Listing spanning trees of outerplanar graphs by pivot-exchanges

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STACS 2025

Introduction

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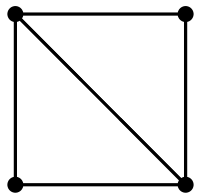
$G =$ 

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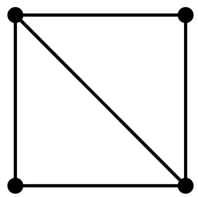

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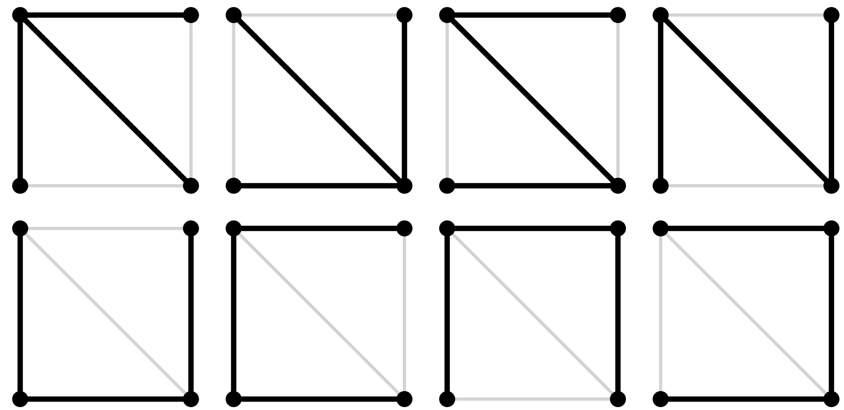

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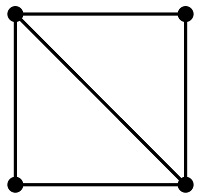
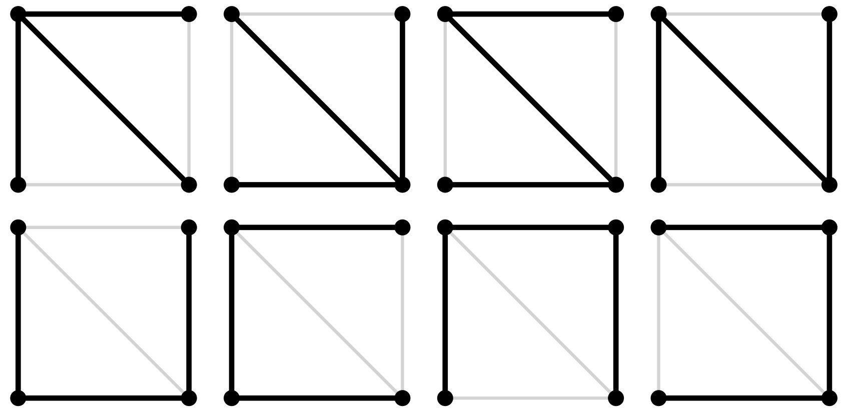
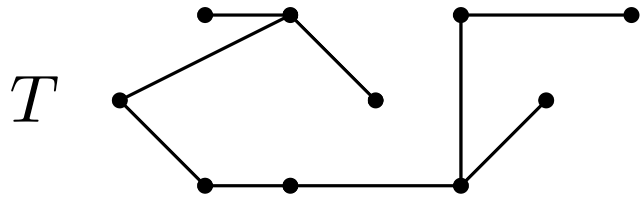
$G =$  $\mathcal{T}(G) =$ 

- of fundamental interest in many CS settings: optimization, counting, random sampling, **exhaustive generation**

Edge exchanges

- edge exchange: $T' = T + e - f = T \triangle \{e, f\}$



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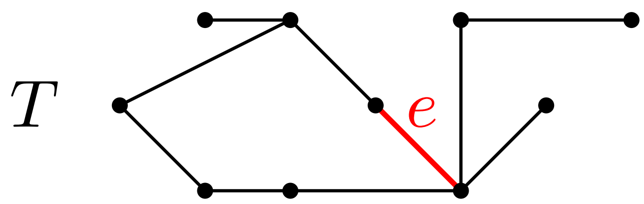

