# Local Enumeration: The Not-All-Equal Case



Mohit Gurumukhani
Cornell University

Ramamohan Paturi
University of California, San Diego

Michael Saks
Rutgers University

Navid Talebanfard
University of Sheffield

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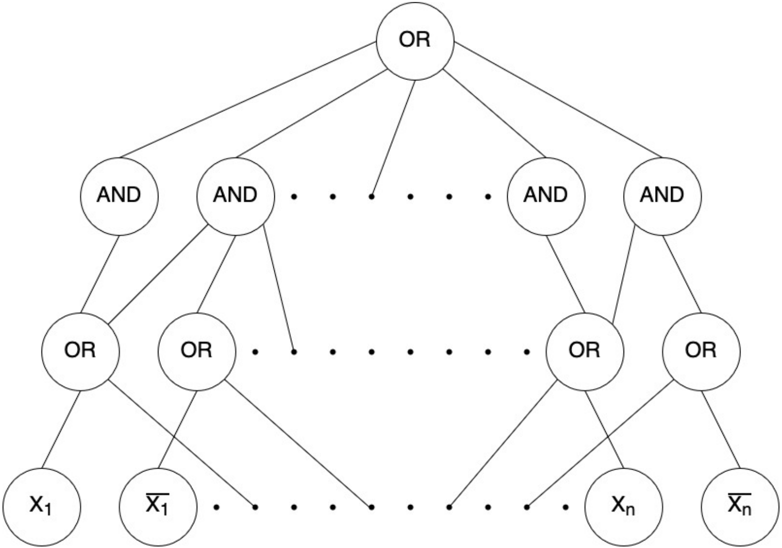

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$\Sigma\Pi\Sigma$ circuits where bottom layer *OR* gates have fan in $k$.

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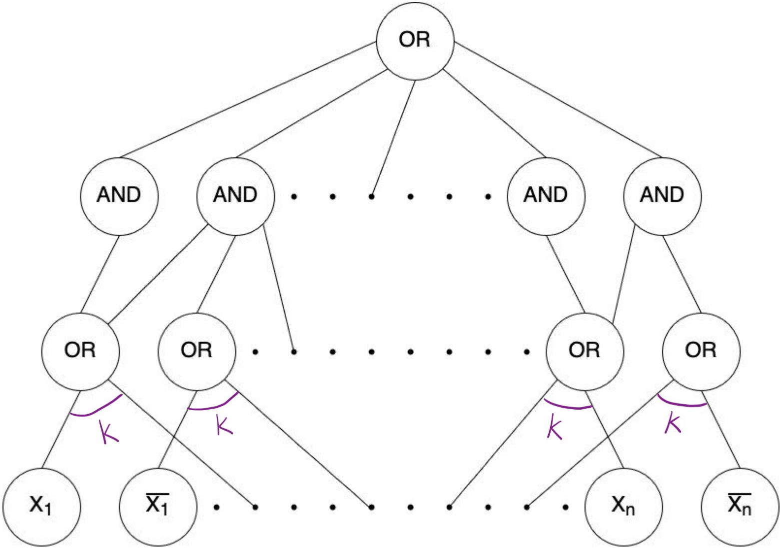

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- Improved $(3.9n)$ general circuit lower bounds [ Golovnev-Kulikov-Williams'21 ].

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## Super Strong Exponential Time Hypothesis (SSETH)

There does not exist a $2^{n-\omega(n/k)}$ time algorithm to solve $k$-SAT [ Vyas-Williams'21 ].

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- Can construct $k$-CNF with $2^{n - O(\log(k)/k) \cdot n}$ satisfying assignments of Hamming weight $n/2$ and no satisfying assignments of Hamming weight $< n/2$.
- $\mathsf{Enum}(k, n/2) \geq 2^{n - O(\log(k)/k) \cdot n}$.

