Identity-preserving Lax Extensions and Where to Find Them

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Plan

This talk is about the **correspondence**:

“There is a **good** way of showing that states in a generalized transition system are behaviourally equivalent.”

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“There is an identity-preserving lax extension of …”

Today, we will provide necessary and sufficient conditions for the existence of identity-preserving lax extensions.

What is Behavioural Equivalence?

Non-deterministic Transition Systems

For non-deterministic transition systems:

Definition (Park 1981; Milner 1989)

Two states in (possibly distinct) systems are bisimilar if there is a relation between the underlying sets of the systems that relates them and preserves and reflects transitions.

"Bisimilarity **is** behavioural equivalence!"

Questions:

- Is there a high-level reason why this definition works?
- Can we transport the definition to other systems?

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Answer: Yes! (But to understand this we need a bit of category theory.)

Rutten 2000

Key ingredients: coalgebra, coalgebra homomorphism and bisimulation.

Let $\mathsf{F}\colon \mathsf{Set} \to \mathsf{Set}$ be a functor. An F-***coalgebra*** (X, α) consists of a set X and a map $\alpha\colon X \to \mathsf{F}X$. A ***morphism*** $f\colon (X, \alpha) \to (Y, \beta)$ of F-coalgebras is a map $f\colon X \to Y$ such that $\beta \cdot f = \mathsf{F}f \cdot \alpha$.

Intuition:

- X is a set of ***states***;
- $\alpha\colon X \to \mathsf{F}X$ is a ***transition map*** $\alpha\colon X \to \mathsf{F}X$ assigning to each state $x \in X$ a collection $\alpha(x)$ of successors, structured according to F;
- $f\colon (X, \alpha) \to (Y, \beta)$ is a map that preserve the ***behaviour*** of states.

Bisimulations Coalgebraically

Let $\mathsf{F}\colon \mathsf{Set} \to \mathsf{Set}$ be a functor.

1. Extend F to a relator L, i.e. to a monotone map on relations s.t. $r\colon X \nrightarrow Y \mapsto \mathsf{L}r\colon \mathsf{F}X \nrightarrow \mathsf{F}Y$.
2. Then, a relation $r\colon X \nrightarrow Y$ is an L-bisimulation from an F-coalgebra (X, a) to an L-coalgebra (Y, β) if for all $x \in X$ and $y \in Y$,

$$x\,r\,y \implies \alpha(x)\,\mathsf{L}r\,\beta(y).$$

Remark

For all F-coalgebras (X, α) and (Y, β) there is a largest L-bisimulation from (X, α) to (Y, β) which is called L-bisimilarity.

Definition

Given a relation $r\colon X \nrightarrow Y$, take a span $(\pi_1\colon A \to X, \pi_2\colon A \to Y)$ such that $r = \pi_2 \cdot \pi_1^{\circ}$. Then, put $\overline{\mathsf{F}}r = \mathsf{F}\pi_2 \cdot (\mathsf{F}\pi_1)^{\circ}$.

The Barr relator of F

- agrees with F on functions;
- preserves converses, i.e. $\overline{\mathsf{F}}(r^{\circ}) = (\overline{\mathsf{F}}r)^{\circ}$, where r° is the converse relation of r.
- in general, it does **not** preserve composition laxly, i.e., $\overline{\mathsf{F}}r \cdot \overline{\mathsf{F}}s \not\leq \overline{\mathsf{F}}(r \cdot s)$.

Example

The Barr relator of the powerset functor $\mathcal{P}$ sends a relation r to the relation given by: $A\,\overline{\mathcal{P}}r\,B \iff (\forall a \in A\, \exists b \in B. a\,r\,b) \wedge (\forall b \in B\, \exists a \in A. b\,r\,a)$.

Example

The powerset functor $\mathcal{P}\colon \mathsf{Set} \to \mathsf{Set}$ sends

- a set X to the set $\mathcal{P}X$ of subsets of X,
- a function $f\colon X \to Y$ to the function $\mathcal{P}f\colon \mathcal{P}X \to \mathcal{P}Y$ that assings $A \subseteq X$ to $f[A]$.

- $\mathcal{P}$-coalgebras are non-deterministic transition systems.
- $\mathcal{P}$-coalgebra homomorphisms are transition preserving and reflecting maps.
- $\overline{\mathcal{P}}$-bisimulations are transition preserving and reflecting relations.

We can transport the definition of bisimilarity from non-deterministic transition systems to other systems! But does it make sense to do so?

Definition

States x and y in coalgebras (X, α) and (Y, β), respectively, are ***behaviourally equivalent*** if there is a coalgebra (Z, γ) and morphisms $f\colon (X, \alpha) \to (Z, \gamma)$, $g\colon (Y, \beta) \to (Z, \gamma)$ such that $f(x) = g(y)$.