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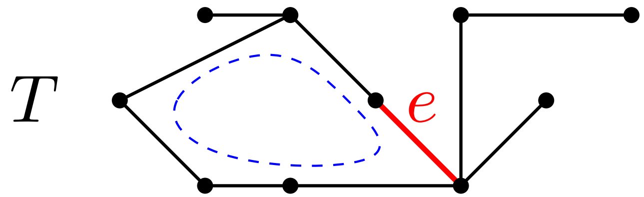

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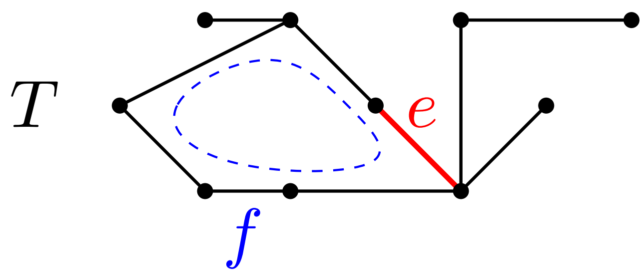

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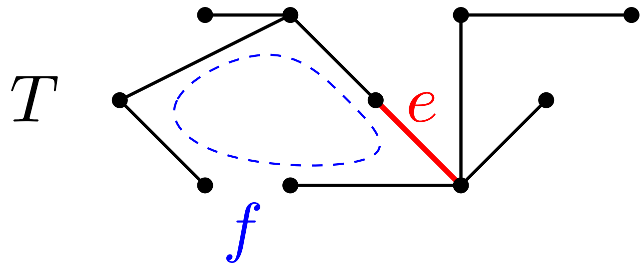

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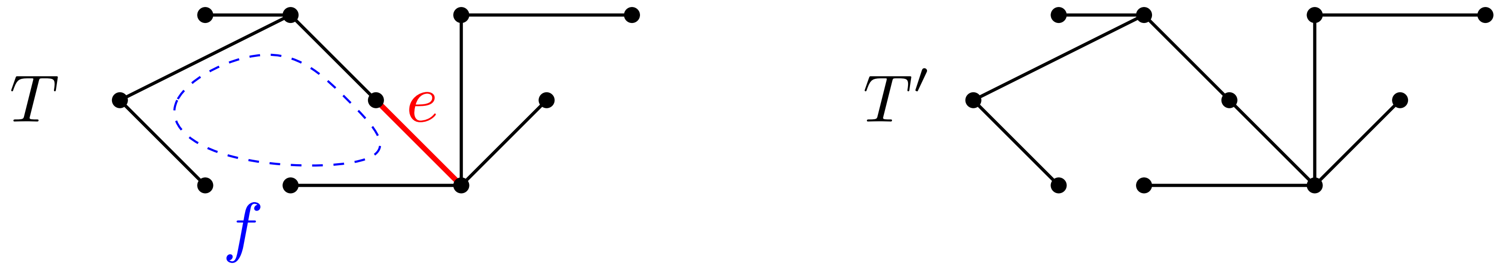

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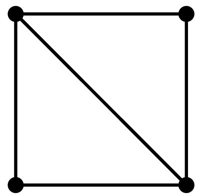
$G =$ 

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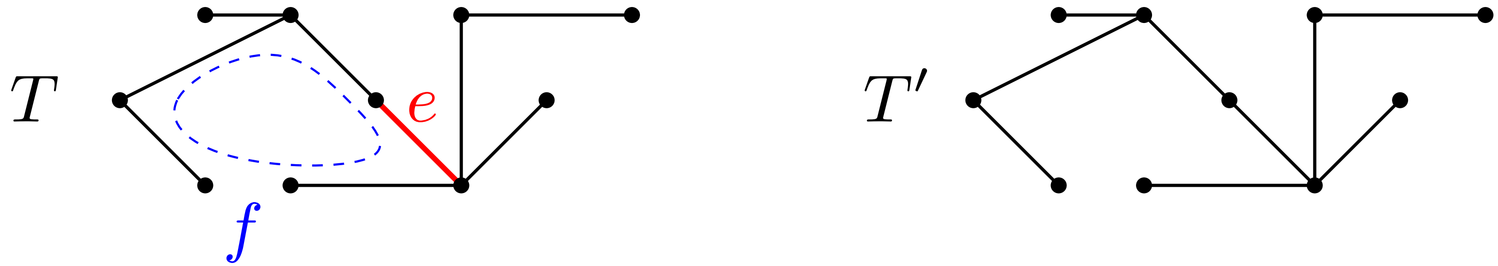

