Independence and Domination on Bounded-Treewidth Graphs:

Integer, Rational, and Irrational Distances

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STACS
March 5th, 2025

Integer and Rational Distances

(Distance) a-Independent Set

Distance a-Independent Set:
Find a maximum subset $S \subseteq V(G)$, s.t. all $u, v \in S$ have distance $\geq a$.

- IndSet $=$ 2-IndSet is NP-hard [Karp 1972]

[1] We assume a tree-decomposition of width tw as part of the input.
[2] Assuming SETH, there is no $\varepsilon > 0$ and an $(a - \varepsilon)^{tw} \cdot n^{O(1)}$ time algorithm.

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- a-IndSet, for $a \geq 3$, is NP-hard (even for planar bipartite subcubic graphs). [Eto, Guo, Miyano 2012]
- a-IndSet in time $a^{tw(G)} \cdot n^{O(1)}$,[1] tight under SETH.[2] [Katsikarelis, Lampis, Paschos 2019]

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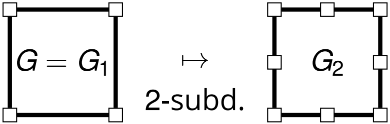
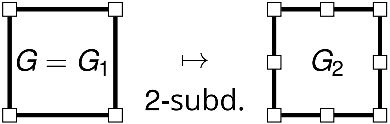
a-Independent Set on b-Subdivided Graphs



a-Independent Set on b-Subdivided Graphs



- [Grigoriev, H, Lendl, Woeginger 2019]:
  2-IndSet on 2-subdivided graphs is in P.
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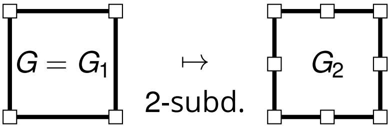
  Actually:
  1-Dispersion and 2-Dispersion is in P.

a-Independent Set on *b*-Subdivided Graphs



b \ a	1	2	3	4	5	6
1						
2						
3						
4						
5						
6						

P, NP-hard

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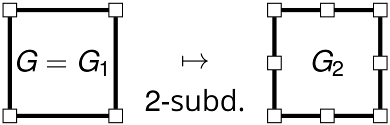
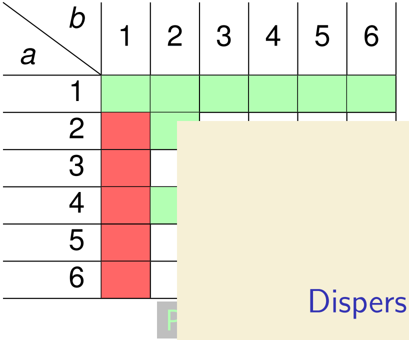
. . . and *a*-IndSet on *b*-subdivided graphs?

a-Independent Set on b-Subdivided Graphs





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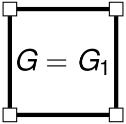
ns?

Dispersing Obnoxious Facilities on a Graph

STACS 2019

a-Indeper



Dispersing Obnoxious Facilities on a Graph

STACS 2019

Alexander Grigoriev, Tim A. Hartmann, Stefan Lendl,
Gerhard J. Woeginger

March 15, 2019

a \ *b*	1
1	
2	
3	
4	
5	
6	

19]:

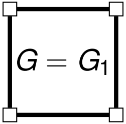
s in P.

s in P.

n P.

hs?

a-Indeper



a \ b	1	
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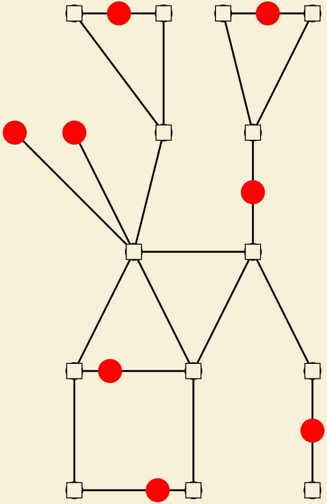
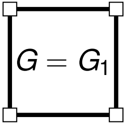
δ-Dispersion, positive $\delta \in \mathbb{R}$



- fixed positive distance $\delta \in \mathbb{R}$
- *Input:* Graph G with **unit length edges**
- *Task:* Place maximum δ-dispersed set in G,
  that is a set of **points** pairwise in distance δ.

