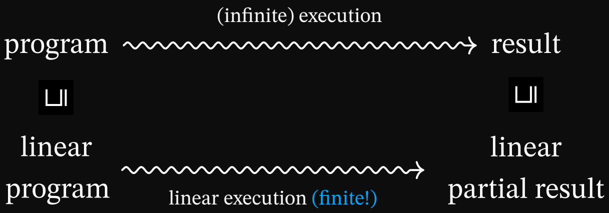
and the result is the *limit* of all partial results:
it's a *continuous* approximation.

Linear Approximation of Functional Programs

This continuous approximation:



can be refined into a *linear* one: [Ehrhard-Regnier, 2000s]



Linear programs: each argument of a function is used *exactly once*.

LINEAR APPROXIMATION OF FUNCTIONAL PROGRAMS

Linear programs: each argument of a function is used *exactly once*.

program := instruction

program … program

Linear programs: each argument of a function is used *exactly once*.

program := instruction

program ... program

linear program := instruction

[linear program , ... , linear program] ... [linear program , ... , linear program]

Linear Approximation of Functional Programs

Programs are **λ-terms**:

$$M, N, \ldots \quad := \quad x \quad | \quad \lambda x.M \quad | \quad MN$$

$$x \mapsto M \qquad M(N)$$

Linear programs are **resource λ-terms**:

$$s, t, \ldots \quad := \quad x \quad | \quad \lambda x.s \quad | \quad s\,[t_1, \ldots, t_n]$$

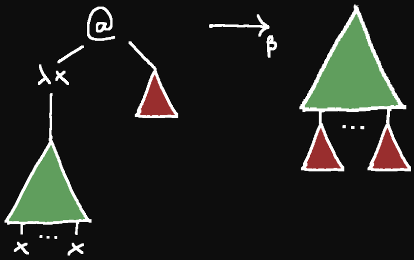
Given a λ-term M, we denote by $\mathcal{T}(M)$ the set of resource λ-terms approximating it. (Its *Taylor expansion*!)

Need a nap? Have it during this slide!

Program execution is the **β-reduction** on λ-terms:

$$(x \mapsto M)(N) \longrightarrow M \text{ where } x \text{ is replaced with } N$$

$$(\lambda x.M)N \longrightarrow_\beta M[N/x]$$



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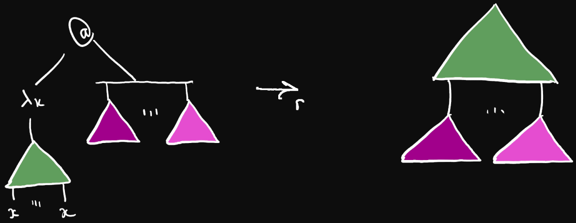
Linear execution is the **resource reduction** on resource λ-terms:



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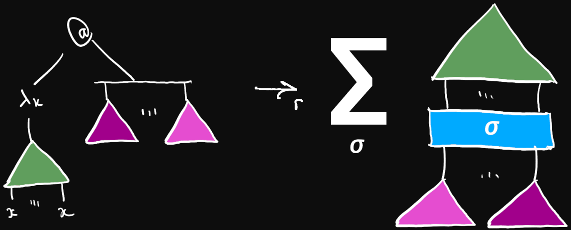
Linear execution is the **resource reduction** on resource λ-terms:



Wake up!

Fact 1 (simulation, finite) [V.A. 2017]

Let M, N be finite λ-terms.

If $M \longrightarrow_\beta^* N$ then $\mathcal{T}(M) \twoheadrightarrow_{\mathrm{r}} \mathcal{T}(N)$.

Problem 1 (conservativity, finite)

Is the converse true?

Theorem 1

Yes it is! If $\mathcal{T}(M) \twoheadrightarrow_{\mathrm{r}} \mathcal{T}(N)$ then $M \longrightarrow_\beta^* N$.

Fact 2 (simulation, infinitary) [C. and V.A. 2023]

Let M, N be infinitary λ-terms.

If $M \longrightarrow_{\beta}^{\infty} N$ then $\mathcal{T}(M) \twoheadrightarrow_{\mathrm{r}} \mathcal{T}(N)$.

Problem 2 (conservativity, infinitary)

Is the converse true?

Theorem 2

No, it isn't!

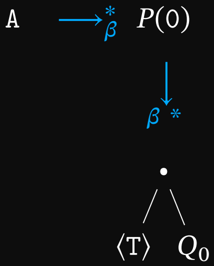
There are terms $\mathtt{A}, \bar{\mathtt{A}}$ such that $\mathcal{T}(\mathtt{A}) \twoheadrightarrow_{\mathrm{r}} \mathcal{T}(\bar{\mathtt{A}})$ but there is no reduction $\mathtt{A} \longrightarrow_{\beta}^{\infty} \bar{\mathtt{A}}$.

