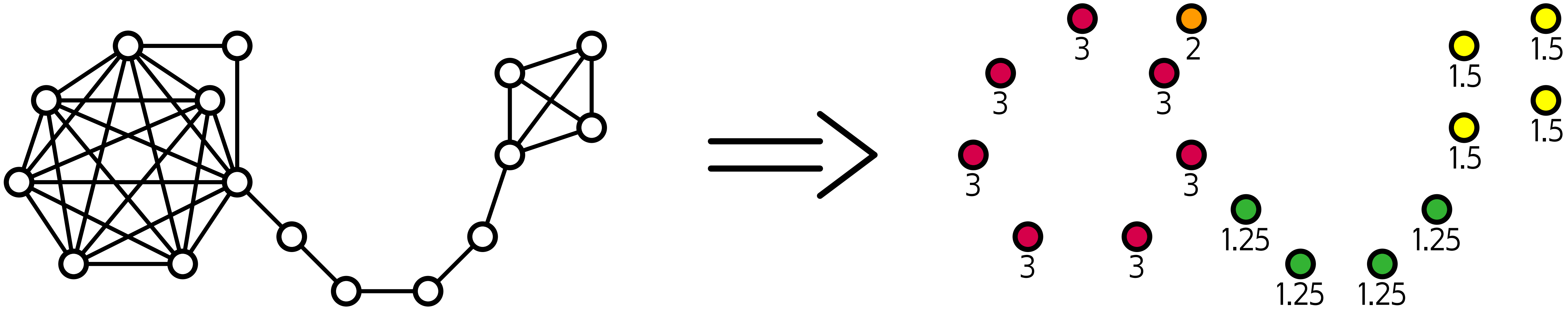
Local Density and its Distributed Approximation



Aleksander B. Christiansen

Ivor van der Hoog

Eva Rotenberg

Maximum Subgraph Density

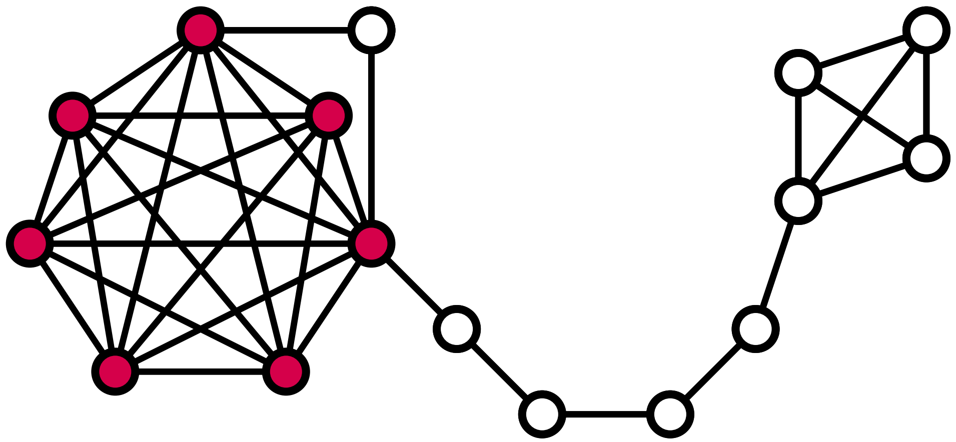
Input: graph $G = (V, E)$.

Given any subset $X \subset V$:

$E[X] := \{(u, v) \in E \mid u, v \in X\}$.

$G[X] := (X, E[X])$.

$$\rho(G[X]) := \frac{|E[X]|}{|X|} = \frac{21}{7} = 3.$$



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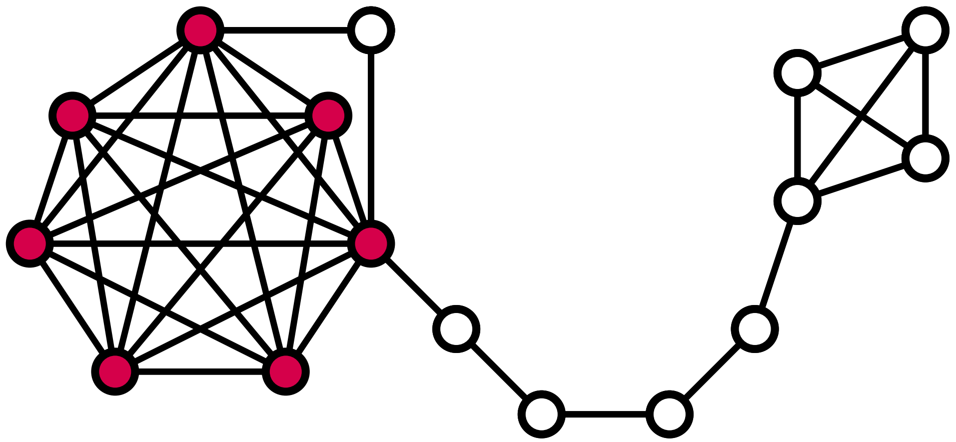


We define the **maximum subgraph density** as:

$$\rho^{\max}(G) := \max_{X \subset V} \rho(G[X])$$

There exist **many** papers that either compute or dynamically maintain $\rho^{\max}(G)$.

Why care about Maximum Subgraph Density?

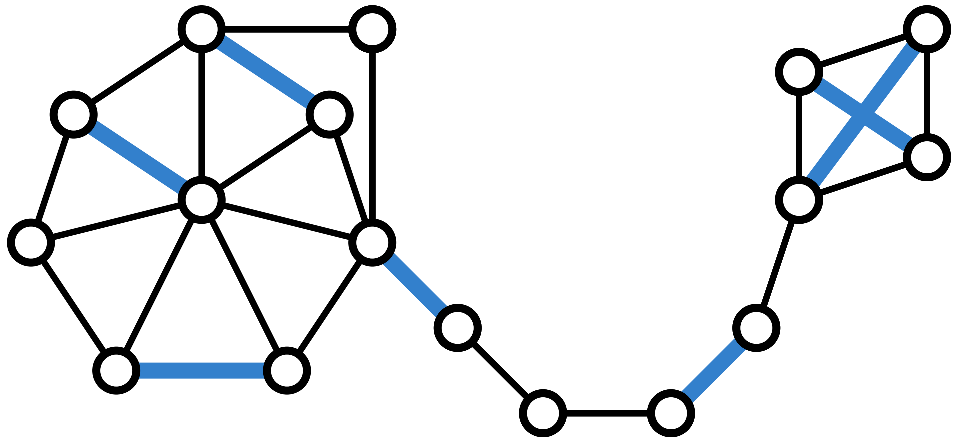
Many algorithms can have their running time be parametrized by $\rho^{\max}(G)$.

E.g., dynamic maximal matching.

Input: dynamic graph $G = (V, E)$.

Edge insertion + deletion.

Maintain a maximal matching.



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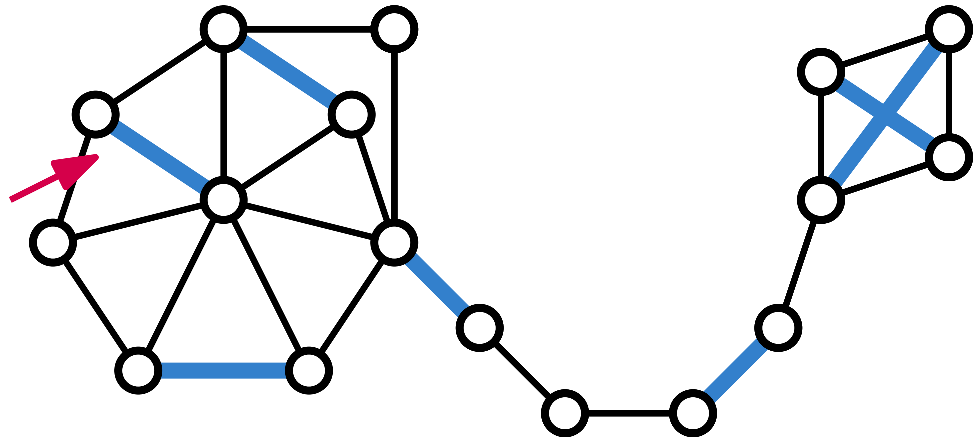

Naive algorithm:

- Upon a deletion (u, v), check the neighborhood of u and v

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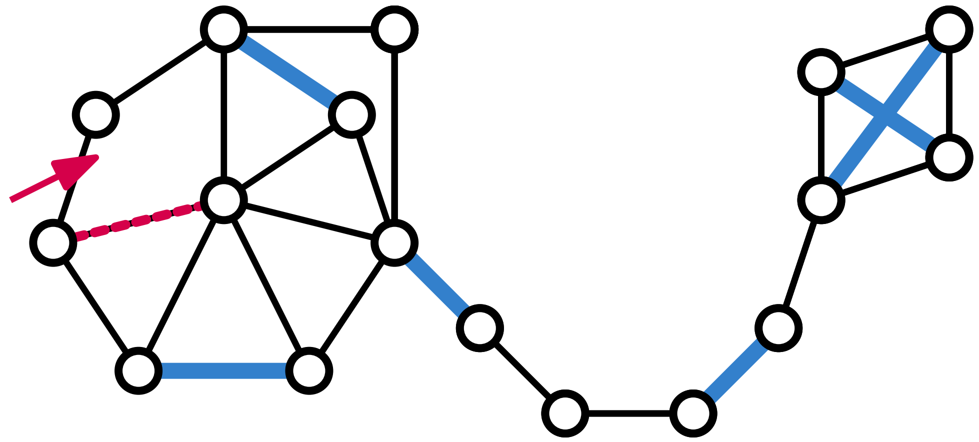

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Runtime: $O(\Delta)$

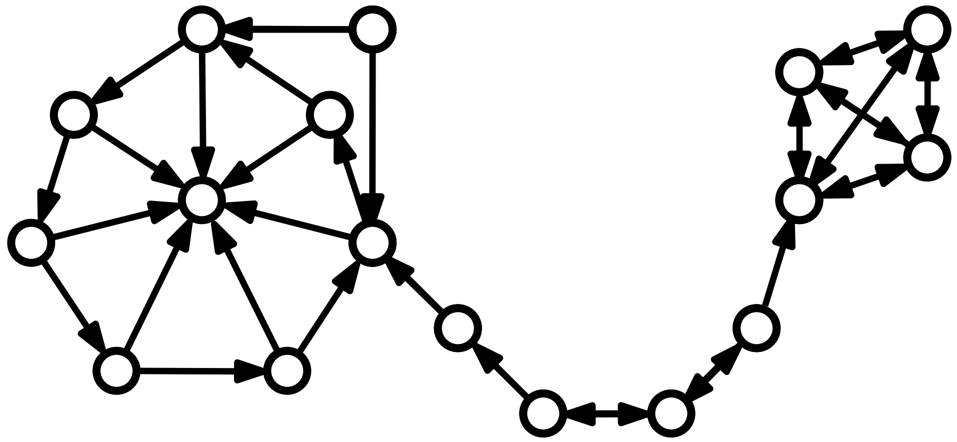
Runtime can be parametrized by $\rho^{\max}(G)$ instead!

Duality: fractional orientations

Input: dynamic graph $G = (V, E)$.

Maximum subgraph density: $\rho^{\max}(G) := \max_{X \subset V} \rho(G[X])$.

Minimum out-degree $\Delta^{\min}(G) := \min_{\vec{G}} \max_{v \in V} \Delta(v)$.



Fractional orientation $\vec{G}$:

$\forall (u, v) \in E$, define: $g(u \rightarrow v), g(v \rightarrow u) \in [0, 1]$.

$\forall (u, v) \in E$, require: $g(u \rightarrow v) + g(v \rightarrow u) = 1$.

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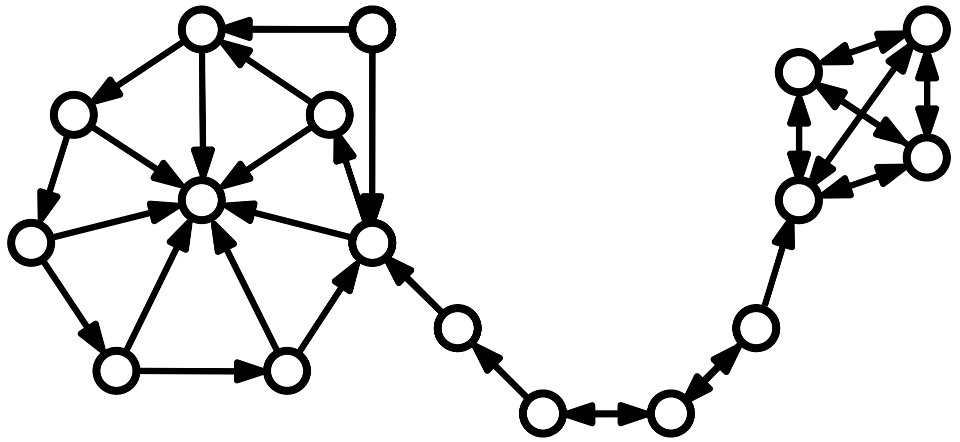

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Minimum out-degree $\Delta^{\min}(G)$:

Minimum over all fractional orientations, of the maximum out-degree $\Delta(v)$ over all $v \in V$.

Duality of linear programs:

$\rho^{\max}(G) = \Delta^{\min}(G)$

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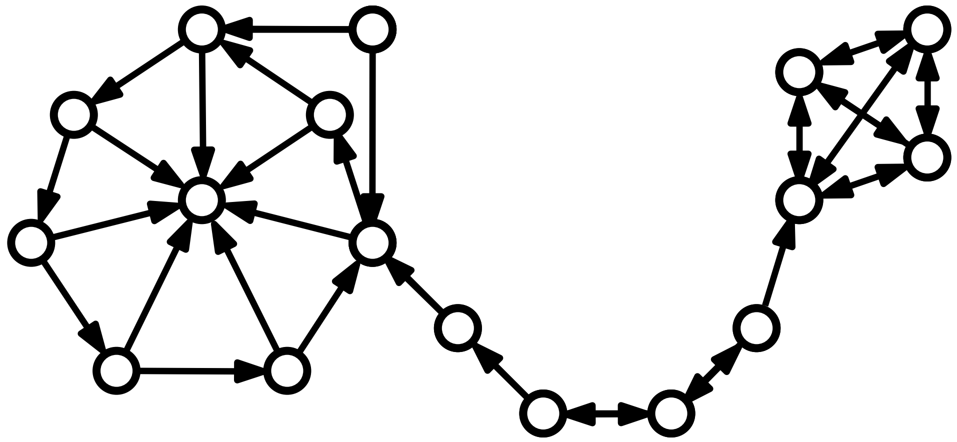


Smart algorithm:

Maintain an orientation with minimum out-degree.

Each node v maintains a list of all un-matched in-neighbors.