19]:
s in P.
s in P.
n P.
hs?

a-Indeper



b / a	1	
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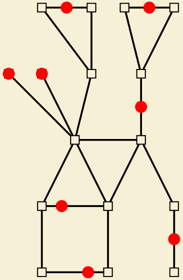
Normalize Solutions



$\mapsto$ $\mapsto$

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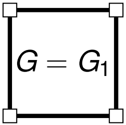
hs?

a-Indeper



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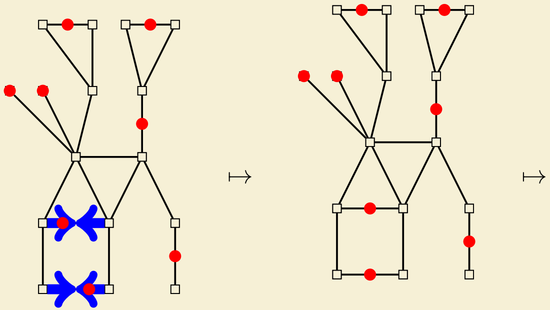
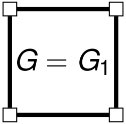


- Wlog. a 2-dispersed set has all its points on a **vertex** or a **midpoint** of an edge.

a-Indeper



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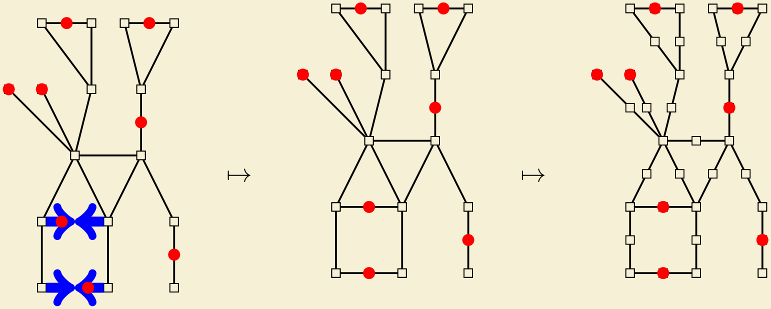

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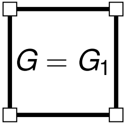
- 2-dispersed set in G = 4-IndSet in G_2.

a-Indeper



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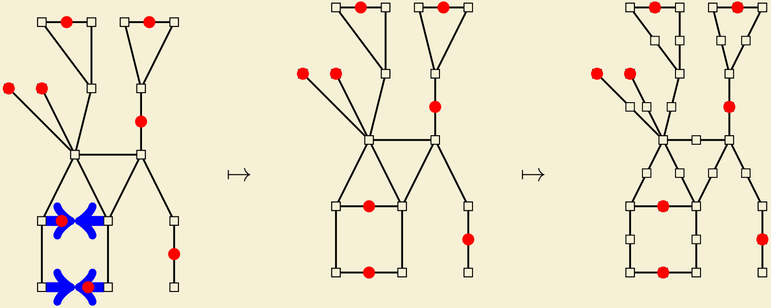
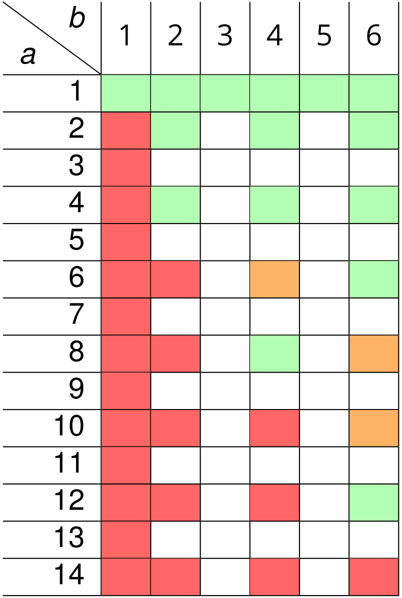


- Wlog. a 2-dispersed set has all its points on a **vertex** or a **midpoint** of an edge.
- 2-dispersed set in G = 4-IndSet in G_2.
- $\frac{a}{b}$-dispersed set in G = $(2a)$-IndSet in G_{2b}.

a-IndSet on *b*-Subdivided Graphs (2019)

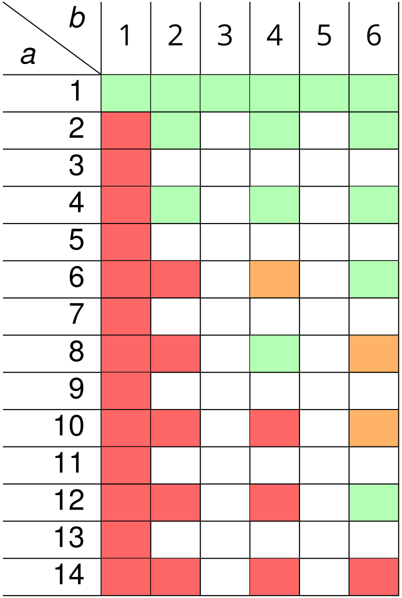


P

NP-hard, FPT in solution size

NP-hard, W[1]-hard

a-IndSet on b-Subdivided Graphs (2019)



