Modal Separation of Fixpoint Formulae

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Jena

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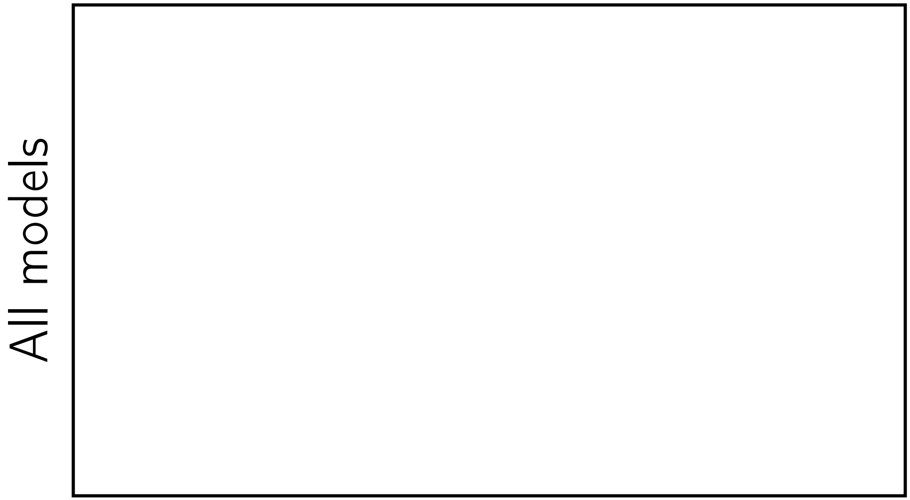

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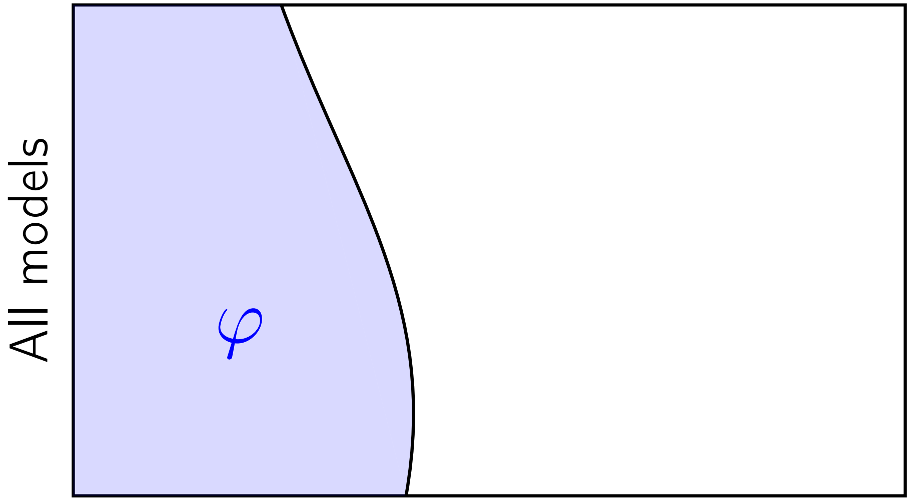

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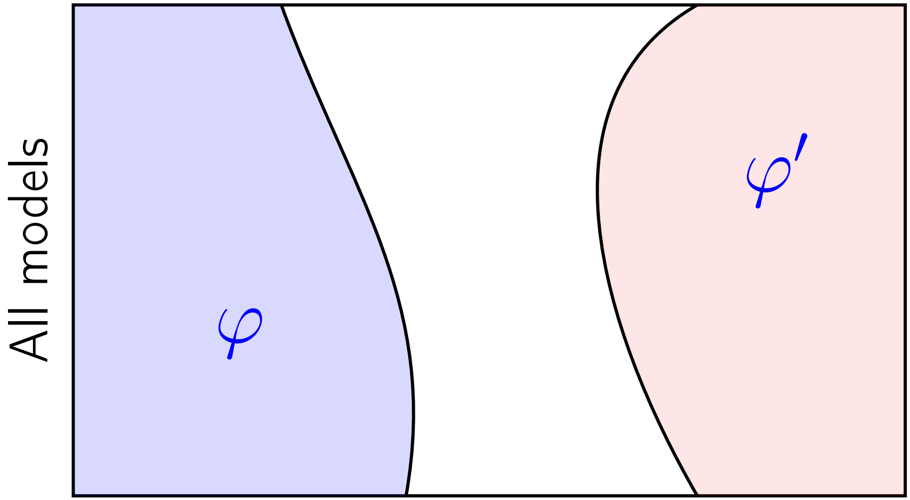