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## This is Not Tight

By construction, Enum$(3, n/2)$ requires time $\geq \mathbf{6^{n/4} \approx 1.565^n}$.

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  2. Add edge from current node to $T_x$ labelled $x$.

# Transversal Tree

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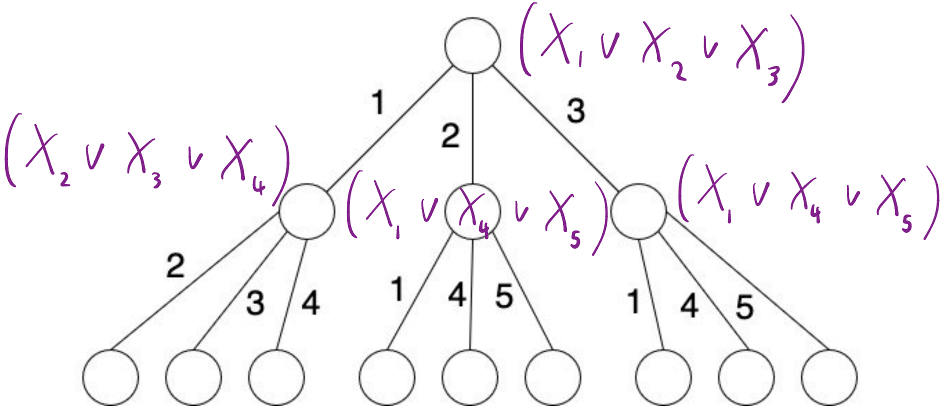
AND (negations of these clauses)



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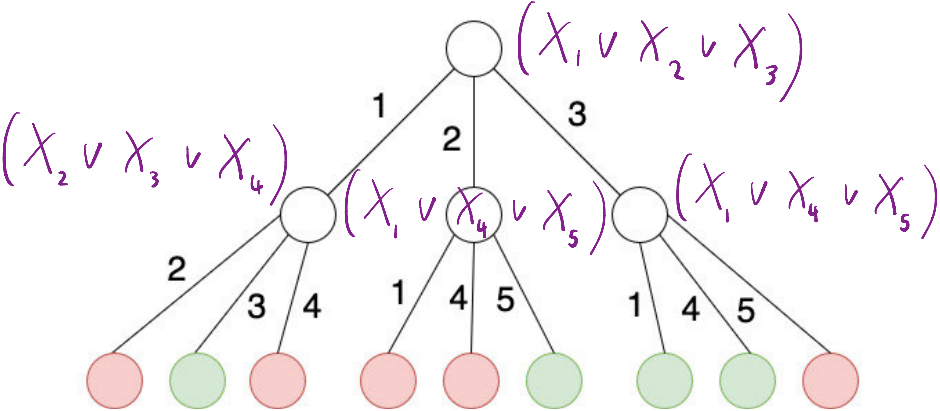

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**Lemma**

*Every "transversal" of $F$ corresponds to some (maybe many) valid leaves of $T$.*

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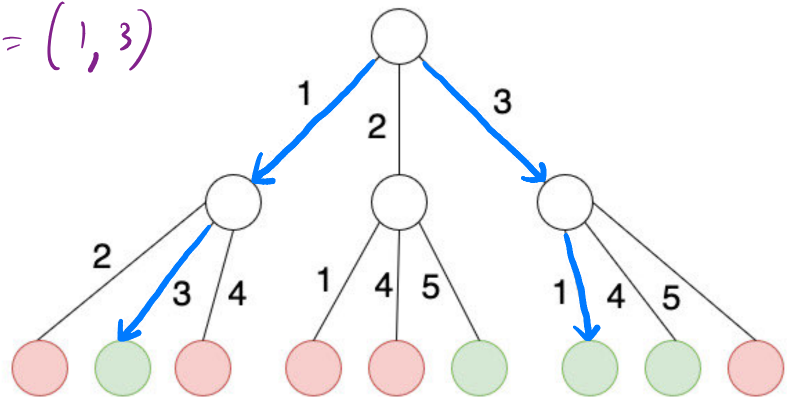

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1. At a node, let outgoing edges be ordered as $x_1, \ldots, x_k$.
2. For $1 \leq i \leq k$:
   2.1 Prune (delete) all edges in $T_{x_i}$ labelled with any of $x_1, \ldots, x_{i-1}$ & search $T_{x_i}$.