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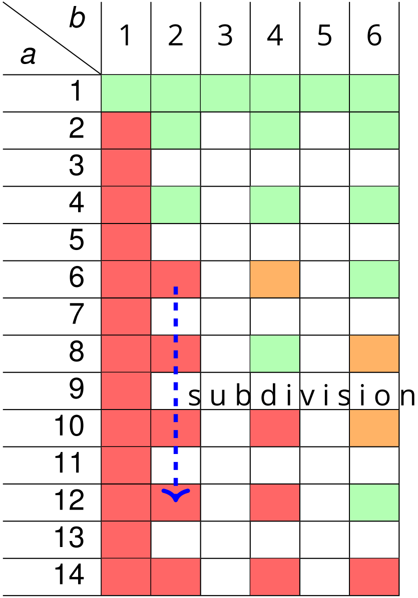
$\frac{a}{b}$-disp$(G) = c\frac{a}{b}$-disp(G)

a-IndSet on b-Subdivided Graphs (2019)



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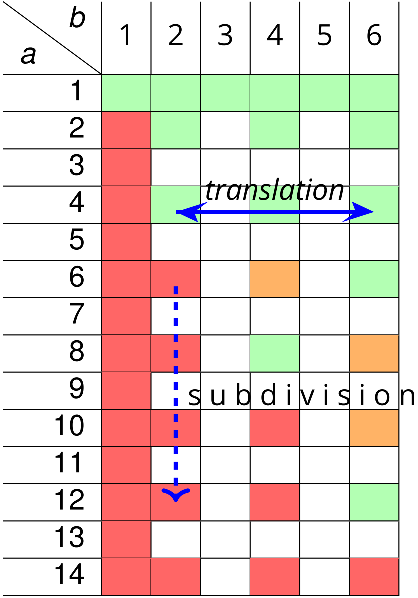
$\frac{a}{b}$-disp$(G) = c\frac{a}{b}$-disp(G)

subdivision

$\frac{a}{b}$-disp$(G) = \frac{ca}{b}$-disp(G_c)

a-IndSet on b-Subdivided Graphs (2019)



P

NP-hard, FPT in solution size

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$\frac{a}{b}$-disp$(G) = c\frac{a}{b}$-disp(G)