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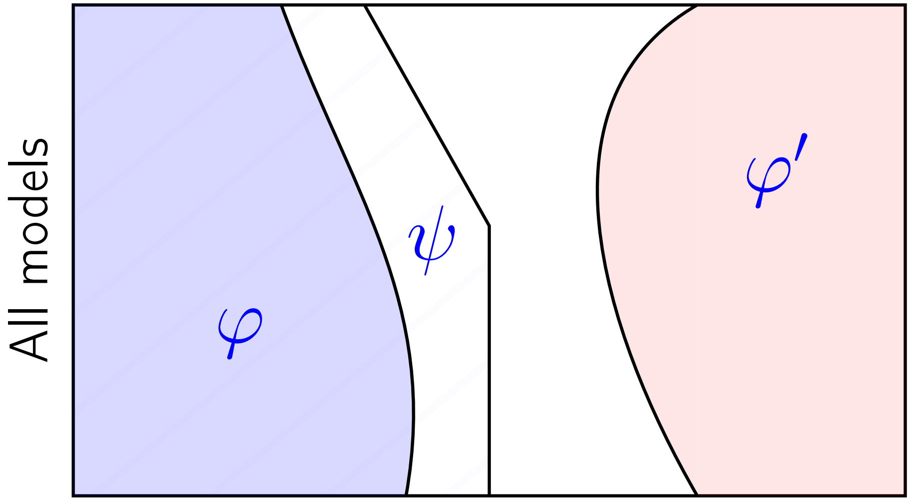

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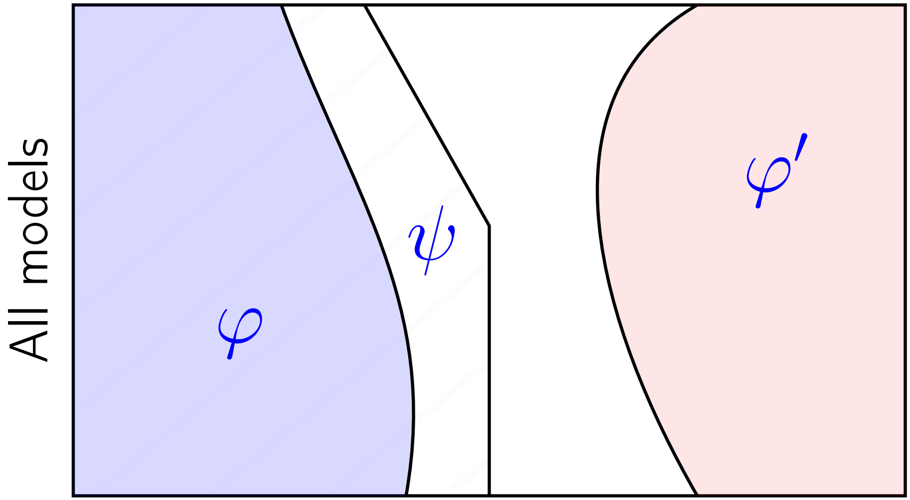

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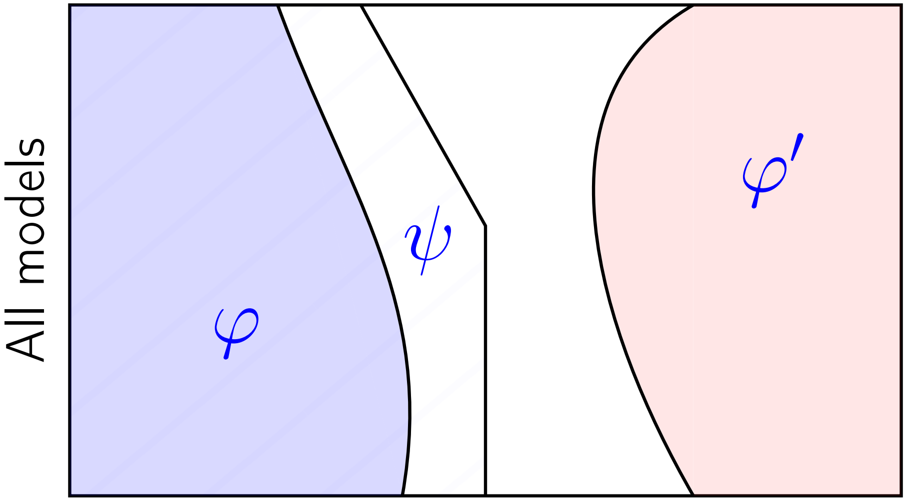