Runtime: $O(\rho^{\max}(G) + \text{polylog } n)$

Delete (u, v):

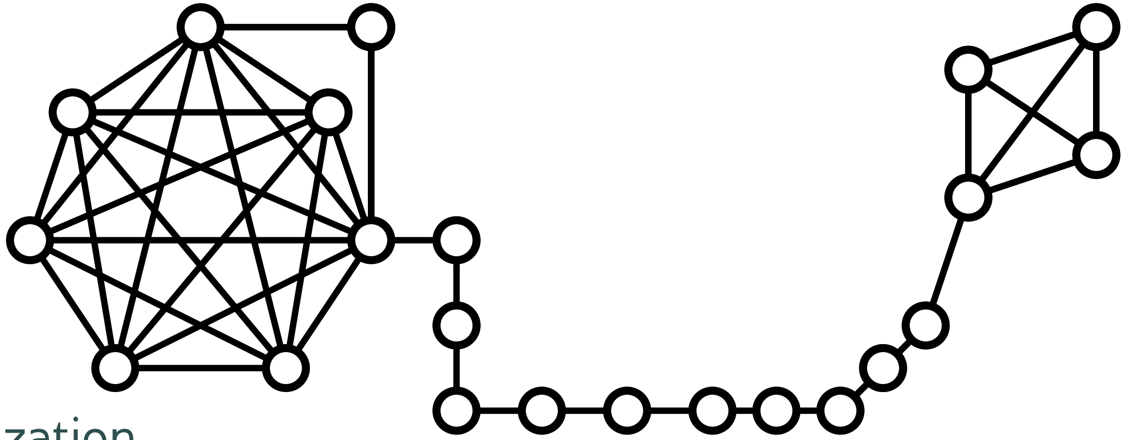
The vertices u, v inform their out-neighbors.

If unmatched, u checks its outneighbors.

If still unmatched, u checks whether its in-neighbor list is empty.

This paper: Local density

Danisch, Chan, and Sozio. [WWW, 2017].



Maximum subgraph density has downsides:

- It is a global measure, not a local one.
- Distributed lower bound: $\Omega(D)$ where D is the diameter.
- Dynamic algorithms get a very crude running time parametrization.

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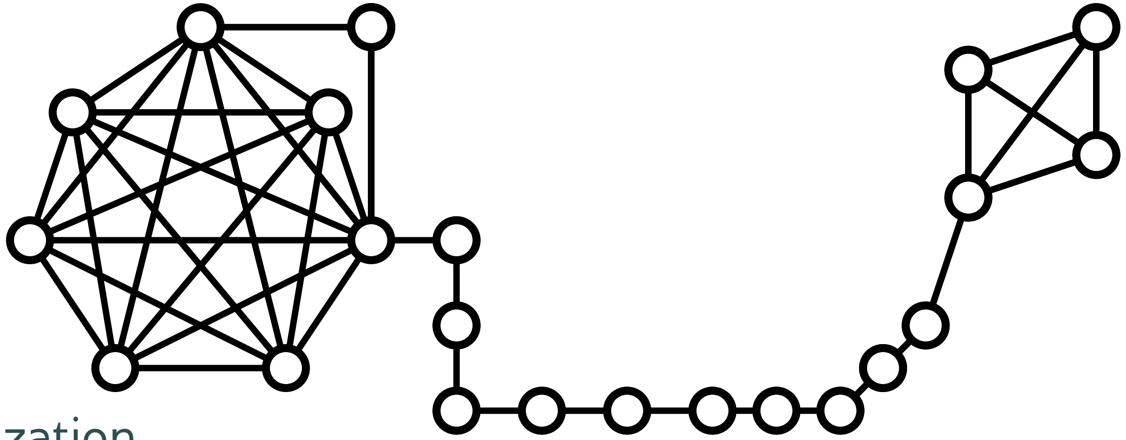

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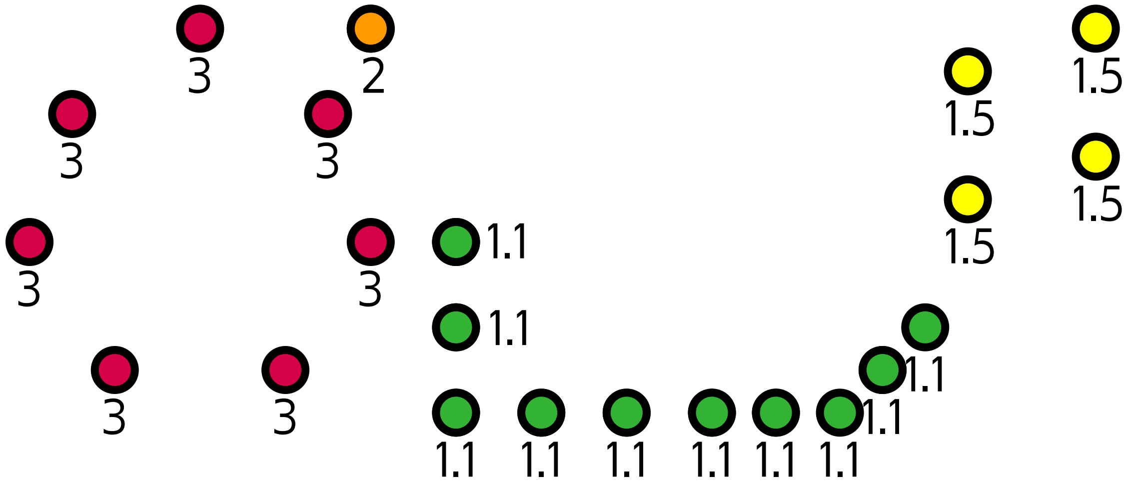
Their idea:

- Define a measure to measure how ‘locally dense’ the graph is around a vertex v.
- Local density assigns to each vertex $v \in V$ a value $\rho^*(v)$,
- Experimental paper.
- They approach $\rho^*(v)$ with no guarantees.



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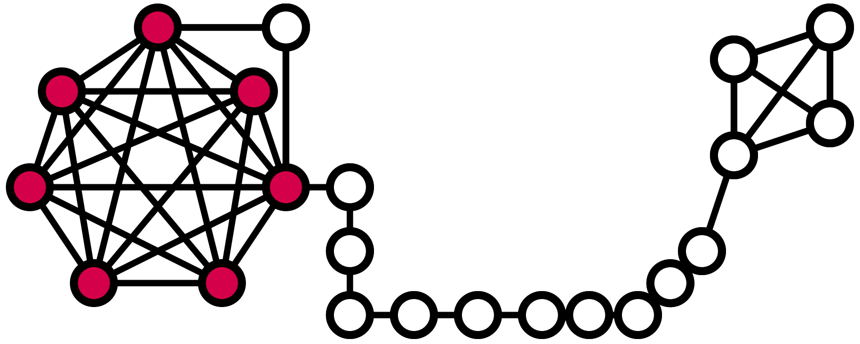

Iterative definition:

1: find $X \subseteq V$ that maximizes $\rho(G[X])$.

2: Remove X and all edges (u, v) with $u, v \in X$.

3: Replace all (u, v) with $u \notin X$ and $v \in X$ with a self-loop.

4: Repeat.

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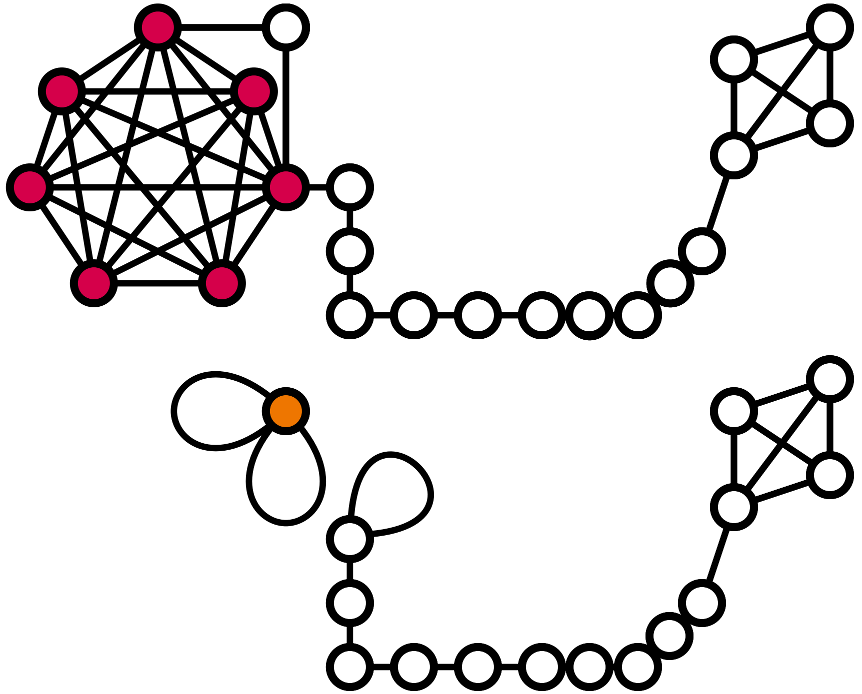

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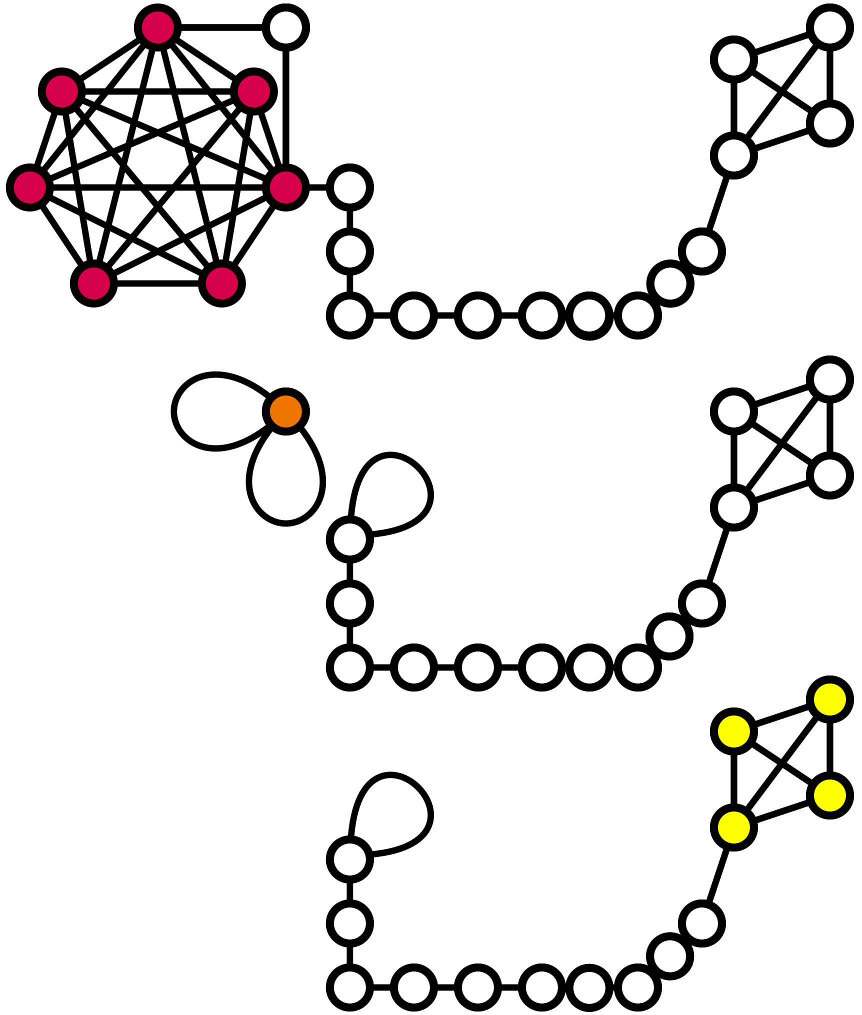

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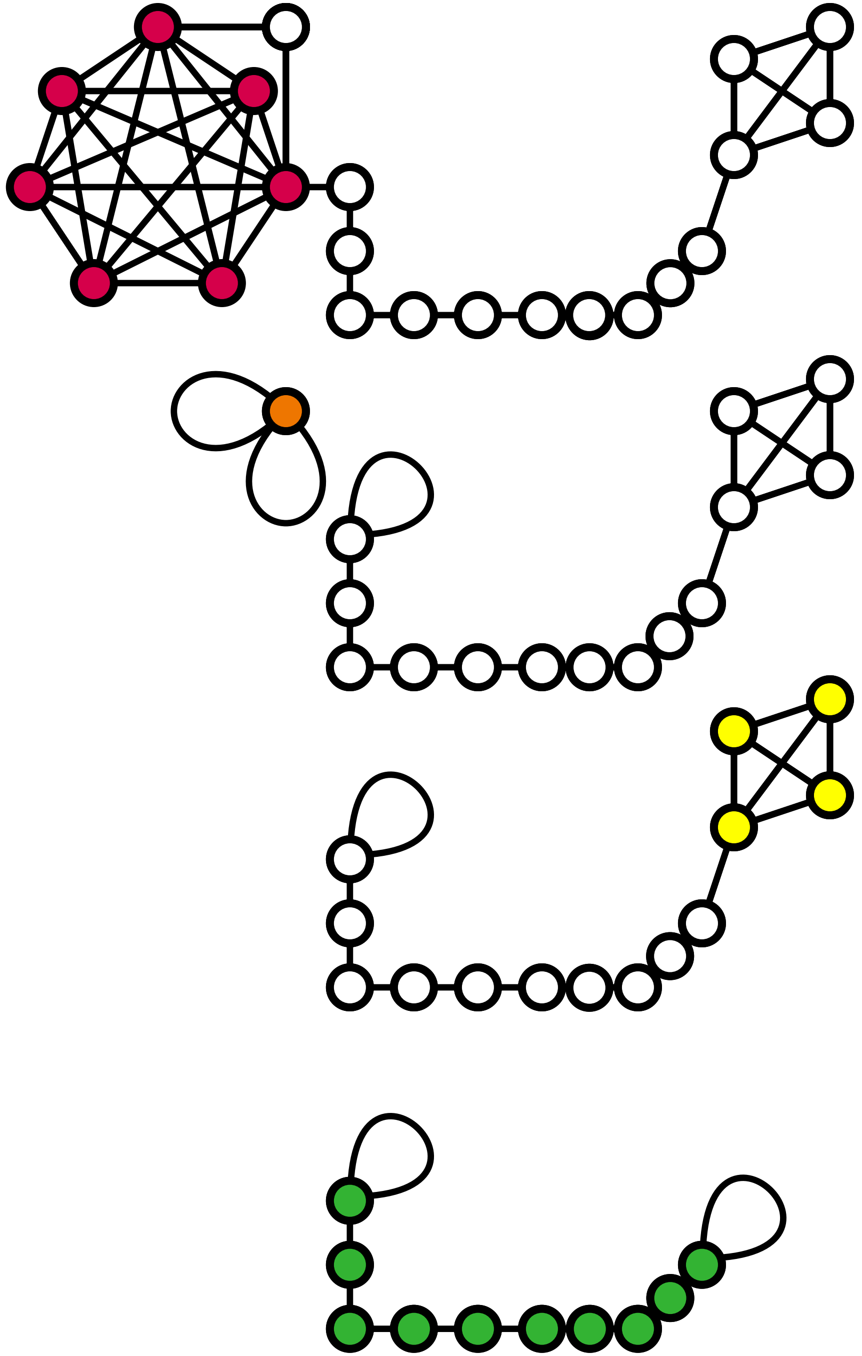