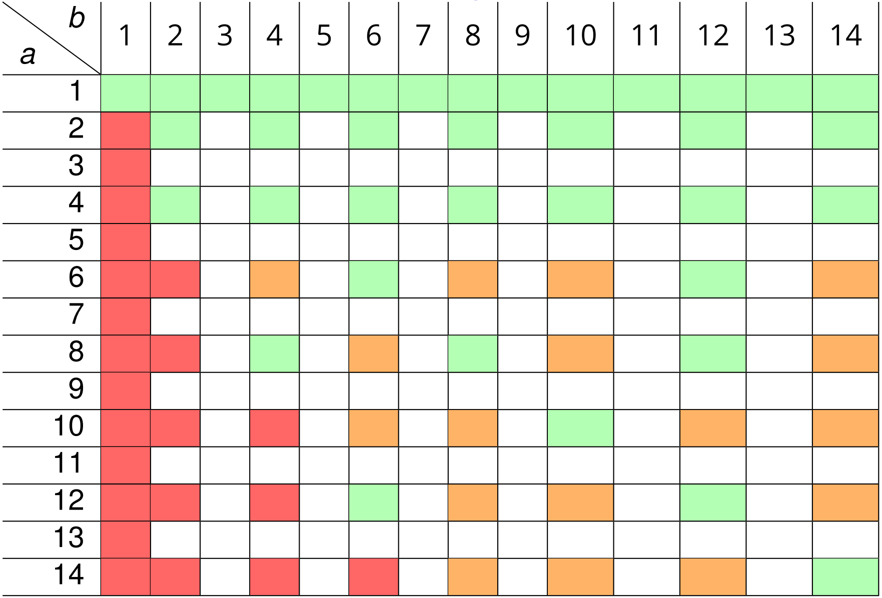
translation

$\frac{a}{b}$-disp(G) = $\frac{a}{a+b}$-disp$(G) + |E|$

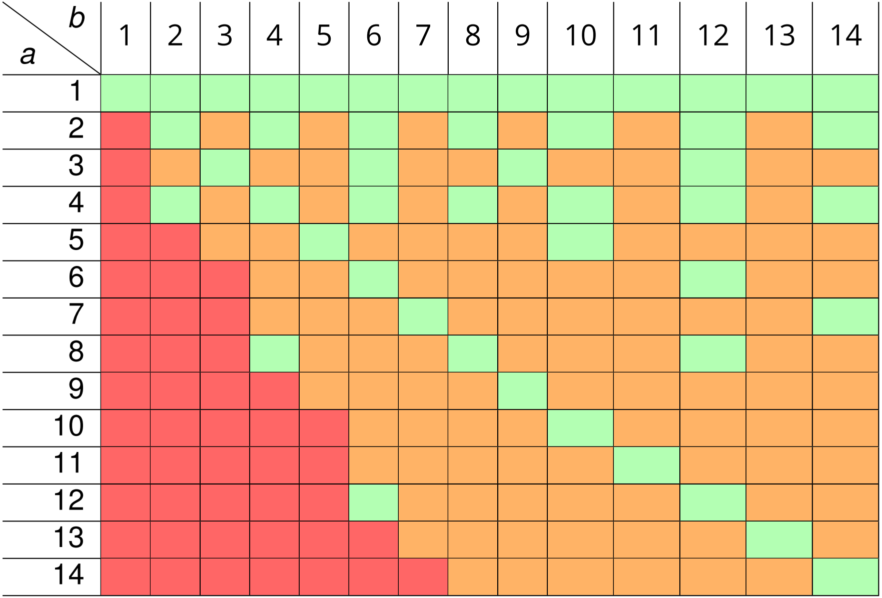
subdivision

$\frac{a}{b}$-disp(G) = $\frac{ca}{b}$-disp(G_c)

a-IndSet on *b*-Subdivided Graphs (2019)



This Work



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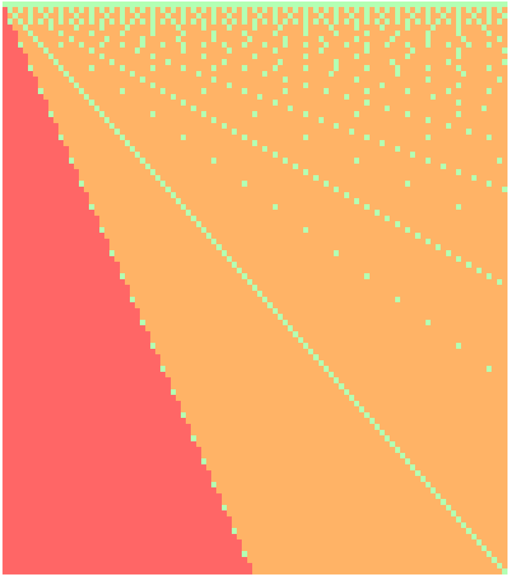


Bounded Treewidth Graphs

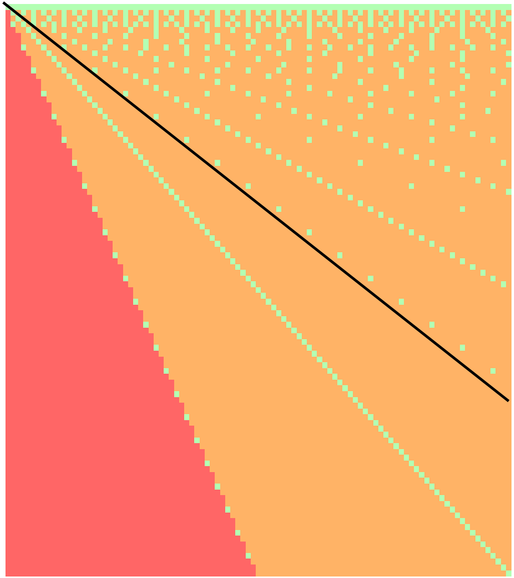
Theorem

Let integers a', b' define $\gcd(a', b') = c$ and $ca = a'$ and $cb = b'$.

- If $\gcd(a', b')$ is odd:
  If $a = 1$, a'-IndSet on b'-subdivided graphs is in P,
  else can be solved in time $a^{tw} \cdot n^{O(1)}$ (tight under SETH).
- If $\gcd(a', b')$ is even:
  If $a \in \{1, 2\}$, a'-IndSet on b'-subdivided graphs is in P,
  else can be solved in time $(2a)^{t} \cdot n^{O(1)}$ (tight under SETH).
- If $a' \in \{1, 2\}$, $\frac{a'}{b'}$-Dispersion is in P;
  else can be solved in $(2a)^{tw} \cdot n^{O(1)}$ time (tight under SETH).