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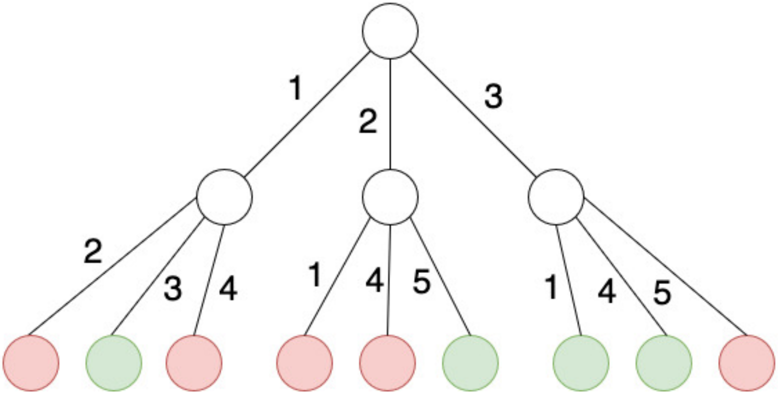

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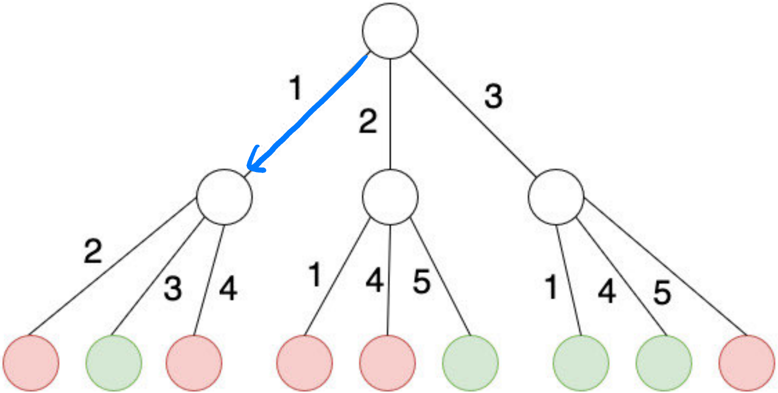

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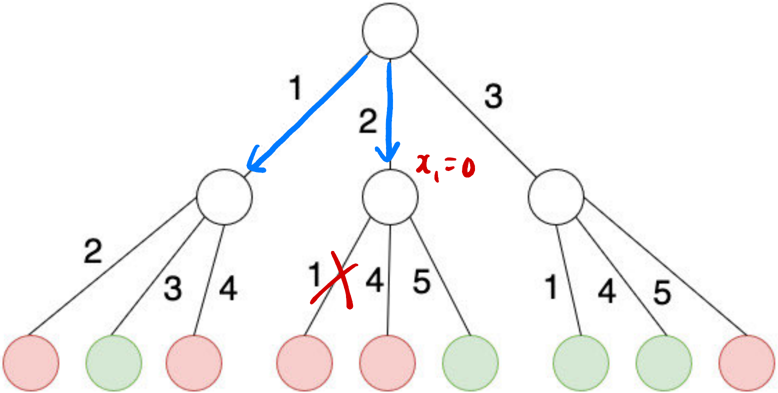