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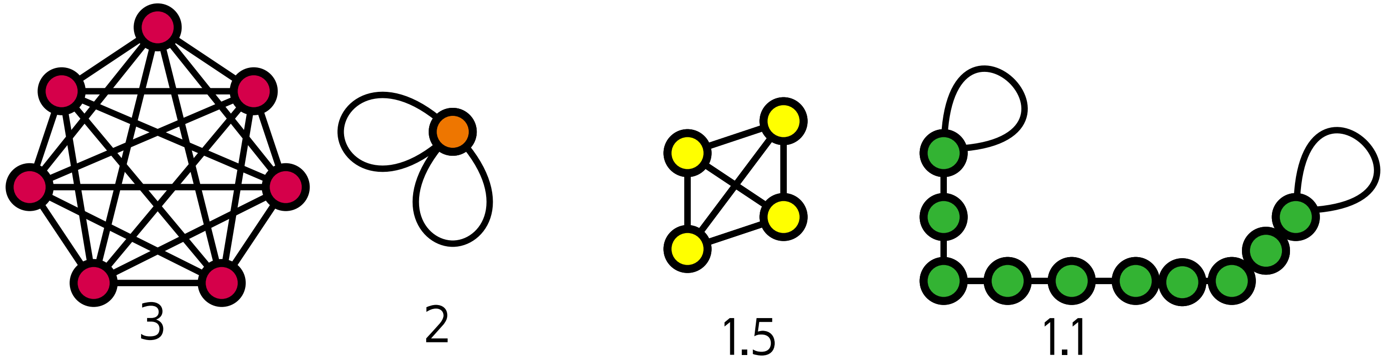
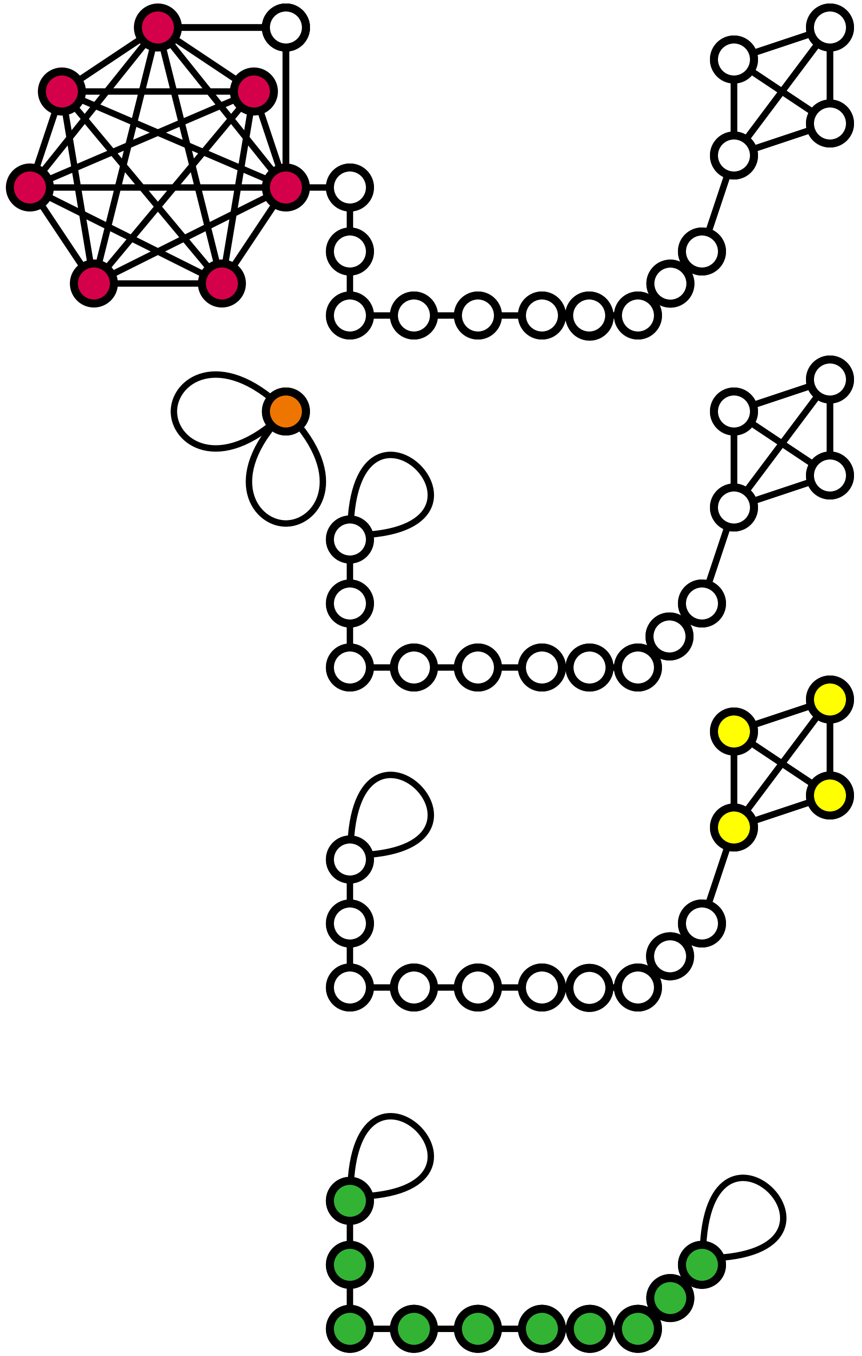


If $v \in H_i$ then $\rho^*(v) = \rho(H_i) = \frac{|\text{Edges}(H_i)|}{|\text{Vertices}(H_i)|}$.



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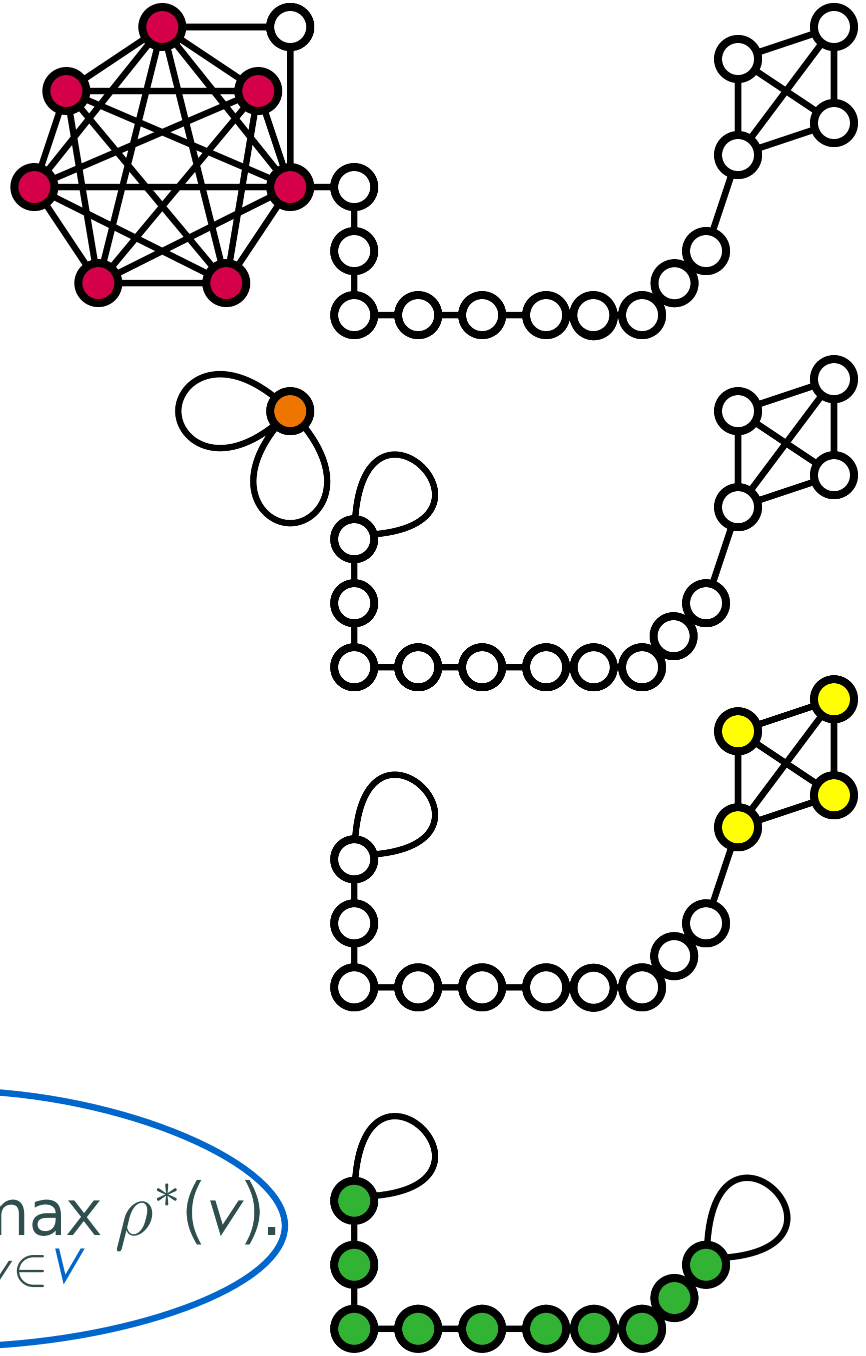

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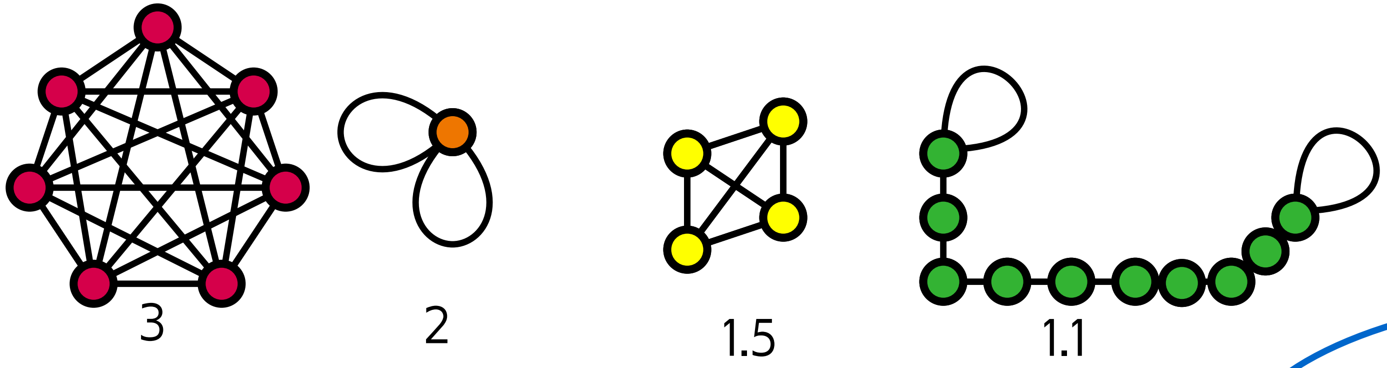

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This paper:

Fractional orientation $\overrightarrow{G}$:

$\forall (u, v) \in E$, require: $\quad g(u \to v) + g(v \to u) = 1.$

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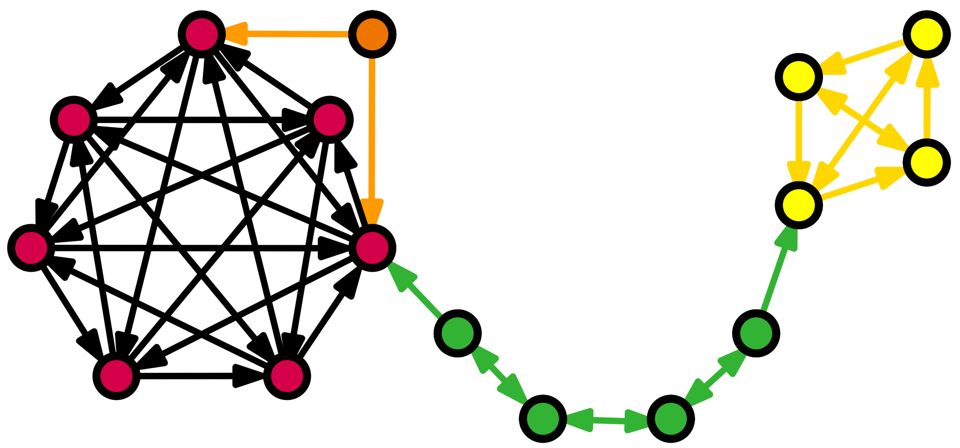
Locally fair orientation $\overrightarrow{G}$:

$$g(u \to v) > 0 \quad \Rightarrow \quad \Delta(u) \leq \Delta(v).$$



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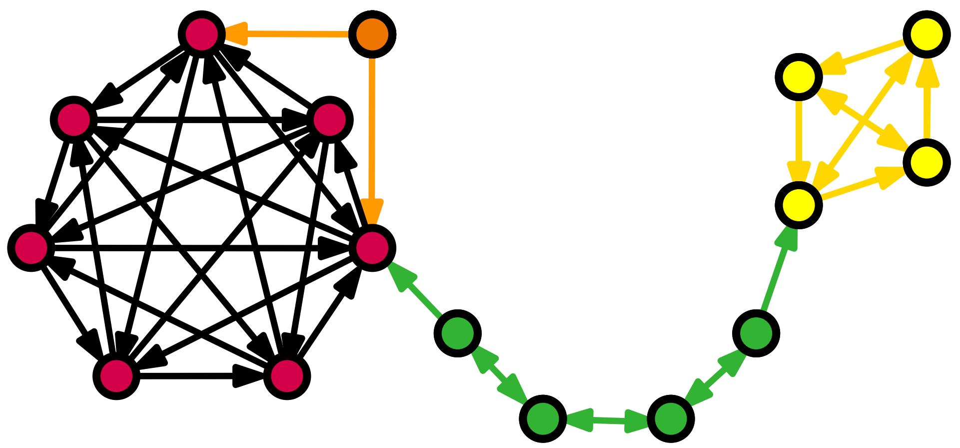

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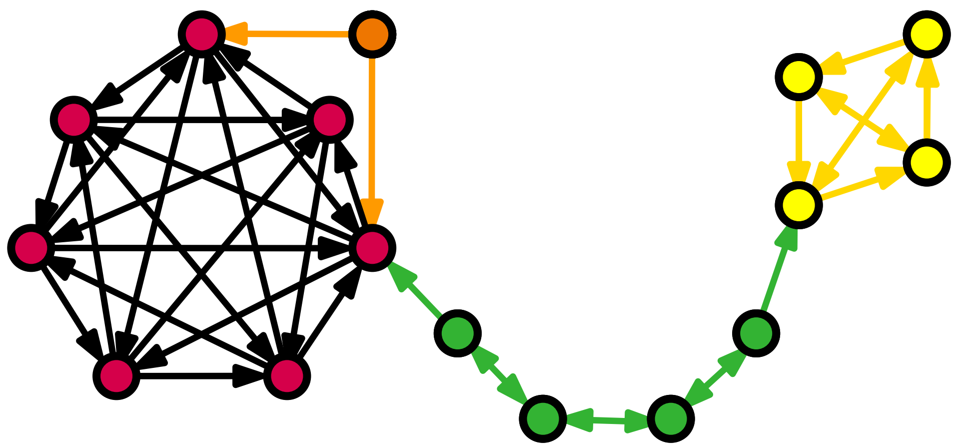

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Cor.: $\exists$ dynamic algorithms to $(1+\varepsilon)$-approximate $\rho^*(v)$!

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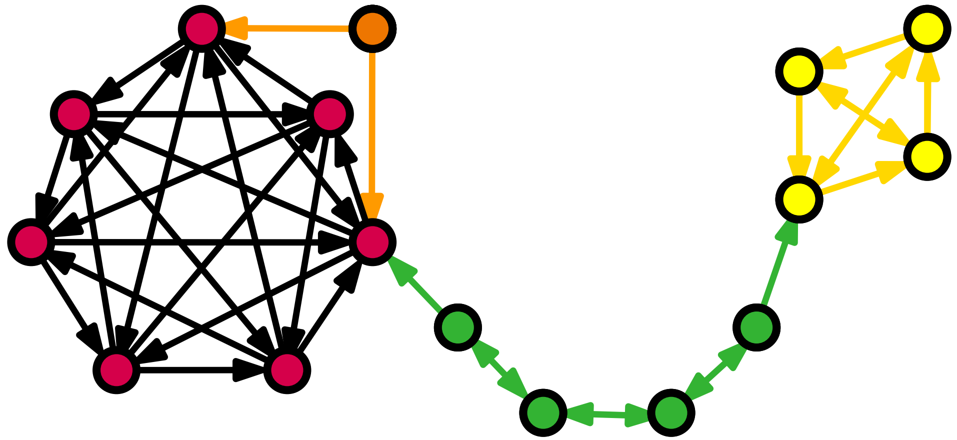

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Result 3:

$O(\varepsilon^{-2} \log^2 n)$ rounds in LOCAL.