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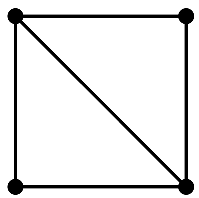
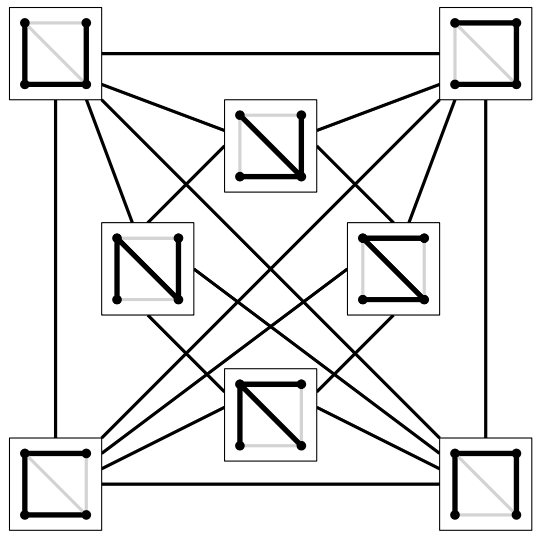
$G =$ 

$\mathcal{F}(G)$ 

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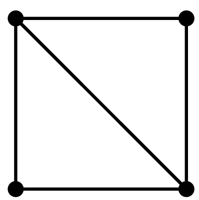

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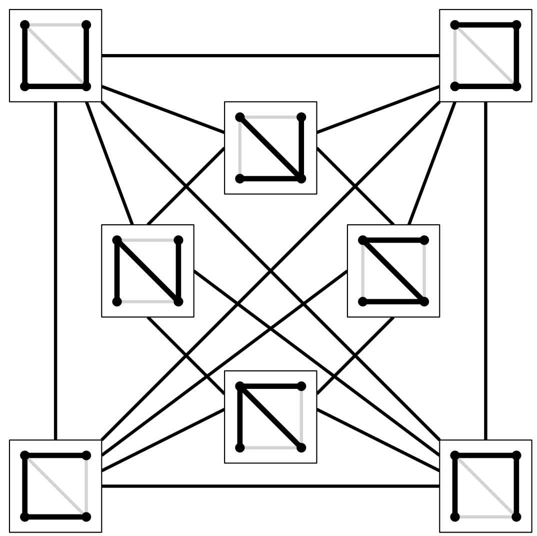

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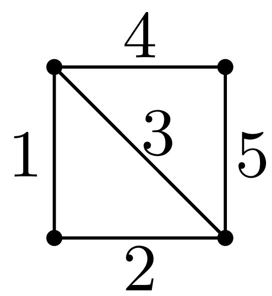

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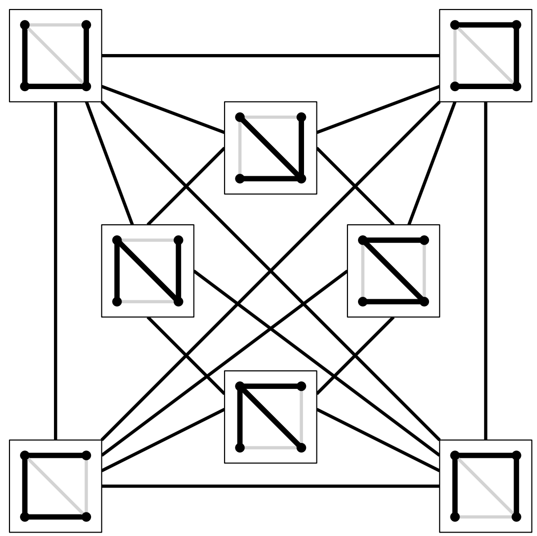