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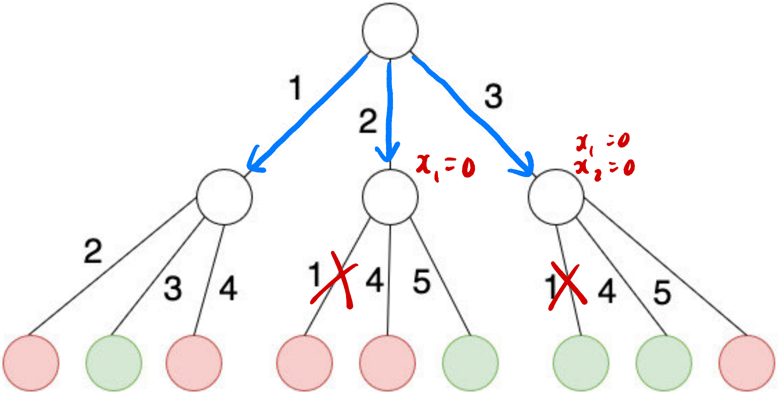

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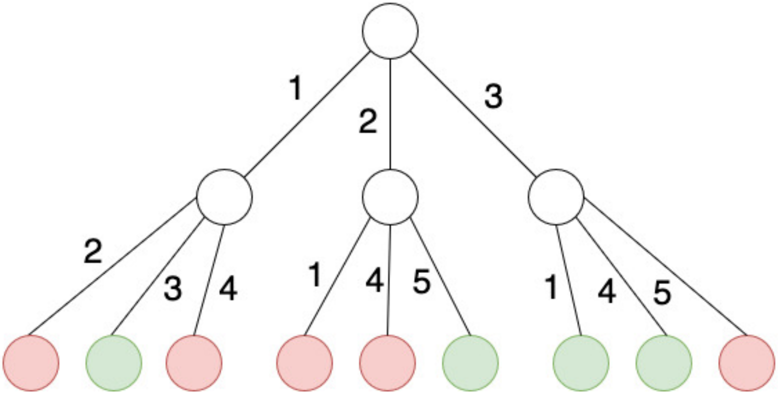

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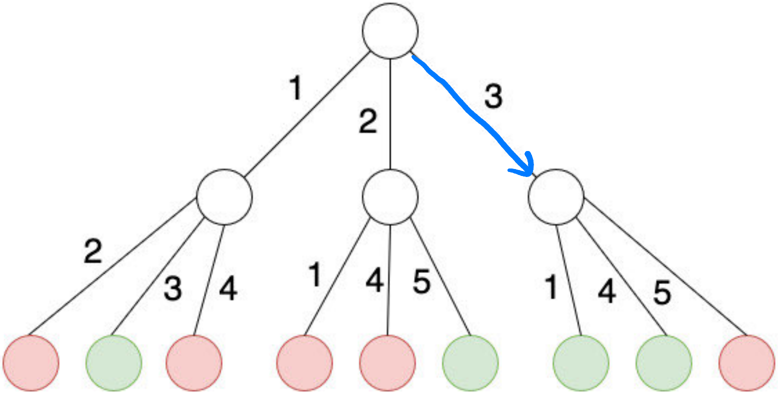

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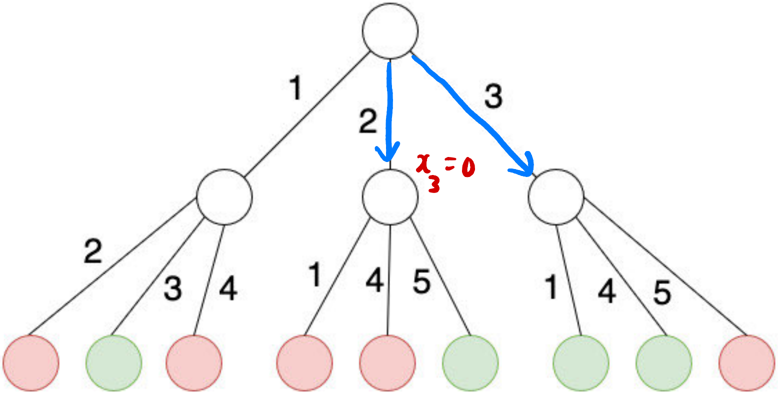