Sublinear rounds in CONGEST.

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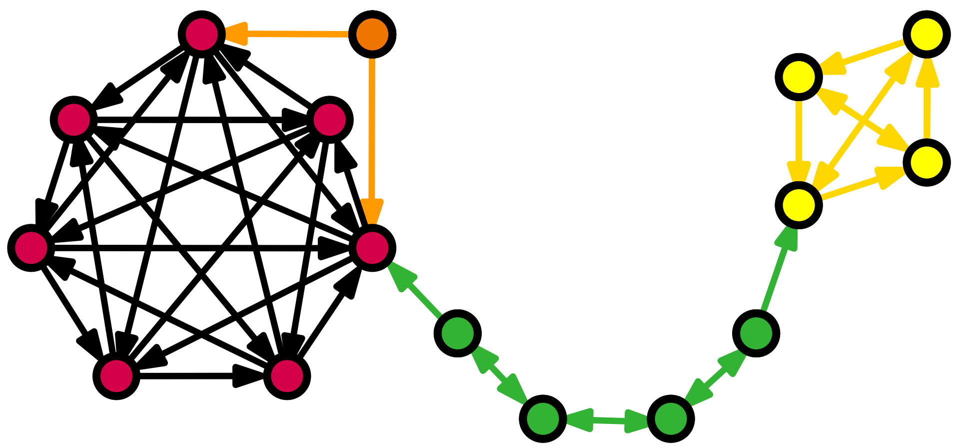

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They use integral orientations, which causes the math to not exactly work out (log-factors in the approximation).

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Dynamic $(1+\varepsilon)$-minimum out-orientation:

Chekuri, Christiansen, Holm, van der Hoog, Quanrud, Rotenberg, and Schwiegelshohn [SODA, 2024].

$(1+\varepsilon)$-approximate $\rho^{\max}(G)$ in $O(\varepsilon^{-6}\log^4 n))$ time.

Result 1: local density is well-defined

Quadratic program FO^2:

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Cost function: minimize $\sum_{v \in V} (\Delta(v))^2$

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Lemma: solution to FO^2 is a locally fair orientation.

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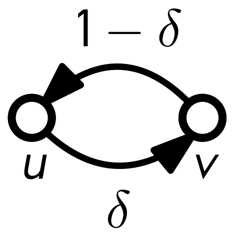

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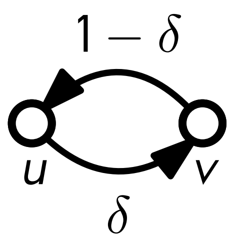

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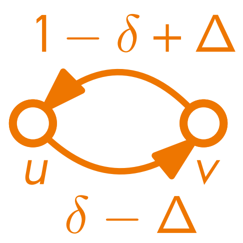

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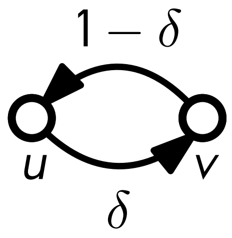

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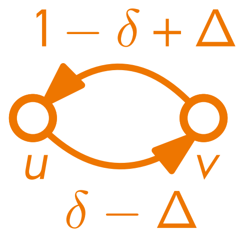

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Cor:

$\Delta^*(v)$ is well-defined.

$\Delta^*(v) = \rho^*(v)$. Local density is a stricter measure than global minimum density.

Result 2: η-fairness $\Rightarrow$ density

Fair orientation $\overrightarrow{G}$:

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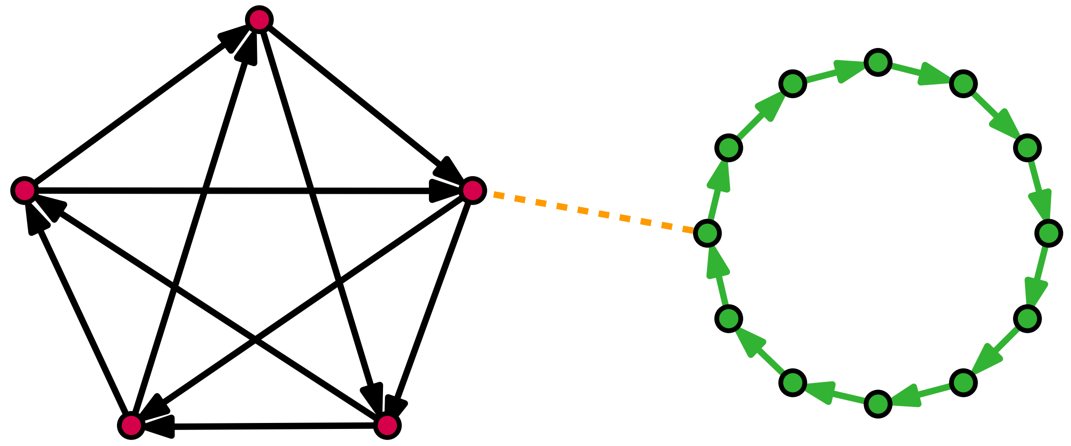

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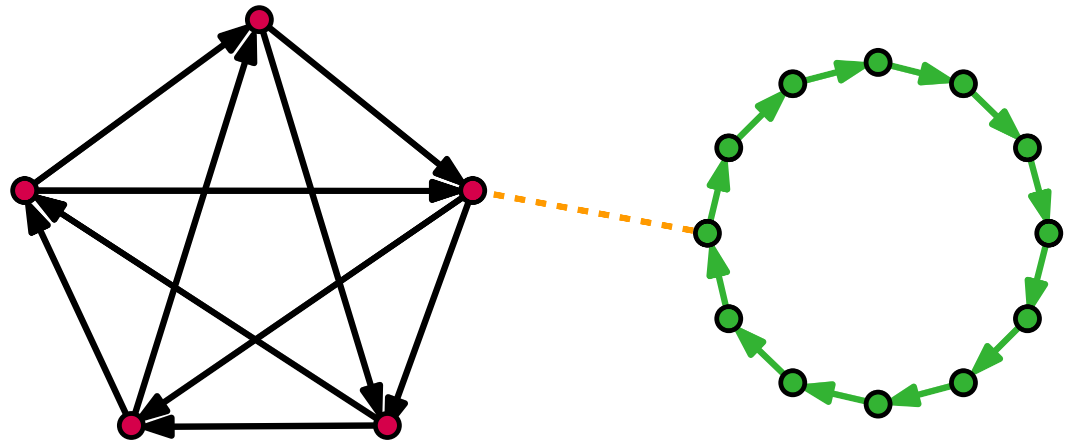

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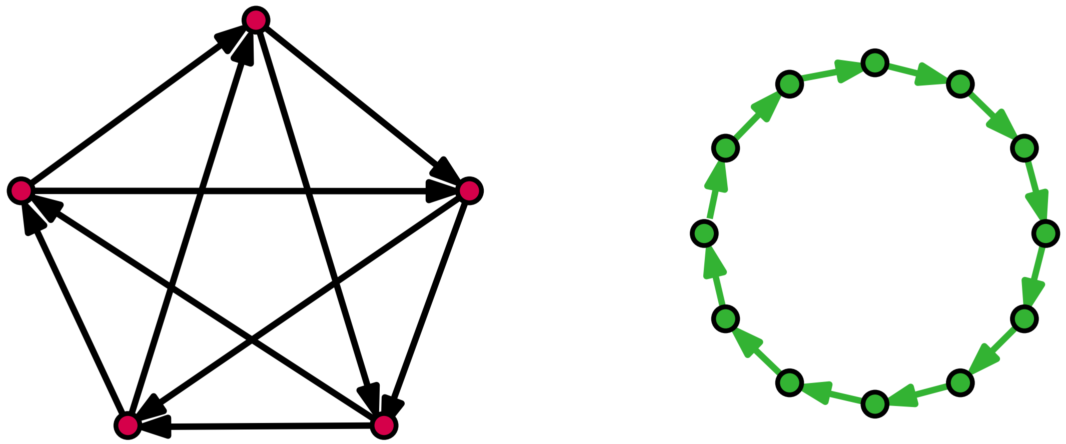

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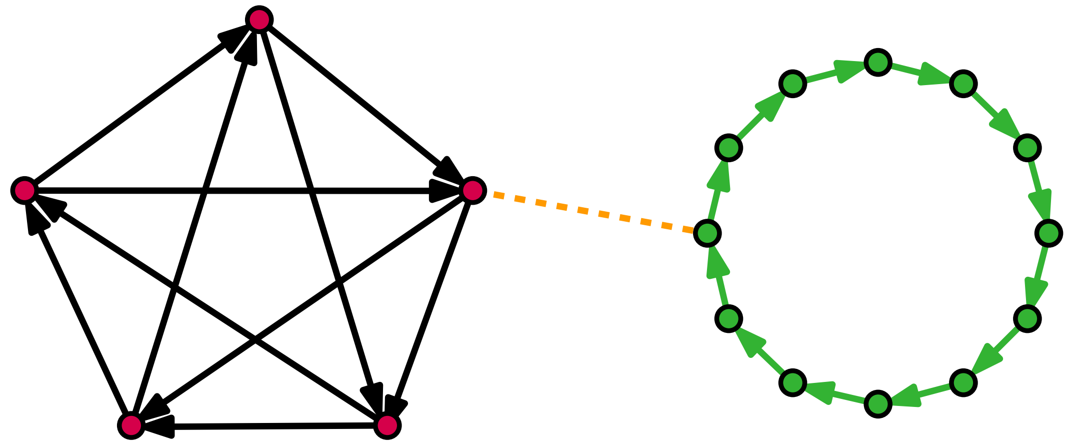

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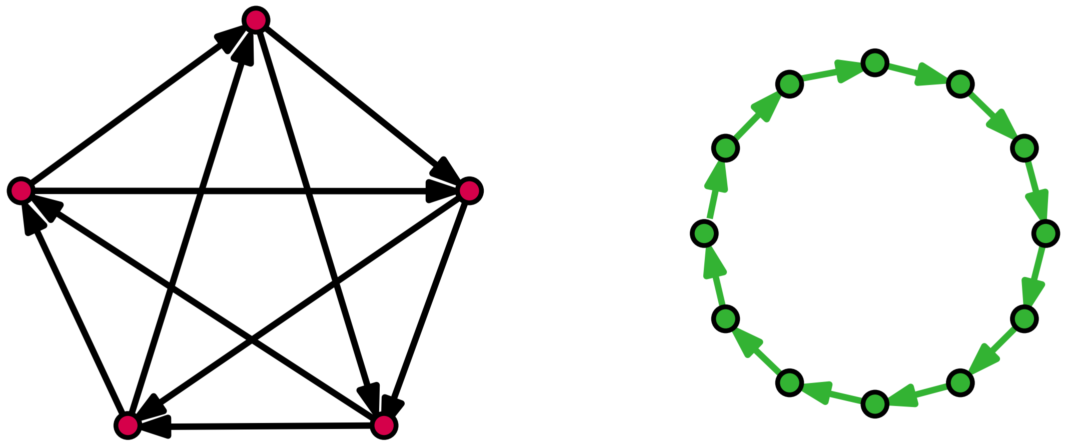
$(1+\varepsilon)$-approximate $\rho^{\max}(G)$ in $O(\varepsilon^{-6} \log^4 n))$ time.



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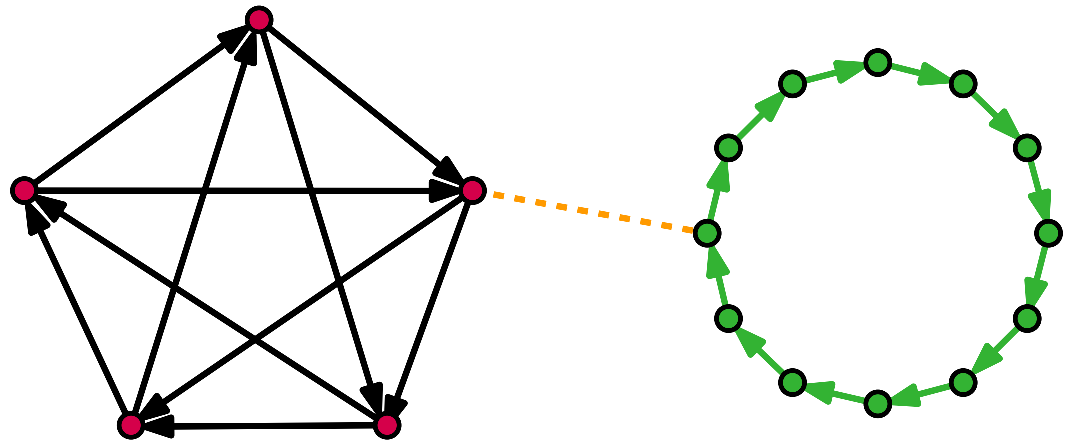

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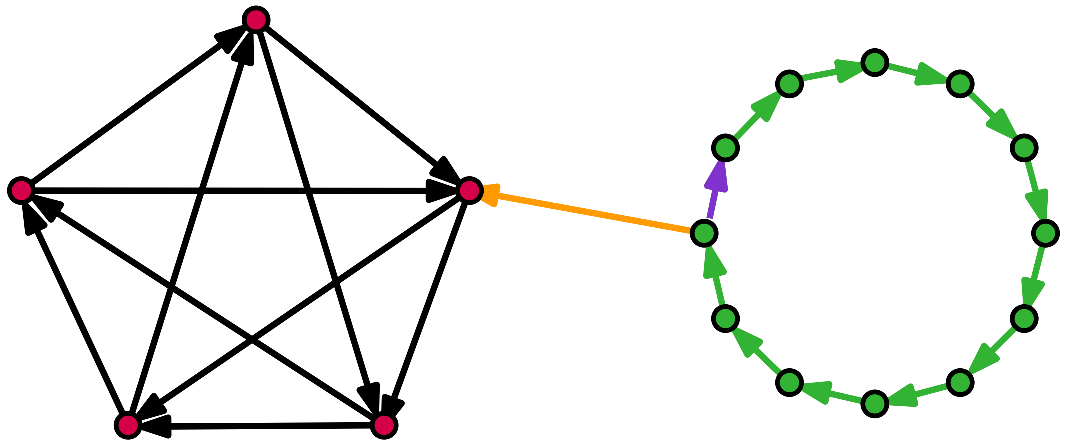

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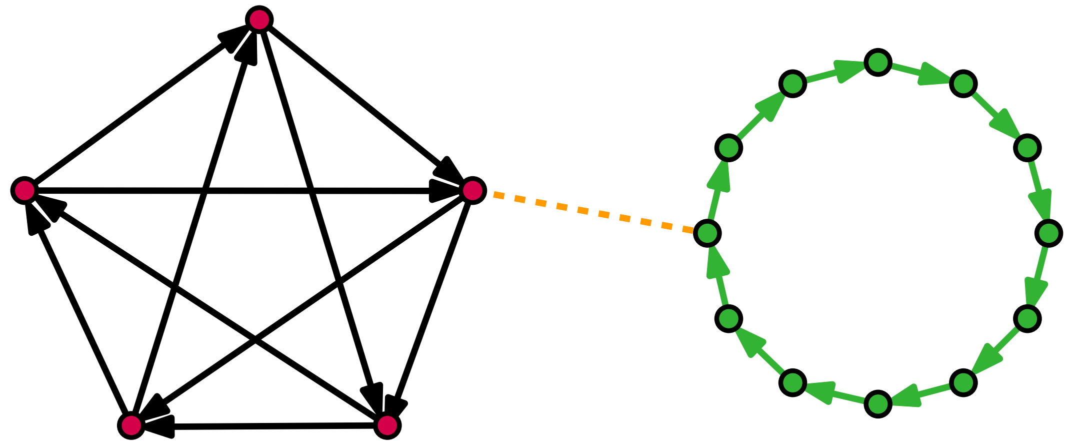