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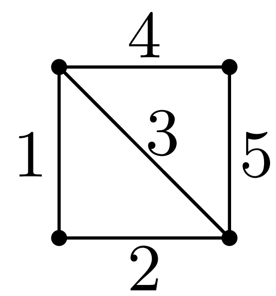

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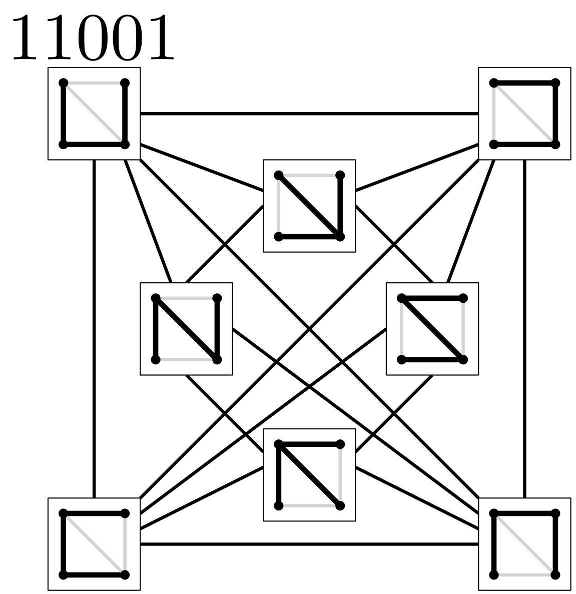

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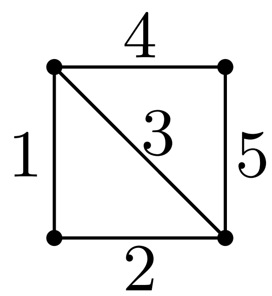

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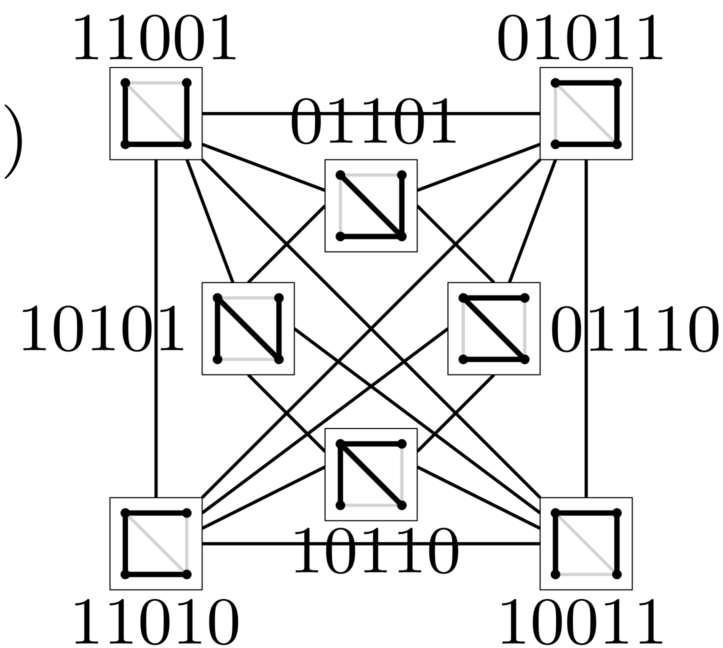

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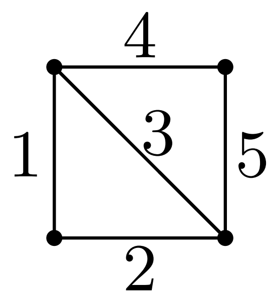