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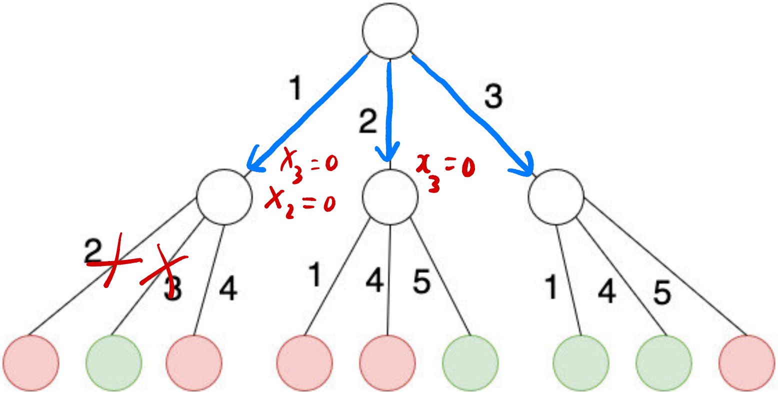

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## Previous and This work: Randomized Ordering

Randomize order of outgoing edges at every node.

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## Lemma (Expected Runtime of Our Algorithm)

*Expected runtime under randomized ordering is* $\sigma(T) = \sum_{\ell \in leaf(T)} \sigma(P_{root,\ell})$

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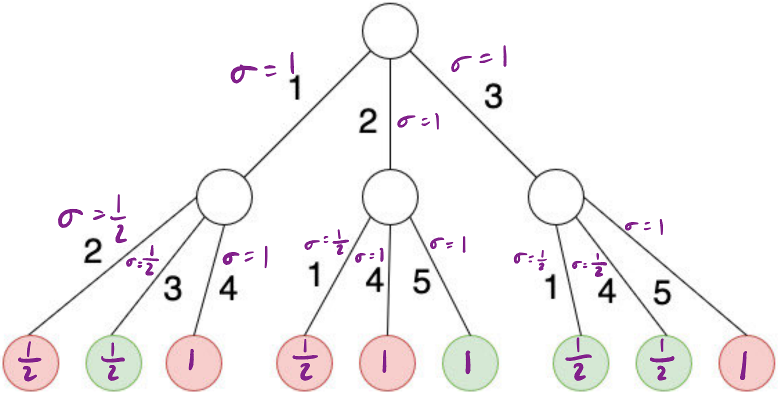
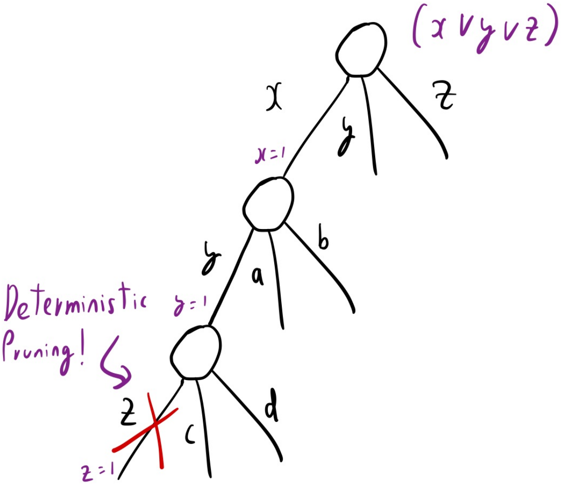


# Why NAE Helps - Deterministic Pruning



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- With careful accounting, conclude

$$\sigma(T) \leq 6^{n/4}.$$

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- Prove that Enum$(3, n/2)$ can be solved in time $\mathbf{6^{n/4}}$.
- Prove that in every $3$-uniform hypergraph with transversal number $n/2$, number of transversals of size $n/2$ is $\leq \mathbf{6^{n/4}}$.