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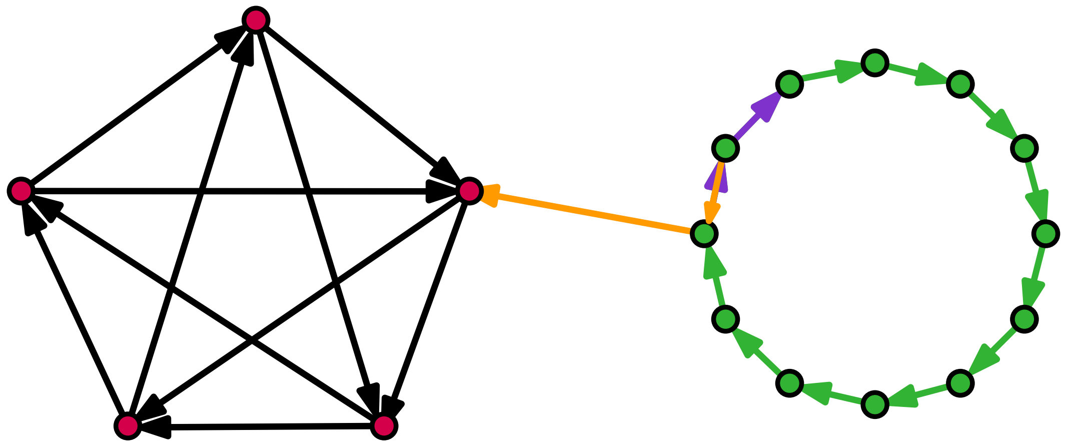

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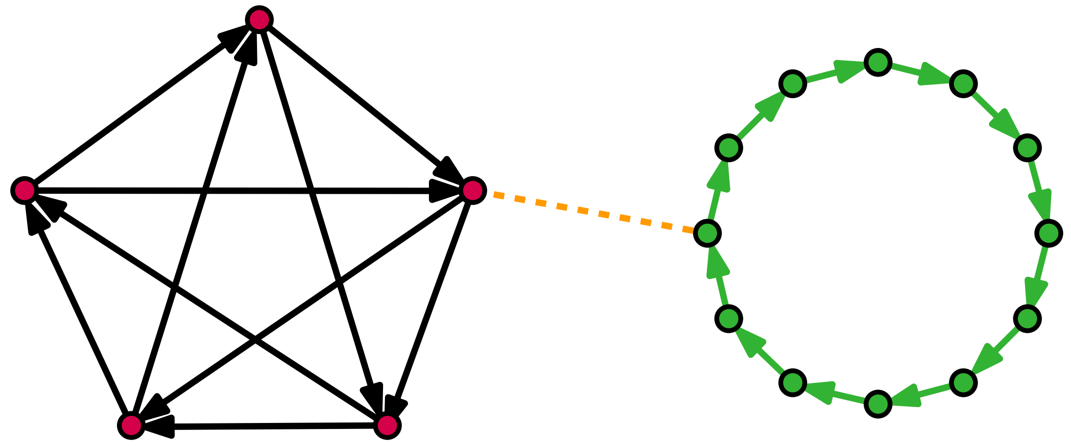

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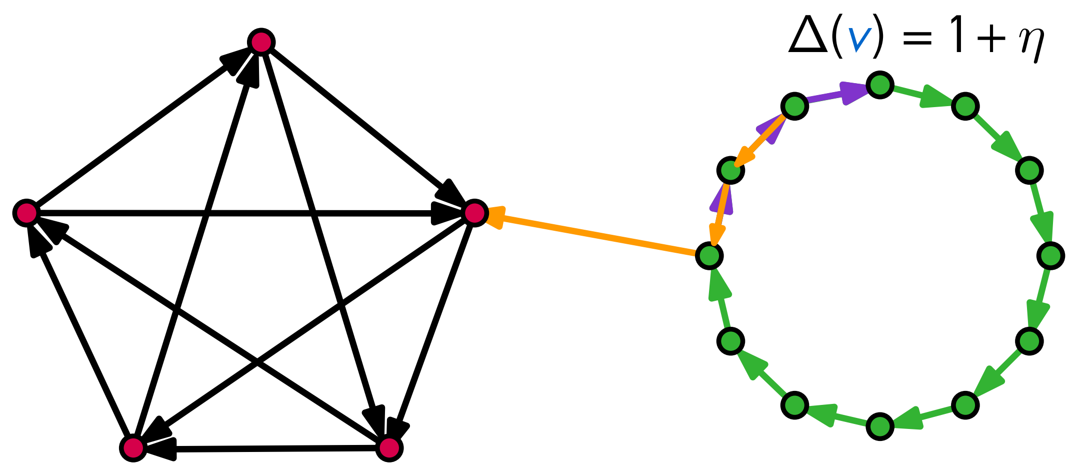

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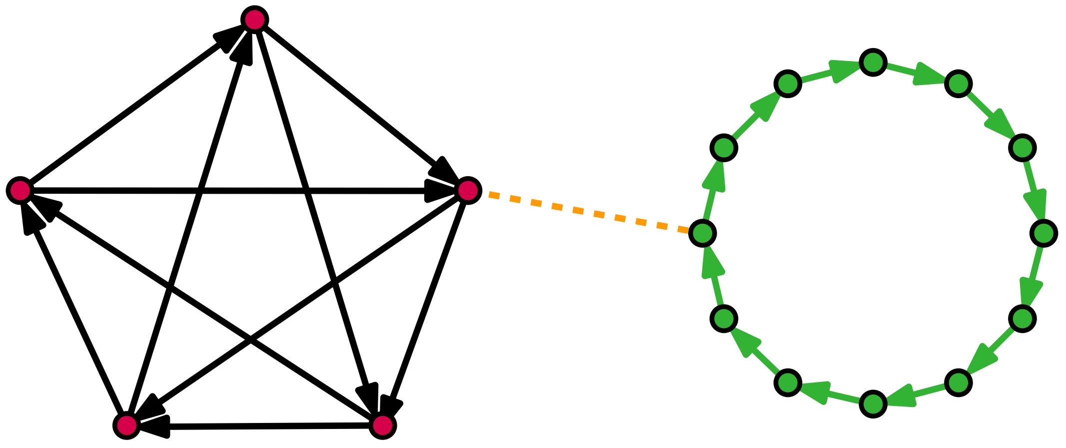

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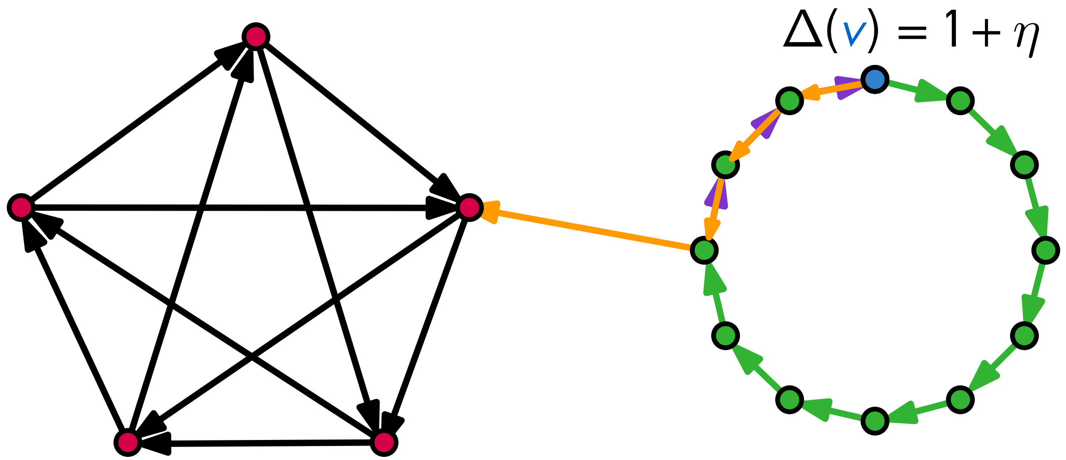

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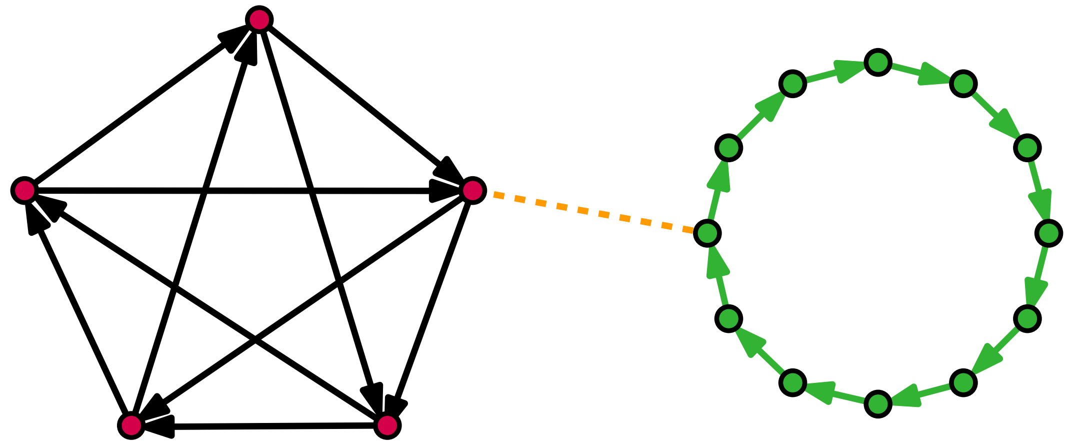

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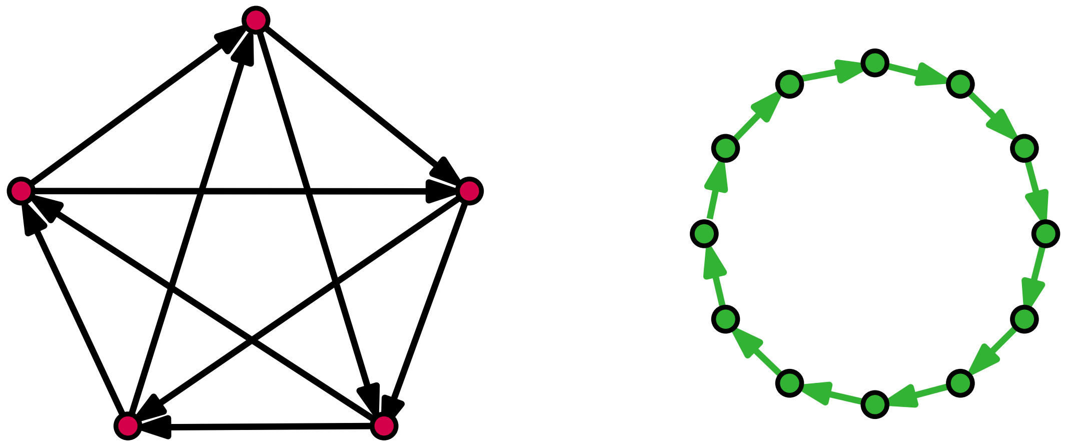

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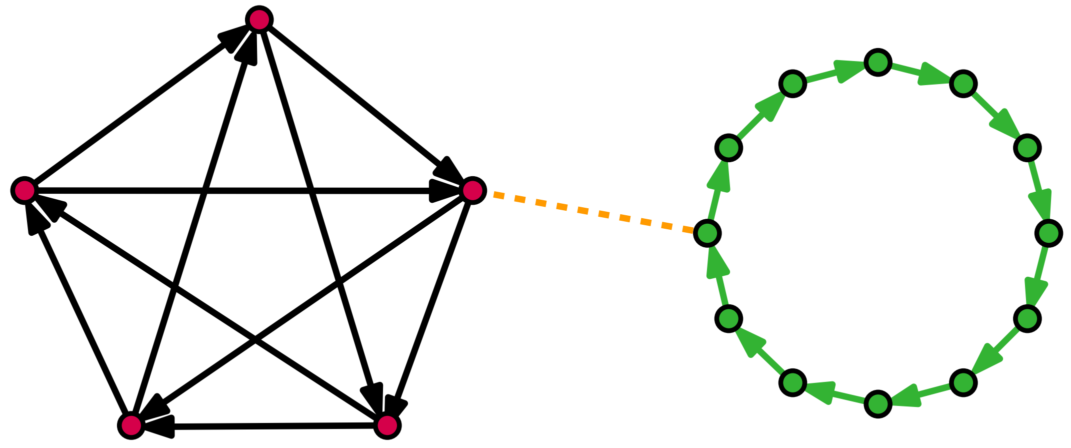

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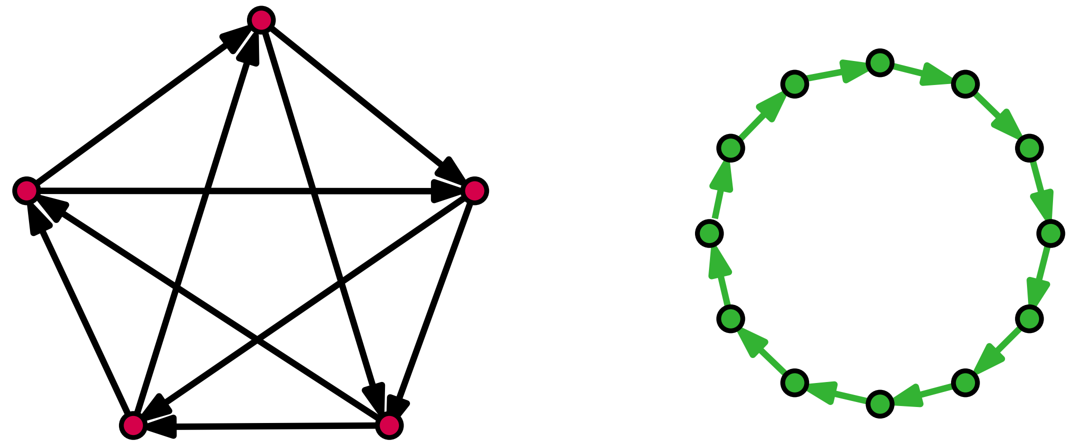

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Cor:

$(1+\varepsilon)$-approximate $\rho^*(v)$ from [WWW, 2017] in $O(\varepsilon^{-6} \log^4 n))$ time.

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Main Theorem:

Let $\overrightarrow{G}$ be an η-fair fractional orientation for $\eta \leq \frac{\varepsilon^2}{128 \cdot \log n}$.