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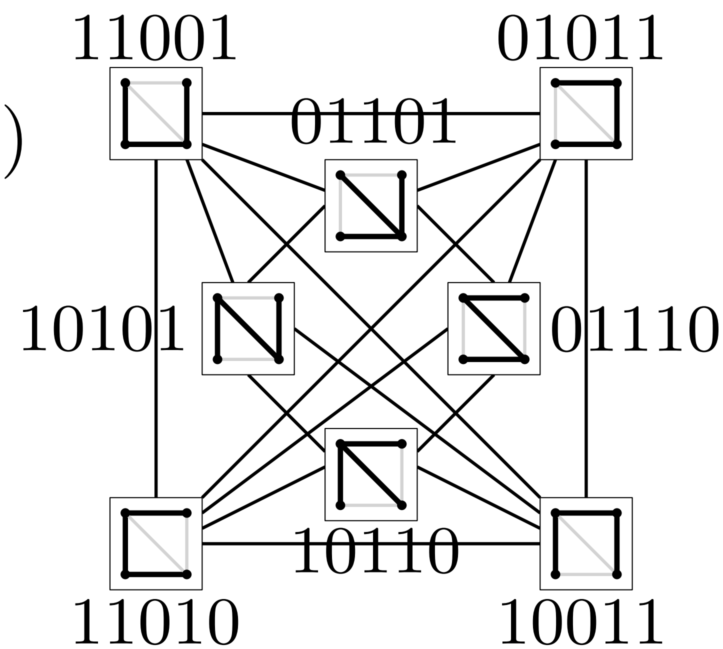

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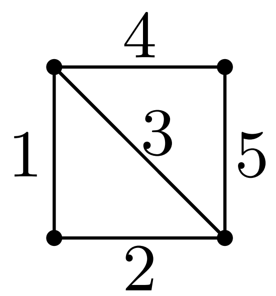

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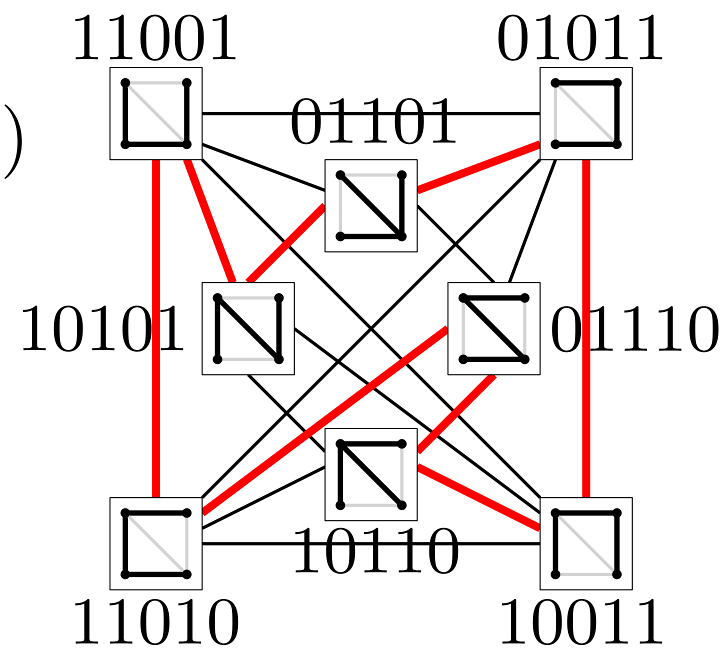

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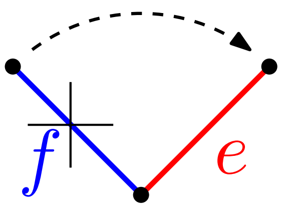
- restricting the allowed edge exchanges?

Pivot-exchanges

- **pivot-exchange** [Cameron, Grubb, Sawada'24]: exchanged edges must have a common end vertex



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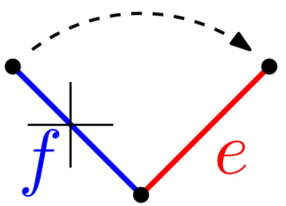

$$T' = T + e - f$$

- corresponds to a subgraph $\mathcal{F}_{\mathrm{p}}(G)$ of $\mathcal{F}(G)$

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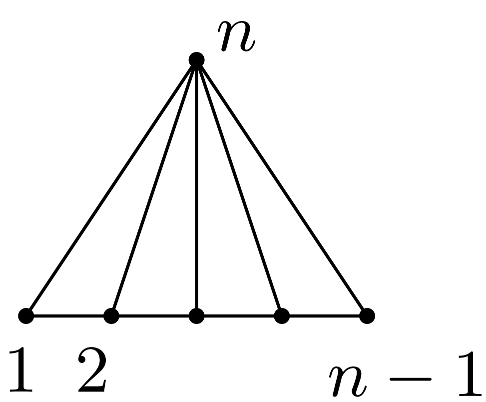
- special case of an even harder problem raised in Knuth's book 'The Art of Computer Programming' (problem 102 in Section 7.2.1.6)
- **Theorem** [Cameron, Grubb, Sawada'24]:
  For $G = F_n$ a **fan graph**, there is a Hamilton cycle in $\mathcal{F}_{\mathrm{p}}(G)$.