Problem: It is hard to work with this definition.

Solution: Bisimulations as witnesses for **coalgebraic** behavioural equivalence!

Now we have an explanation:

Example

For the powerset functor, Barr-bisimilarity **coincides** with behavioural equivalence.

Good Notions of Bisimulation

Good Notions of Bisimulation

We are interested in relators whose class of L-bisimulations

- contains all coalgebra homomorphisms.

  because coalgebra homomorphisms are behaviour-preserving maps.
- is closed under converses.

  because behavioural equivalence is symmetric.
- is closed under composition.
  - because in many situations this is important to show that L-bisimilarity coincides with behavioural equivalence (and some other nice properties).

We have shown that **these properties** essentially **characterize lax extensions**.

Definition

A (symmetric) ***lax extension*** $\mathsf{L}\colon \mathsf{Rel} \to \mathsf{Rel}$ of a functor $\mathsf{F}\colon \mathsf{Set} \to \mathsf{Set}$ is an F-relator s.t.

1. $\mathsf{F}f \leq \mathsf{L}f$,
2. $\mathsf{L}(r^\circ) = (\mathsf{L}r)^\circ$,
3. $\mathsf{L}s \cdot \mathsf{L}r \leq \mathsf{L}(s \cdot r)$,

for all $r\colon X \nrightarrow Y$, $s\colon Y \nrightarrow Z$ and $f\colon X \to Y$. A lax extension is ***identity-preserving*** or ***normal*** if $\mathsf{L}1_X = 1_{\mathsf{F}X}$, for every set X.

The Barr relator is a lax extension iff the functor preserves weak pullbacks!

Let $\mathsf{L}\colon \mathsf{Rel} \to \mathsf{Rel}$ be a lax extension of a functor $\mathsf{F}\colon \mathsf{Set} \to \mathsf{Set}$.

Theorem (MV15)

If L *is identity-preserving, then* L*-bisimilarity coincides with behavioural equivalence.*

Conversely, we have shown:

Theorem

In practise, *if* L*-bisimilarity coincides with behavioural equivalence, then* L *is identity-preserving.*

Existence of Identity-preserving Lax Extensions

Since the smallest lax extension of a functor is given by the laxification of its Barr relator:

Corollary

A functor $\mathsf{F}\colon \mathsf{Set} \to \mathsf{Set}$ *admits a identity-preserving lax extension iff for every set* X *and every composable sequence of relations* $r_1, \ldots, r_n$

$$r_n \cdot \ldots \cdot r_1 = 1_X \implies \overline{\mathsf{F}}r_n \cdot \ldots \cdot \overline{\mathsf{F}}r_1 = 1_{\mathsf{F}X}.$$

So, we have a characterization…but we are looking for something simpler!

Preservation of Weak Pullbacks

Definition

A commutative square

$$\begin{array}{ccc} W & \xrightarrow{q} & Y \\ {\scriptstyle p}\downarrow & = & \downarrow{\scriptstyle g} \\ X & \xrightarrow[f]{} & Z \end{array}$$

is a weak pullback if for all $x \in X$ and $y \in Y$ s.t. $f(x) = g(y)$ there is $w \in W$ s.t. $x = p(w)$ and $y = q(w)$.

Example

For every function $f : X \to Z$ the following diagram – the inverse image of the empty set w.r.t f – is a weak pullback.

$$\begin{array}{ccc} \varnothing & = & \varnothing \\ \downarrow & & \downarrow \\ X & \xrightarrow[f]{} & Z \end{array}$$

Definition

We say that a functor $F\colon \mathrm{Set} \to \mathrm{Set}$ preserves ***weak pullbacks***, ***1/4-iso (weak) pullbacks***, ***1/4-mono (weak) pullbacks*** and ***4/4-epi weak pullbacks*** if it sends weak pullbacks of the following forms, respectively, to weak pullbacks

$$\begin{array}{ccc} W & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X & \longrightarrow & Z \end{array} \qquad \begin{array}{ccc} W & \xrightarrow{\simeq} & Y \\ \downarrow & & \downarrow \\ X & \longrightarrow & Z \end{array} \qquad \begin{array}{ccc} W & \rightarrowtail & Y \\ \downarrow & & \downarrow \\ X & \longrightarrow & Z \end{array} \qquad \begin{array}{ccc} W & \twoheadrightarrow & Y \\ \twoheaddownarrow & & \twoheaddownarrow \\ X & \twoheadrightarrow & Z, \end{array}$$

with arrows $\rightarrowtail$, $\twoheadrightarrow$ and $\xrightarrow{\simeq}$ indicating injectivity, surjectivity and bijectivity, correspondingly.