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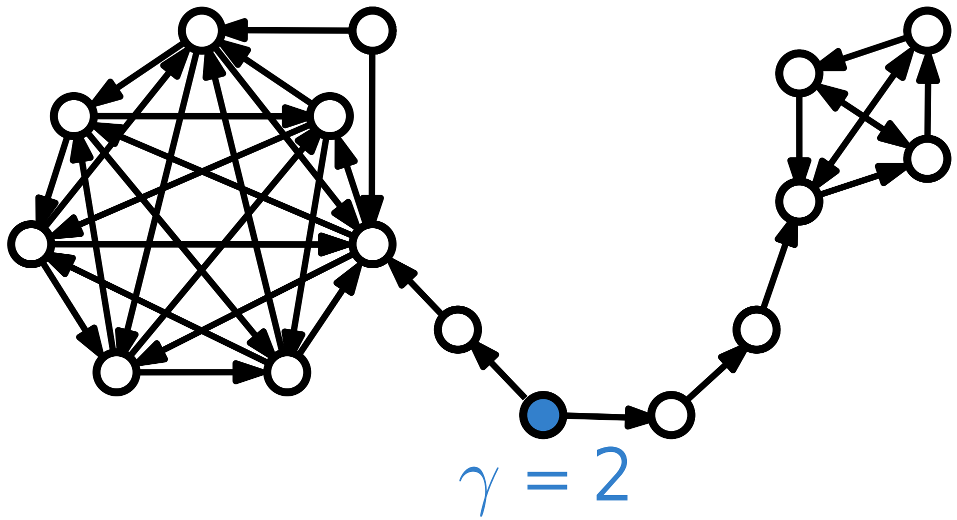

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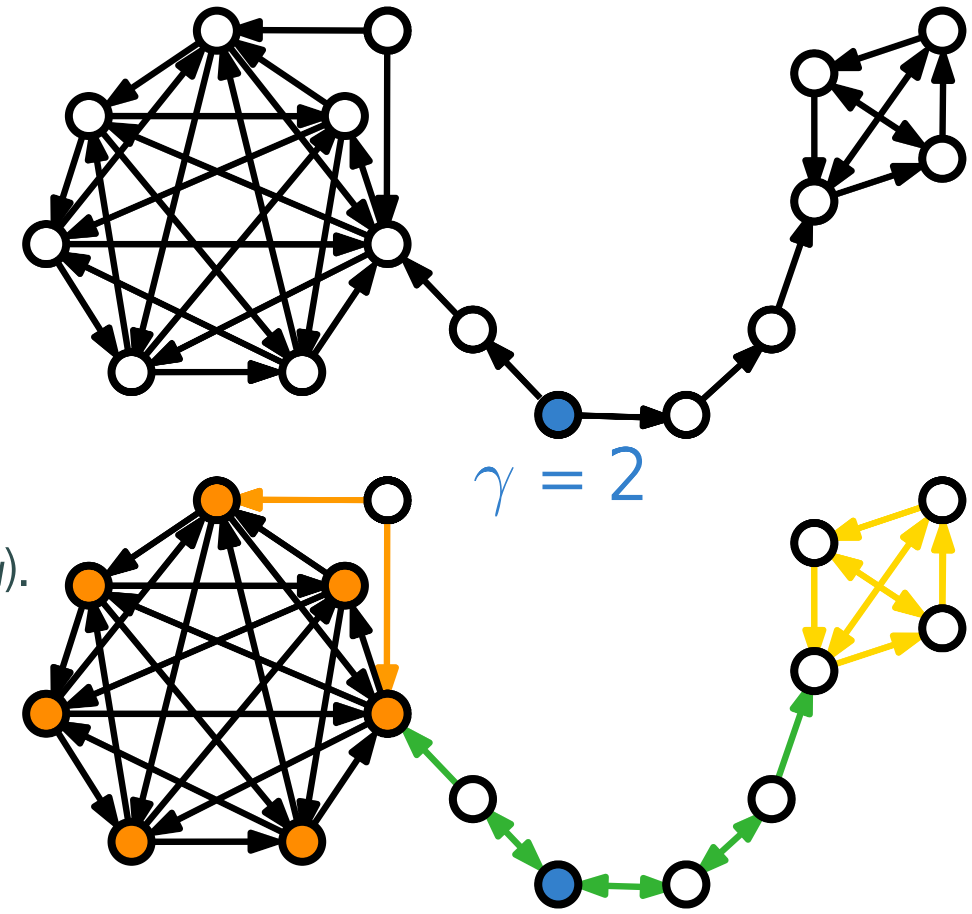

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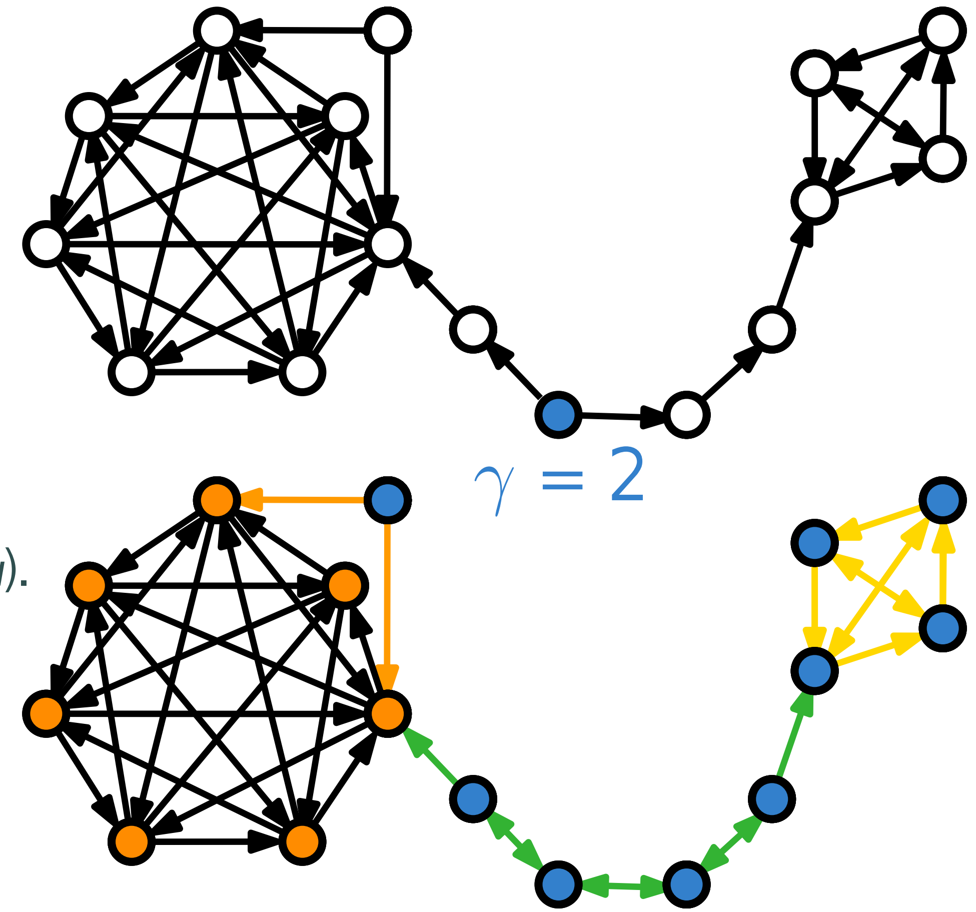

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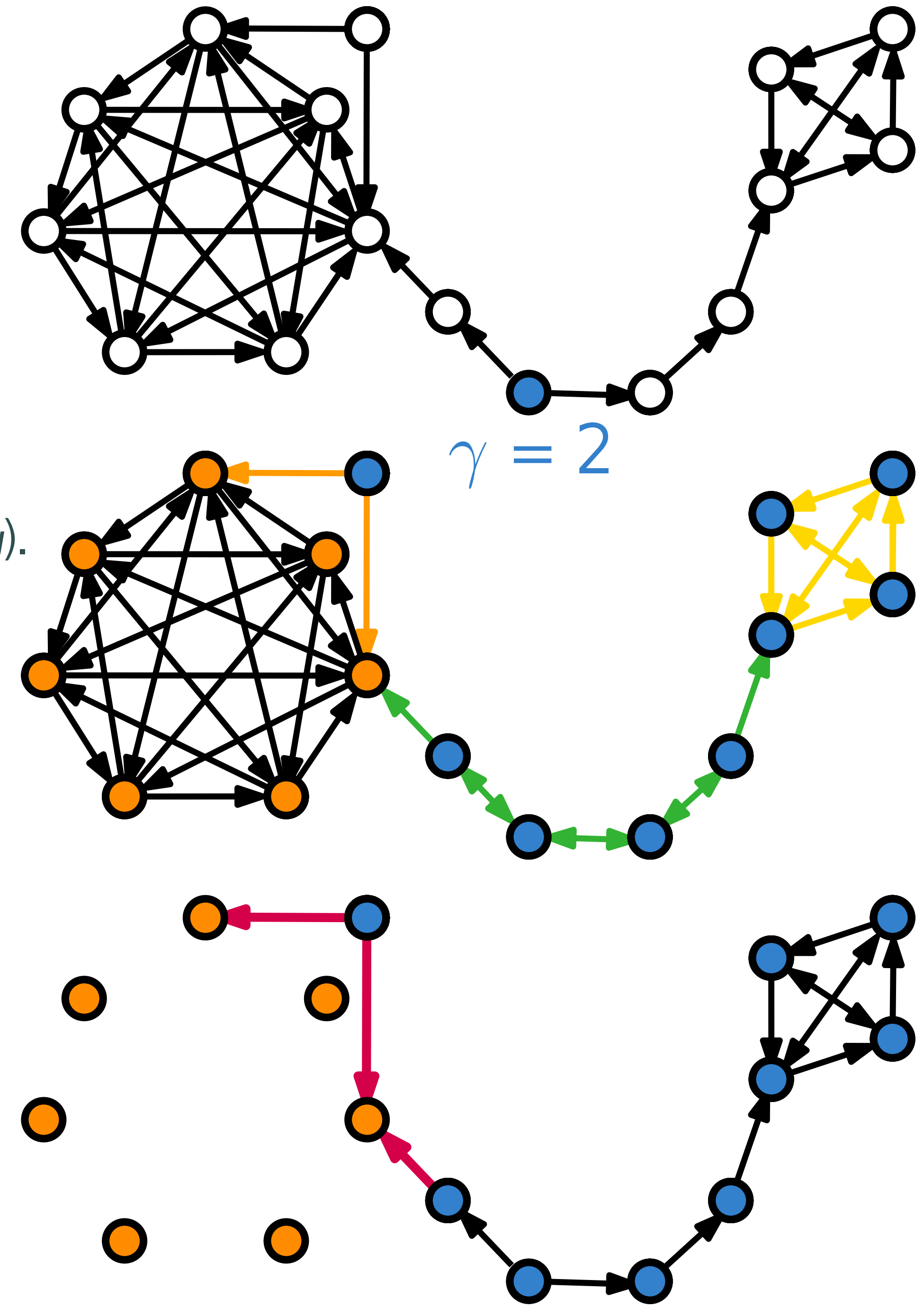

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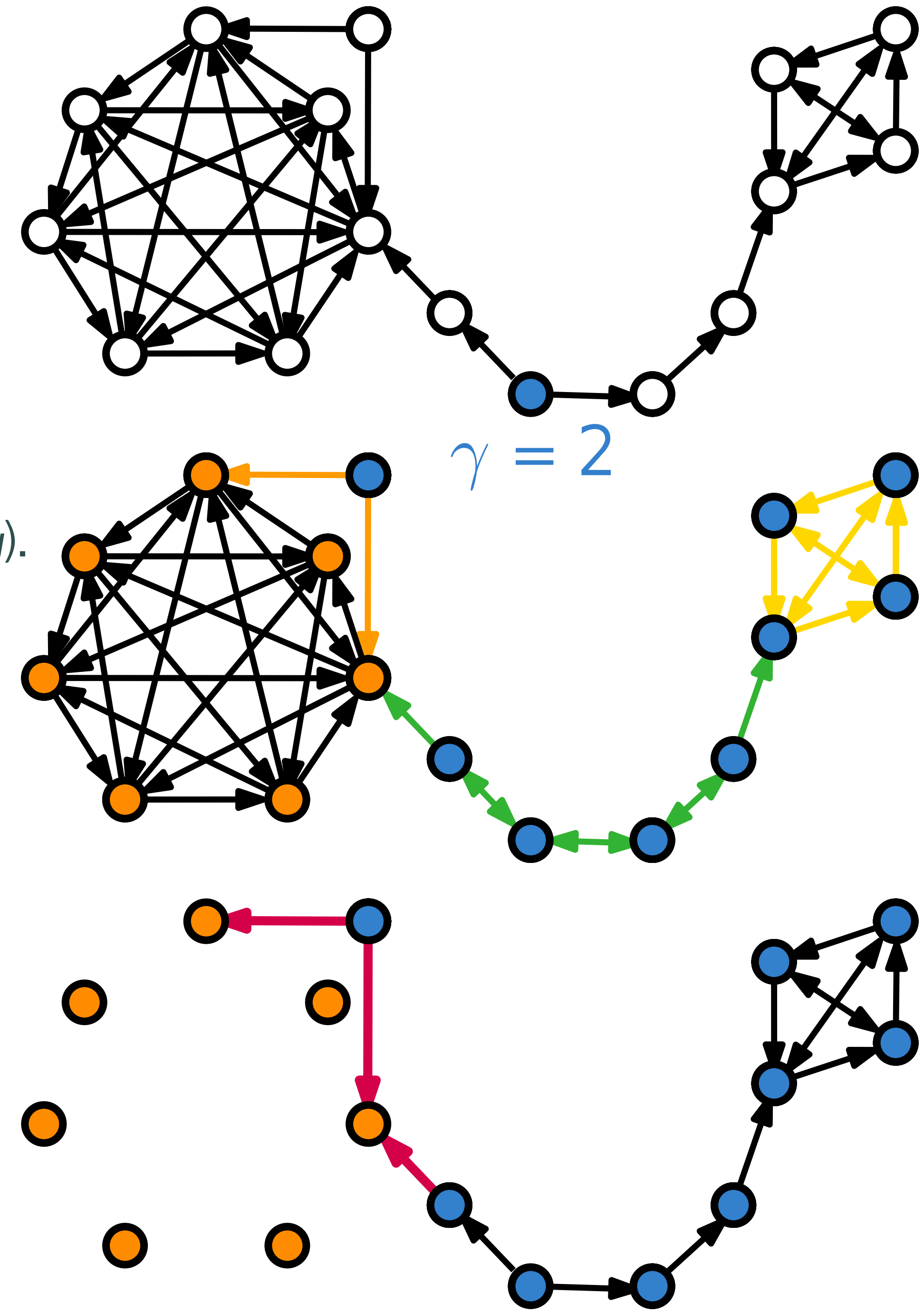