$F_n =$ 

Our results

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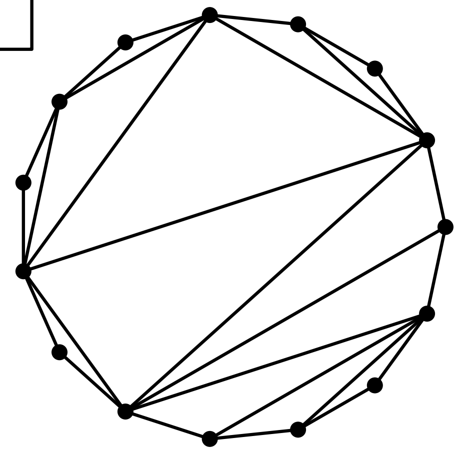
$G =$ 

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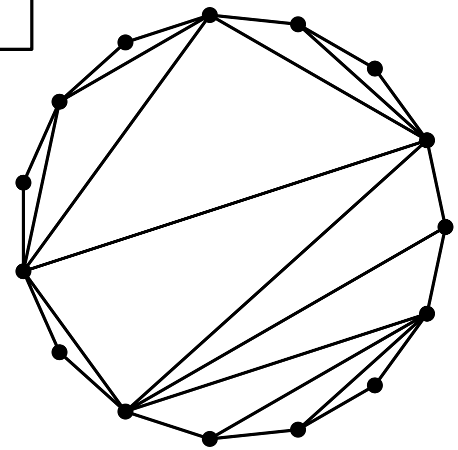
$G =$ 

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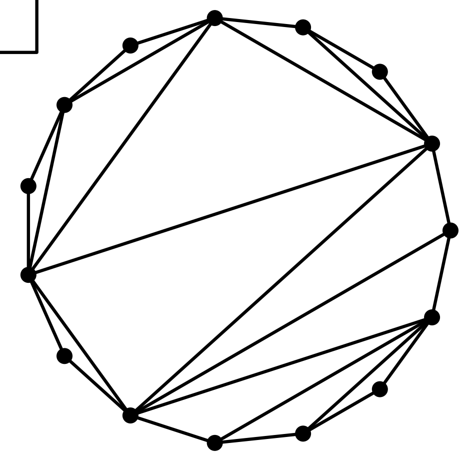
$G =$ 

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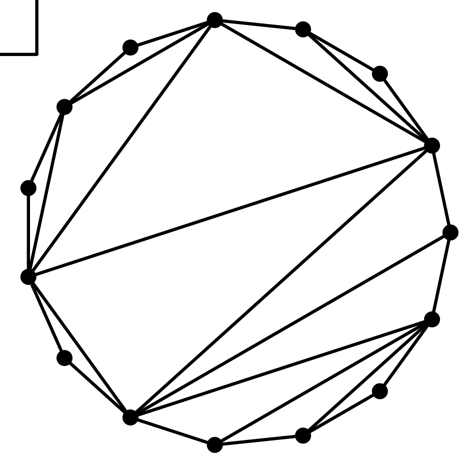
$G =$ 

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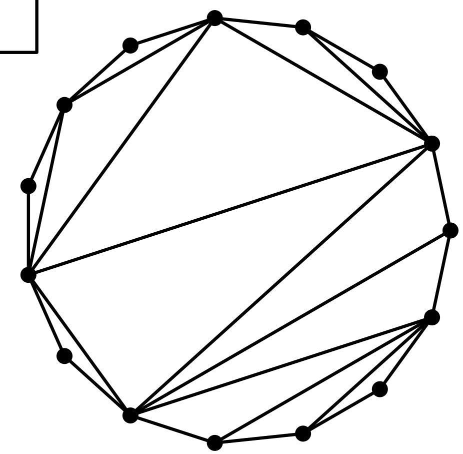