$\mathtt{A}$ is the *Accordion* λ-term.

The Accordion

$$\texttt{A} \longrightarrow_{\beta}^{*} P(0)$$

THE ACCORDION



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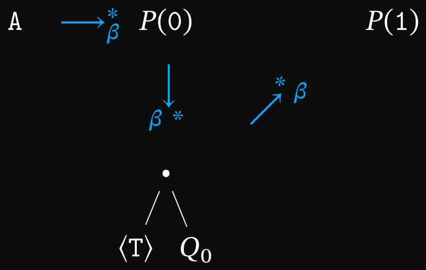

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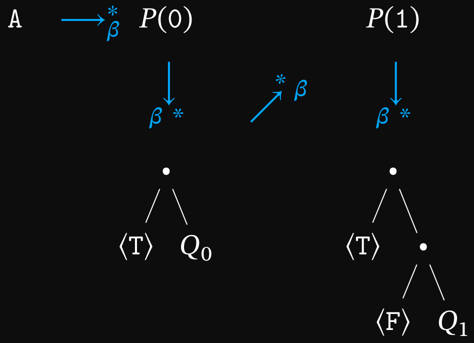

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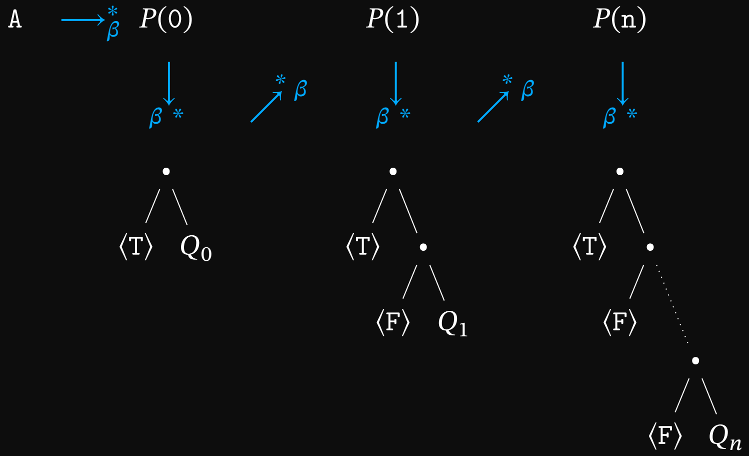

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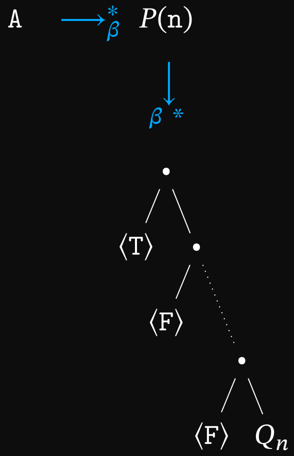

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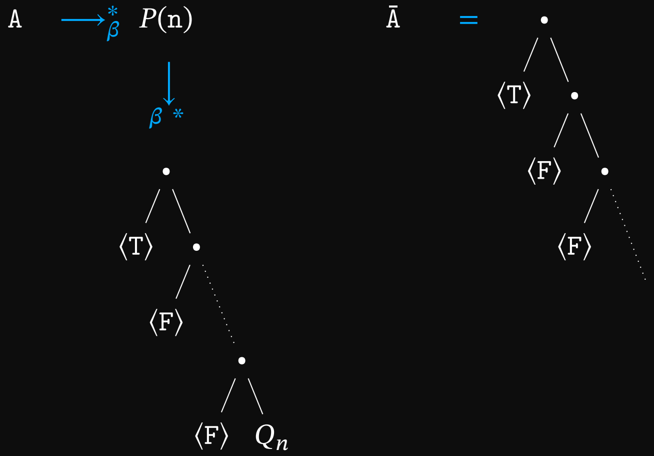

Problem 3 (restoring conservativity)

Can we restrict $\twoheadrightarrow_{\mathrm{r}}$ to obtain a conservative approximation?

Theorem 3

Yes, thanks to the *uniform* lifting of the resource reduction $\overset{\frown}{\longrightarrow}{}_{\mathrm{r}}^{\infty}$!

If $\mathcal{T}(M) \overset{\frown}{\longrightarrow}{}_{\mathrm{r}}^{\infty} \mathcal{T}(N)$ then $M \longrightarrow_{\beta}^{\infty} N$.

In particular, there is no reduction $\mathcal{T}(\mathtt{A}) \overset{\frown}{\longrightarrow}{}_{\mathrm{r}}^{\infty} \mathcal{T}(\bar{\mathtt{A}})$.

Thanks for your attention!
Any questions?

btw, please offer me a postdoc position