Neighbourhood Functors

Example

- The neighbourhood functor $\mathcal{N}\colon \mathsf{Set} \to \mathsf{Set}$ sends
  - a set X to the set $\mathcal{P}\mathcal{P}X$ of neighbourhood systems of X,
  - a function $f\colon X \to Y$ to the function $\mathcal{N}f\colon \mathcal{N}X \to \mathcal{N}Y$ that assigns to every element $\mathcal{A} \in \mathcal{N}X$ the set $\{B \subseteq Y \mid f^{-1}B \in \mathcal{A}\}$.
- The monotone neighbourhood functor $\mathcal{M}$ is the subfunctor of $\mathcal{N}$ that sends a set X to the set $\mathcal{M}X$ of **monotone** neighbourhood systems of X, $\forall A \in \mathcal{A}.\ (A \subseteq B \implies B \in \mathcal{A})$.
- Coalgebras for the (monotone) neighbourhood functor are (monotone) neighbourhood frames.

Examples

- The powerset functor $\mathcal{P}$ preserves weak pullbacks.
- The monotone neighbourhood functor $\mathcal{M}$
  - preserves 1/4-iso pullbacks and 4/4-epi weak pullbacks;
  - does **not** preserve 1/4-mono pullbacks.
- The neighbourhood functor $\mathcal{N}$
  - preserves 4/4-epi weak pullbacks;
  - does **not** preserve 1/4-iso pullbacks.

    Proof: it does not preserve the inverse image of the empty set w.r.t the map $!_2 \colon \{a, b\} \to \{*\}$.

What Was Known

- Every functor that preserves weak pullbacks admits an identity-preserving lax extension.

  Ex: The powerset functor $\mathcal{P}$

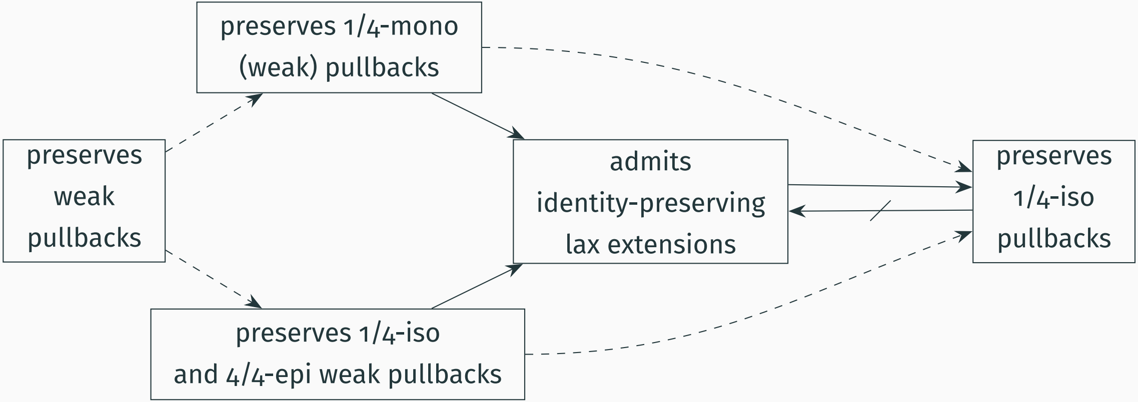
- There are functors that do not preserve weak pullbacks that admit identity-preserving lax extensions.

  Ex: The monotone neighourhood functor $\mathcal{M}$.

- There are functors that do not admit identity-preserving lax extensions.

  The neighbourhood functor $\mathcal{N}$ [Marti and Venema 2015].

What Is Known



Solid arrows are contributions, dashed arrows are trivial. All implications are non-reversible.

Example: M-weighted Transition Systems

Example

Given a commutative monoid $(M, +, 0)$ (or just M), the *monoid-valued functor* $M^{(-)}$ maps a set X to the set $M^{(X)}$ of functions $\mu\colon X \to M$ with *finite support*, i.e. $\mu(x) \neq 0$ for only finitely many x.

$M^{(-)}$-coalgebras correspond to M-weighted transition systems, and we have:

Corollary

A (commutative) monoid-valued functor admits an identity-preserving lax extension iff the monoid is positive.

Since the additive monoid $\mathbb{Z}$ id **not** positive, there is **no** "good" notion of bisimulation for $\mathbb{Z}$-weighted transition systems!

Open Questions

- Useful characterization of the functors that admit an identity-preserving lax extension.
    - We need new strategies to go beyond our sufficient conditions.
    - Not clear how to unify both cases.
- Useful characterization of the functors that admit a greatest identity-preserving lax extension.
    - Maximally permissive notion of bisimulation.