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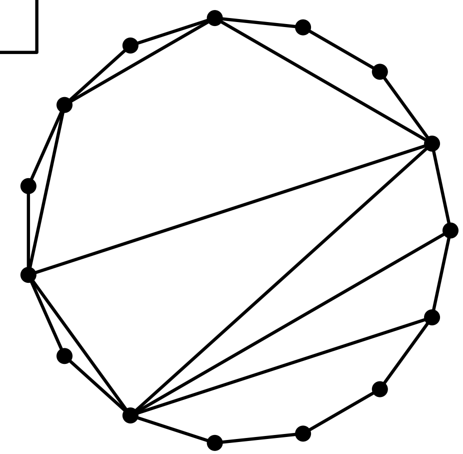
$G =$ 

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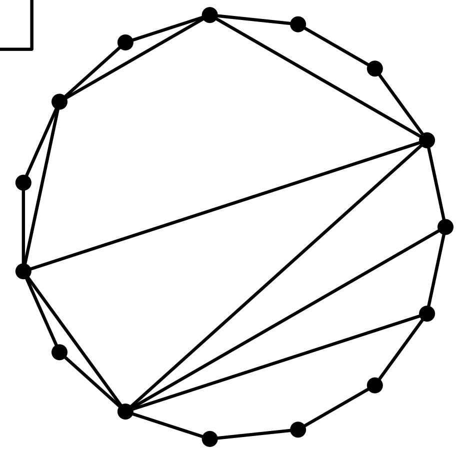
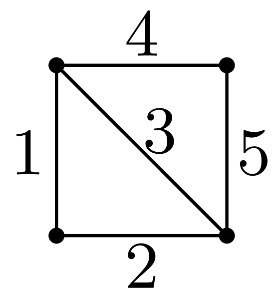
$G =$ 

- can be computed by a simple algorithm in time $\mathcal{O}(n \log n)$ per spanning tree
- general outerplane graphs G? Still true for exchanges $\{e, f\}$ such that e and f share a **common end vertex** **or** **a common face** (=pivot- or face-exchanges)

A simple greedy algorithm

edge-labeling of G: bijection $\ell : E(G) \to \{1, \ldots, m\}$

$G =$ 

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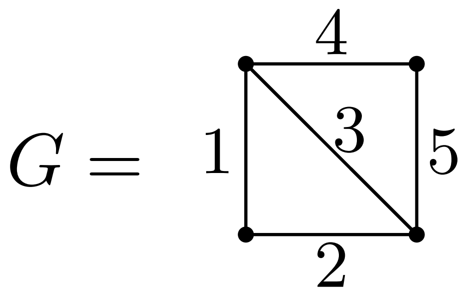

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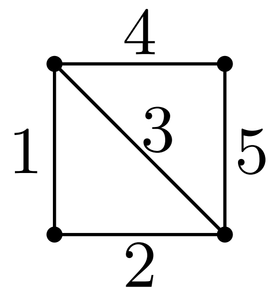
$G =$ 

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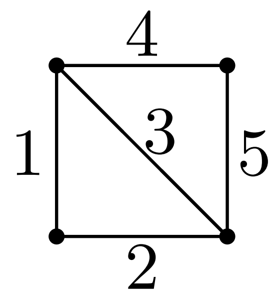
$G =$ 

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ties may arise (same value $\max\{e, f\} = \max\{e', f\}$)

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- **Theorem** [Mütze,Merino,Williams'22]:
  $\forall G$, $\forall \ell$, $\forall \widetilde{T}$, $\forall \tau$:
  Algorithm G computes a Hamilton path in $\mathcal{F}(G)$.

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- **Theorem**:
  $\forall$ outerplane triangulation G, $\exists \ell$, $\forall \widetilde{T}$, $\exists \tau$:
  Algorithm G computes a Hamilton path in $\mathcal{F}_{\mathrm{p}}(G)$.

Open problems

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- Knuth [TAOCP]: Is there a 'nice' Hamilton cycle in $\mathcal{F}(K_n)$?

Thank you!