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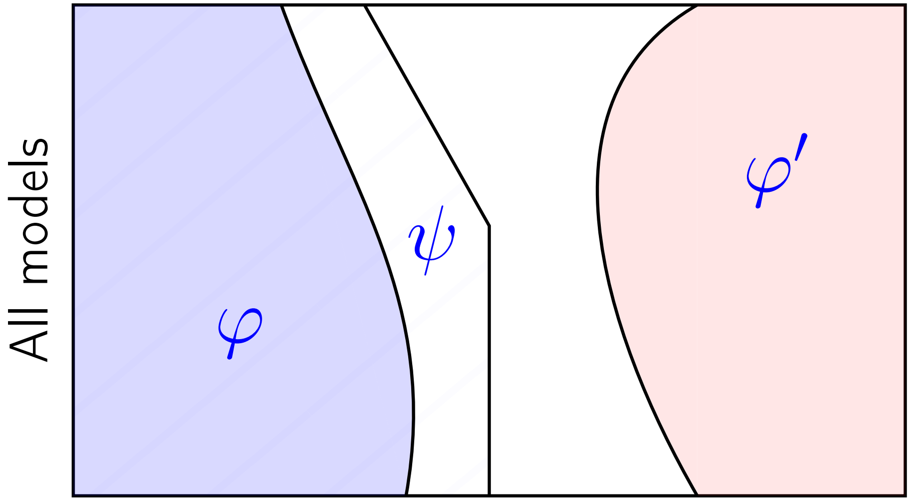

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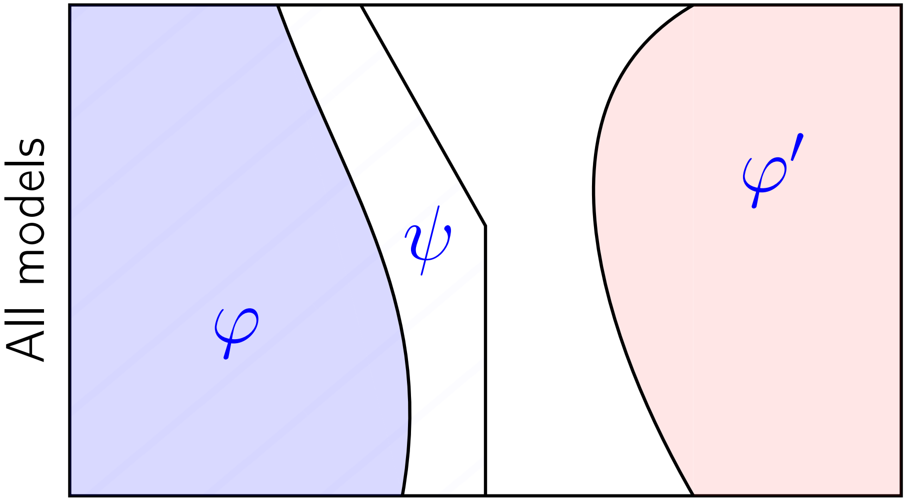

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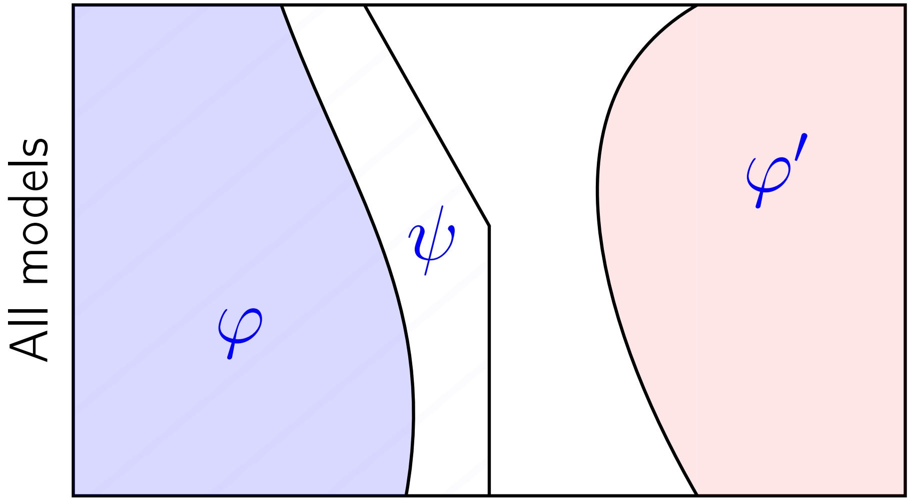