Irrational Distances

Irrational Distance



- For every non-rational δ: δ-Dispersion is NP-hard.

  [H,Lendl 2022]

- What about graphs of bounded treewidth?

Irrational Distance

Theorem (Reminder)

Let integers a', b' define $\gcd(a', b') = c$ and $ca = a'$ and $cb = b'$. If $a' \in \{1, 2\}$, $\frac{a'}{b'}$-Dispersion is in P; else can be solved in $(2a)^{tw} \cdot n^{O(1)}$ time (tight under SETH).

What about a fixed **irrational** δ?

Irrational Distance

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e.g. $\delta = 0.01011 \ldots i \ldots$ with $i = 1$ iff the i-th TM halts on ε.

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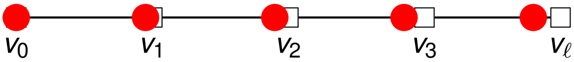

δ-Dispersion is not computable, ... even on paths!

Above δ is not **efficiently comparable** to rationals.

Irrational but Efficiently Comparable

consider δ that is **efficiently comparable** to rationals:
given x, y, if $\frac{x}{y} \leq \delta$ is decidable in $poly(\log x + \log y)$.

By a rounding argument:
δ-dispersion $= \frac{a}{b}$-dispersion with $a, b \leq 2n$ with polytime comp.

[H, Lendl 2022]

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Corollary (XP algorithm)

δ-Dispersion is computable in ($a^{O(tw)} n^{O(1)} =$) $n^{O(tw)}$ time.

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Theorem (W[1]-hardness

There is an **efficiently comparable irrational**
$\delta = (4 \sum_{j=1}^{\infty} 2^{-2^j})^{-1} \approx 0.790085 \dots$
for which δ-Dispersion is W[1]-hard in *treewidth*.

Hardness Reduction

COLORFULCLIQUE k color classes each of size n	$\leq_p$	δ-DISPERSION tw $O(k)$

Hardness Reduction

c_n-Dispersion
tw $O(k)$

ColorfulClique
k color classes
each of size n

$\leq_P$

δ-Dispersion
tw $O(k)$

Hardness Reduction

c_n-DISPERSION
tw $O(k)$

COLORFULCLIQUE
k color classes
each of size n

$\leq_p$

same

$\leq_p$

$(c_n - 1)$- DISPERSION
tw $O(k)$

δ-DISPERSION
tw $O(k)$

Step 1: Leeway
(but c_n depends on n)

Hardness Reduction

c_n-DISPERSION tw $O(k)$ $\leq_p$ δ_n-DISPERSION tw $O(k)$

$\|$

COLORFULCLIQUE: k color classes each of size n $\leq_p$ (same) δ-DISPERSION tw $O(k)$

$\|$

$(c_n - 1)$- DISPERSION tw $O(k)$ $\leq_p$ γ_n-DISPERSION tw $O(k)$

Step 1: Leeway

(but c_n depends on n)

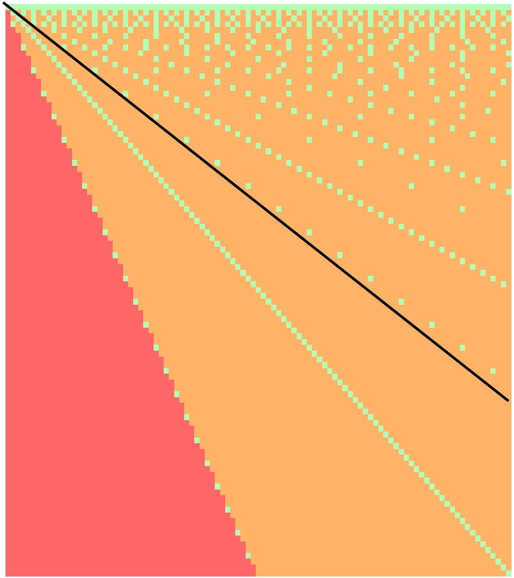
Step 2: Approximating δ

$\gamma_n < \gamma_{n+1} < \delta < \delta_{n+1} < \delta_n$

(by *translations* and s u b d i v i s i o n s)

Summary

Independence



- a-IndSet on b-subdivided graphs:
  P vs NP,
  FPT vs W[1]-hard in solution size
- If not in P, in time of form $a^{tw} \cdot n^{O(1)}$
  (tight under SETH).
- irrational but efficiently comparable
  $0.790085\ldots$-Dispersion
  is W[1]-hard in treewidth

Independence and Domination