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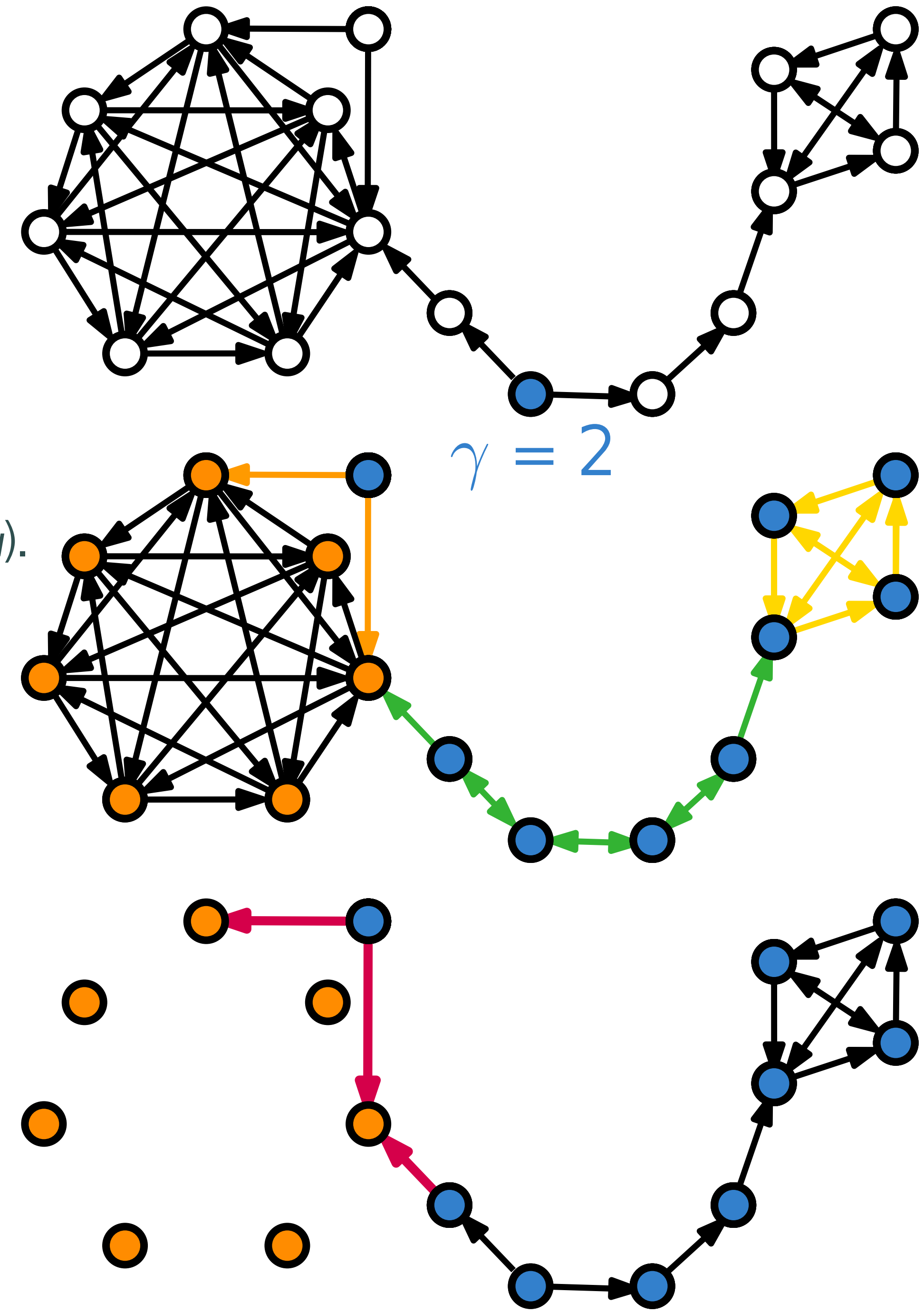

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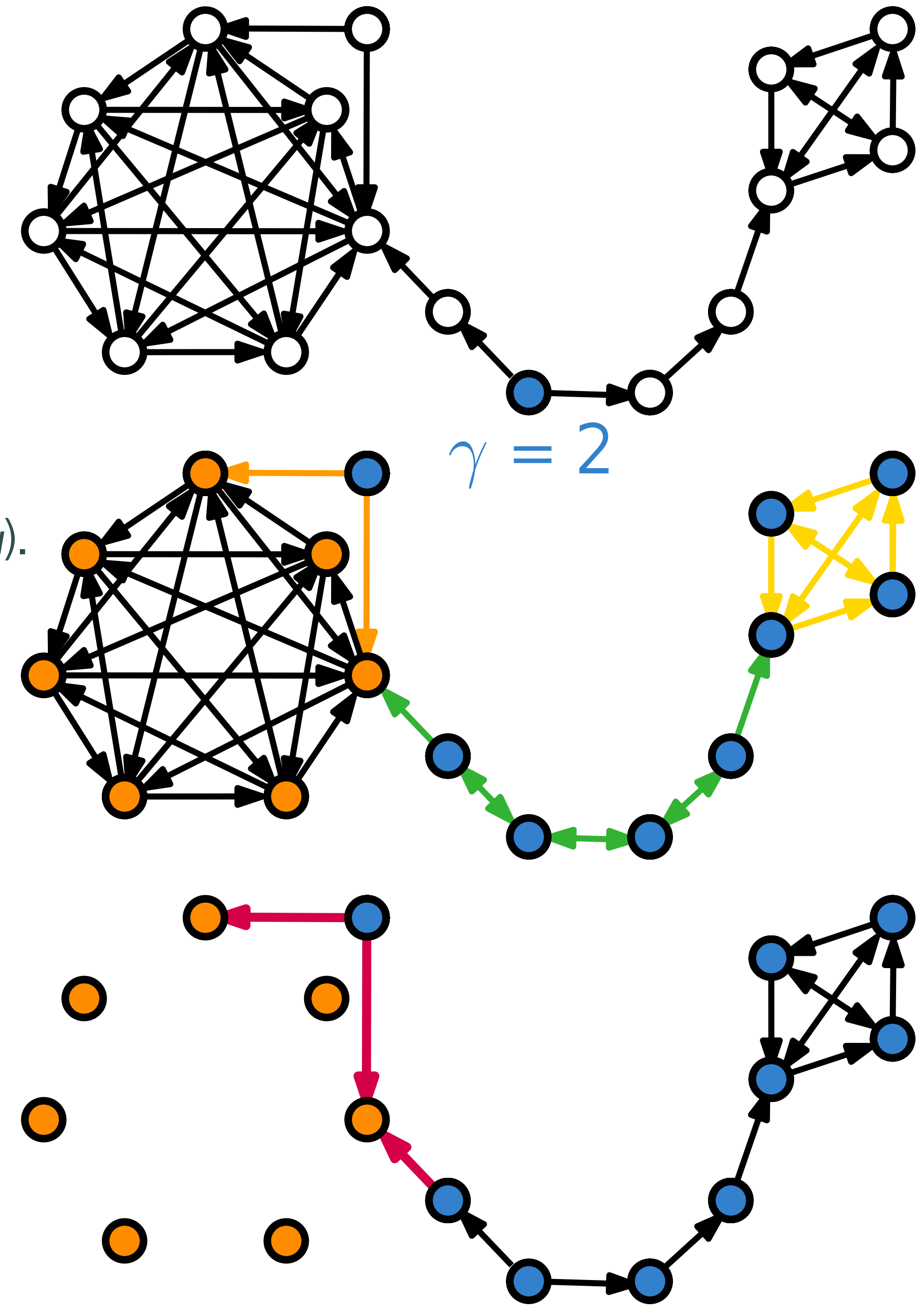
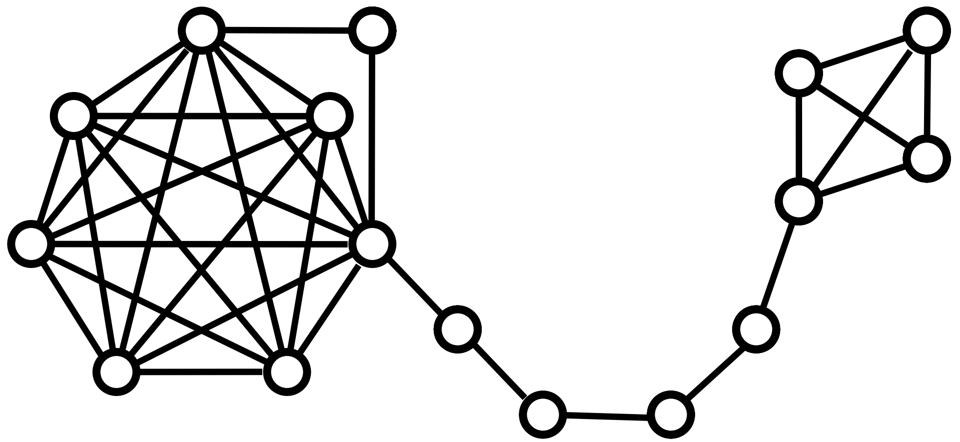


Result 3: Distributed algorithms

This talk:

Compute $\rho^*(v)$ for all v in LOCAL in $O(\varepsilon^{-2} \log^2 n)$ rounds.



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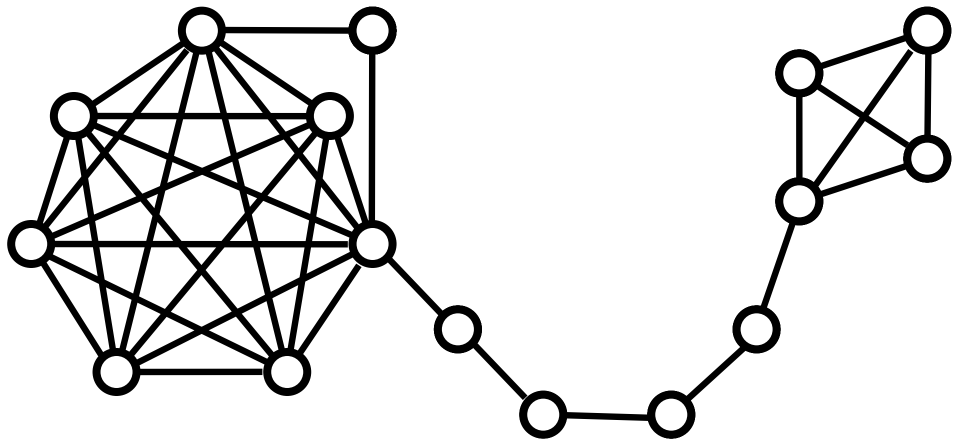

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Thm:

Fix $\eta \leq \frac{\varepsilon^2}{128\cdot\log n}$. $\exists$ dynamic algorithm to maintain an η-fair orientation, with $O(\varepsilon^{-2}\log^2 n)$ **recursive depth**.

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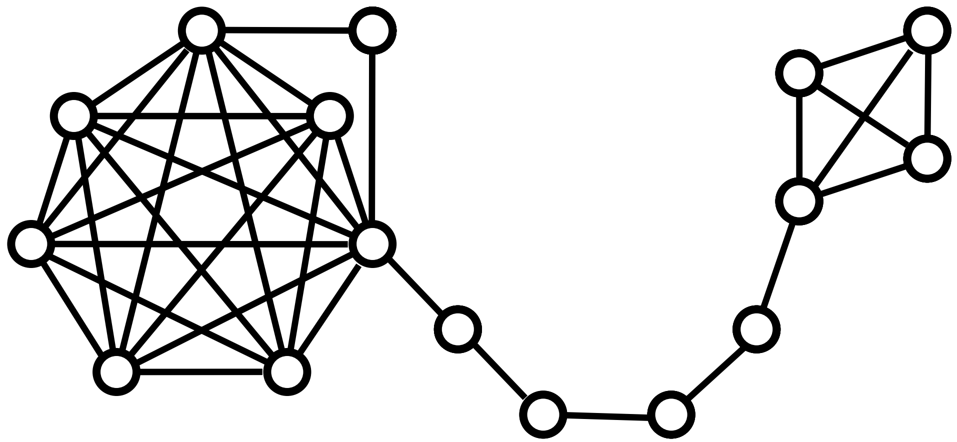

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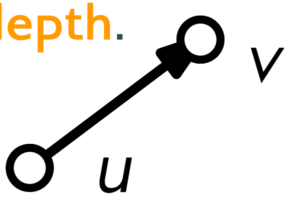

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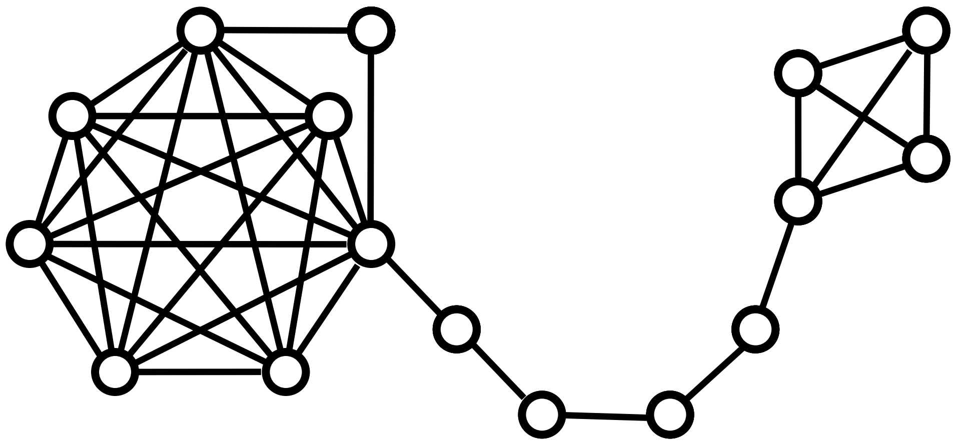

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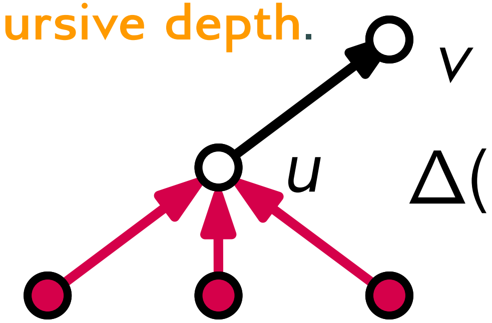

$\Delta(u) - -$

$\Delta(w) > (1+\eta)\Delta(u)$

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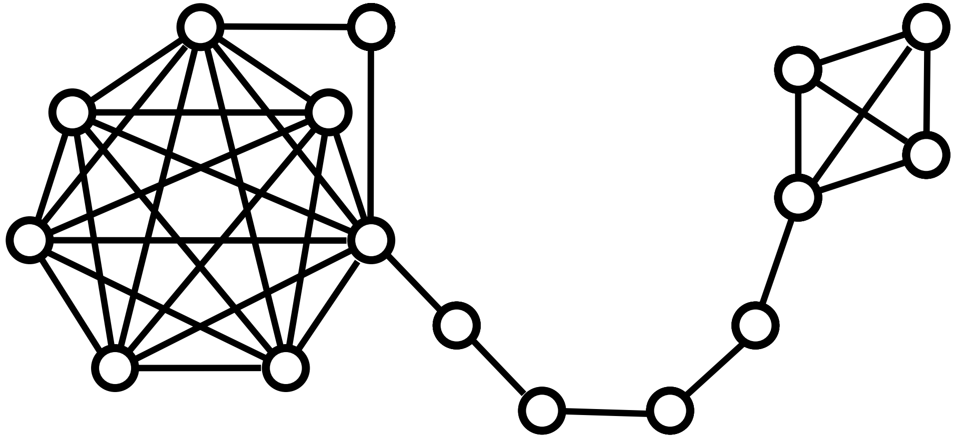

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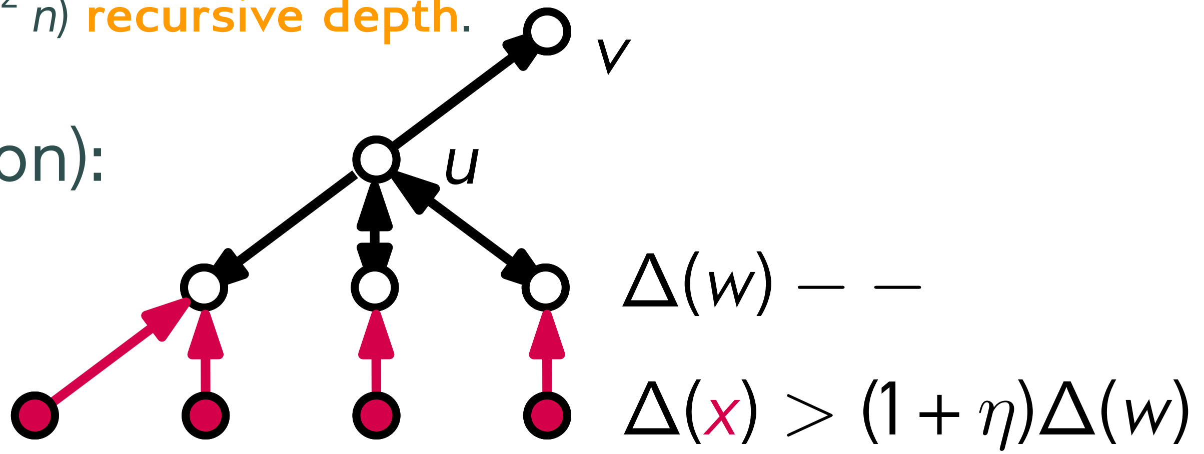

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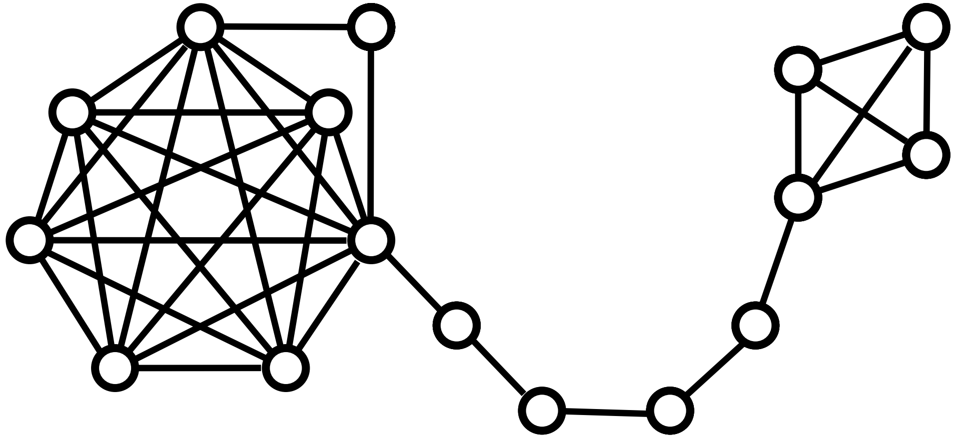

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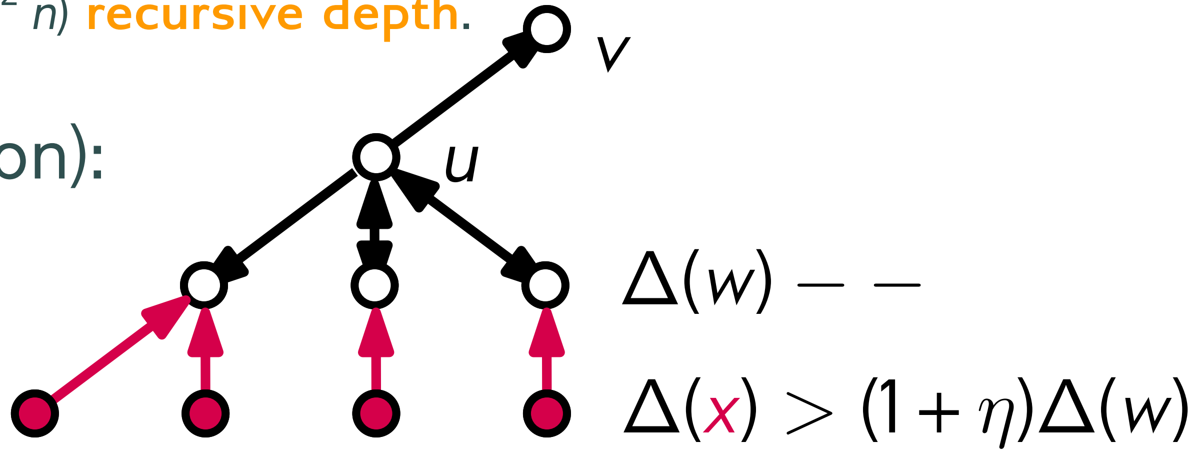

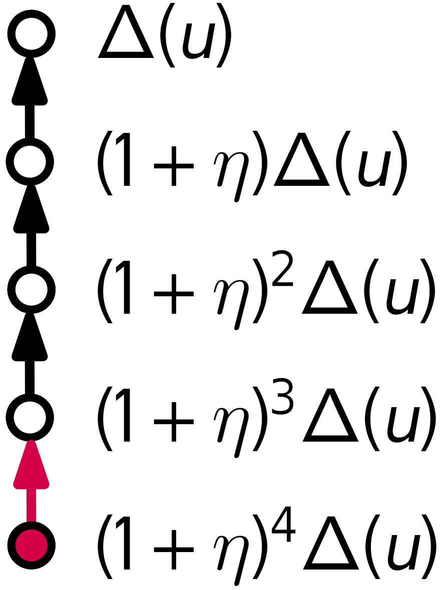

Recursive Depth: $\frac{\log n}{\log(1+\eta)} \in O(\varepsilon^{-2}\log^2 n)$

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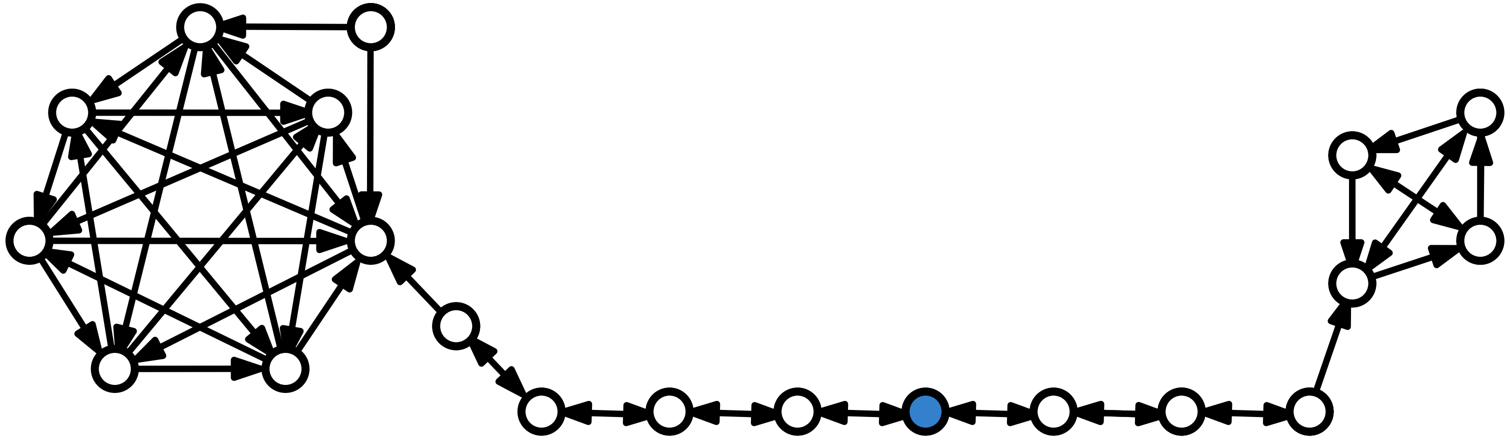

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$\Delta(v) = \rho^*(v)$.

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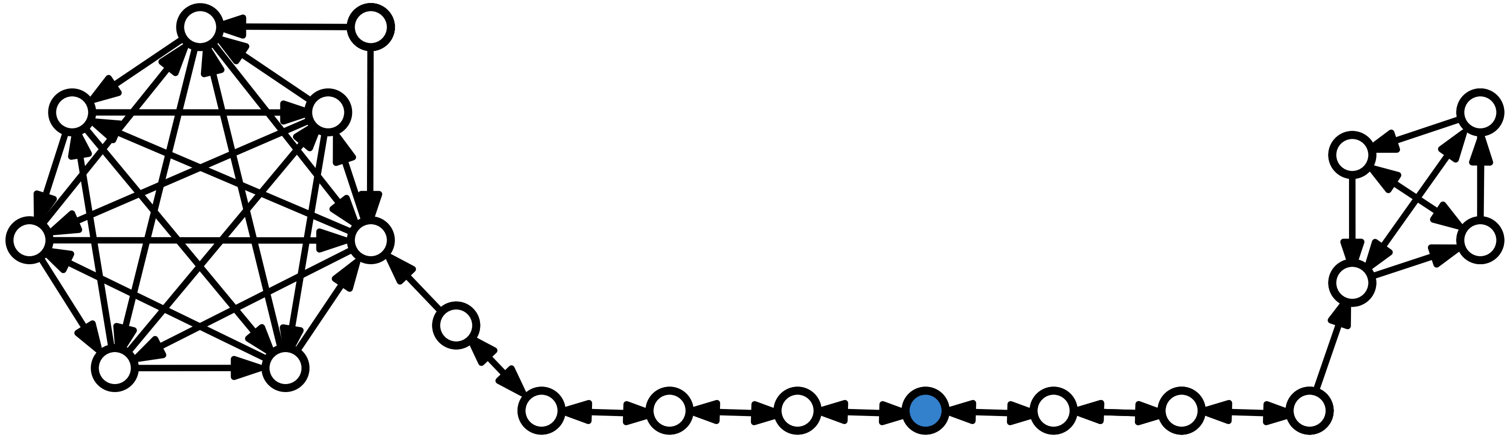

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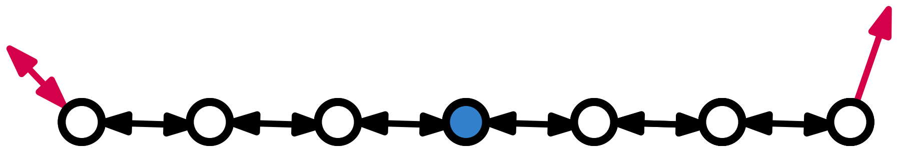
Disconnect the $O(\varepsilon^{-2}\log^2 n)$-hop neighborhood of v from G



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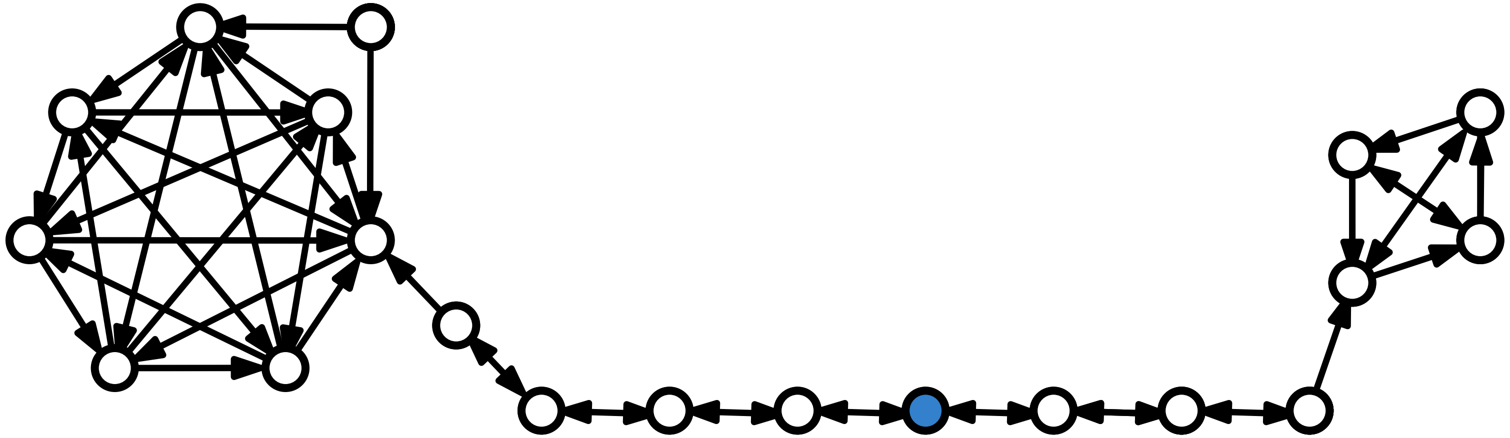
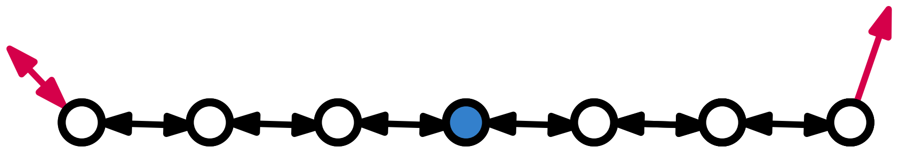


Fix a vertex v and a fair orientation.

$\Delta(v) = \rho^*(v)$.



Disconnect the $O(\varepsilon^{-2}\log^2 n)$-hop neighborhood of v from G

The out-degree $\Delta(v)$ remains unchanged after the update.

$$\Delta(v) \approx_{1+\varepsilon} \rho^*_{G'}(v) \quad \Rightarrow \quad \rho^*_{G'}(v) \approx_{1+\varepsilon} \rho^*(v) \quad \square$$

This paper:

Local out-degree $\Delta^*(v)$.

Fix any locally fair orientation $\overrightarrow{G}$,

We define $\forall v \in V$, $\quad \Delta^*(v) := \Delta(v)$.

Result 1:

Local out-degree is well-defined.

Duality: $\Delta^*(v) = \rho^*(v)$.

Result 2:

η-fair orientations $(1+\varepsilon)$-approximate $\rho^*(v)$ for all vertices v.

Cor.: $\exists$ dynamic algorithms to $(1+\varepsilon)$-approximate $\rho^*(v)$!

Result 3:

$O(\varepsilon^{-2} \log^2 n)$ rounds in LOCAL.

Sublinear rounds in CONGEST.

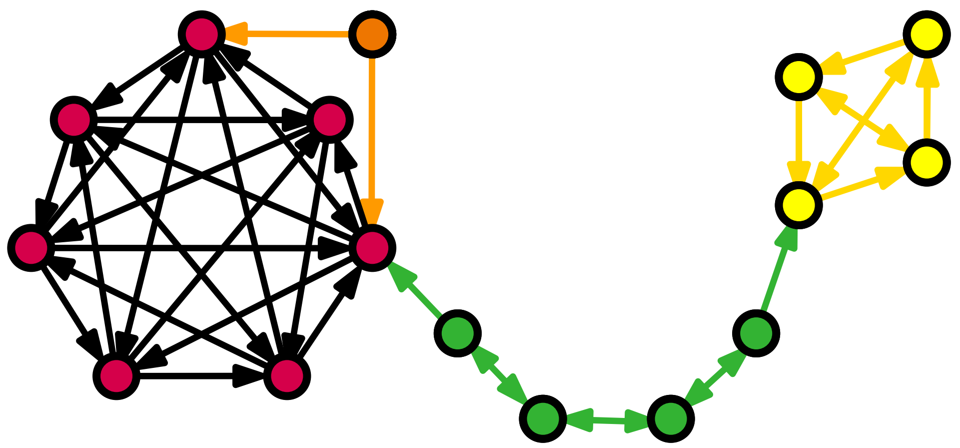
Fractional orientation $\overrightarrow{G}$:

$\forall (u,v) \in E$, require: $\quad g(u \to v) + g(v \to u) = 1.$

$\forall v \in V$, define: $\quad \Delta(u) = \sum_{(u,v) \in E} g(u \to v).$

Locally fair orientation $\overrightarrow{G}$:

$g(u \to v) > 0 \quad \Rightarrow \quad \Delta(u) \leq \Delta(v).$



η-fair orientation $\overrightarrow{G}$:

$g(u \to v) > 0 \quad \Rightarrow \quad \Delta(u) \leq (1+\eta)\Delta(v).$