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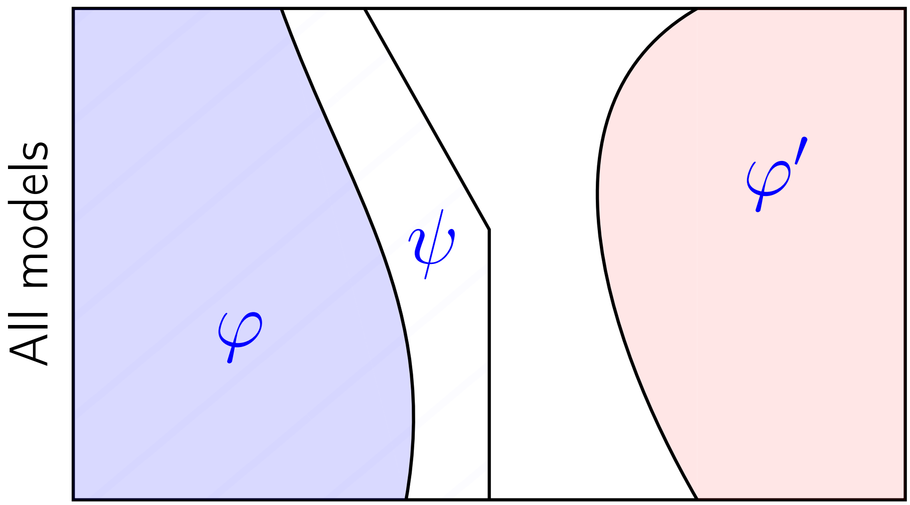

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simple explanation of contradiction

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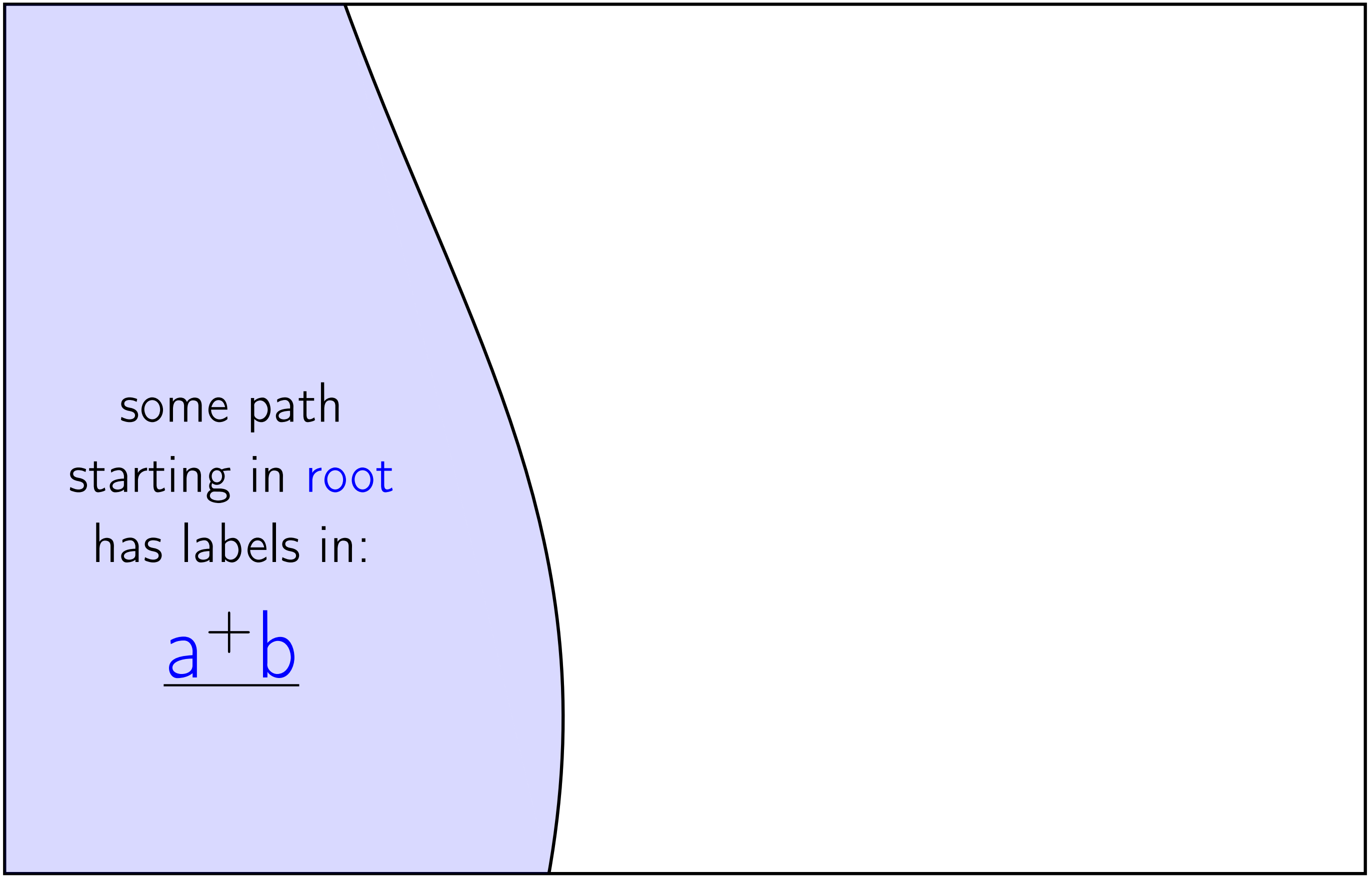

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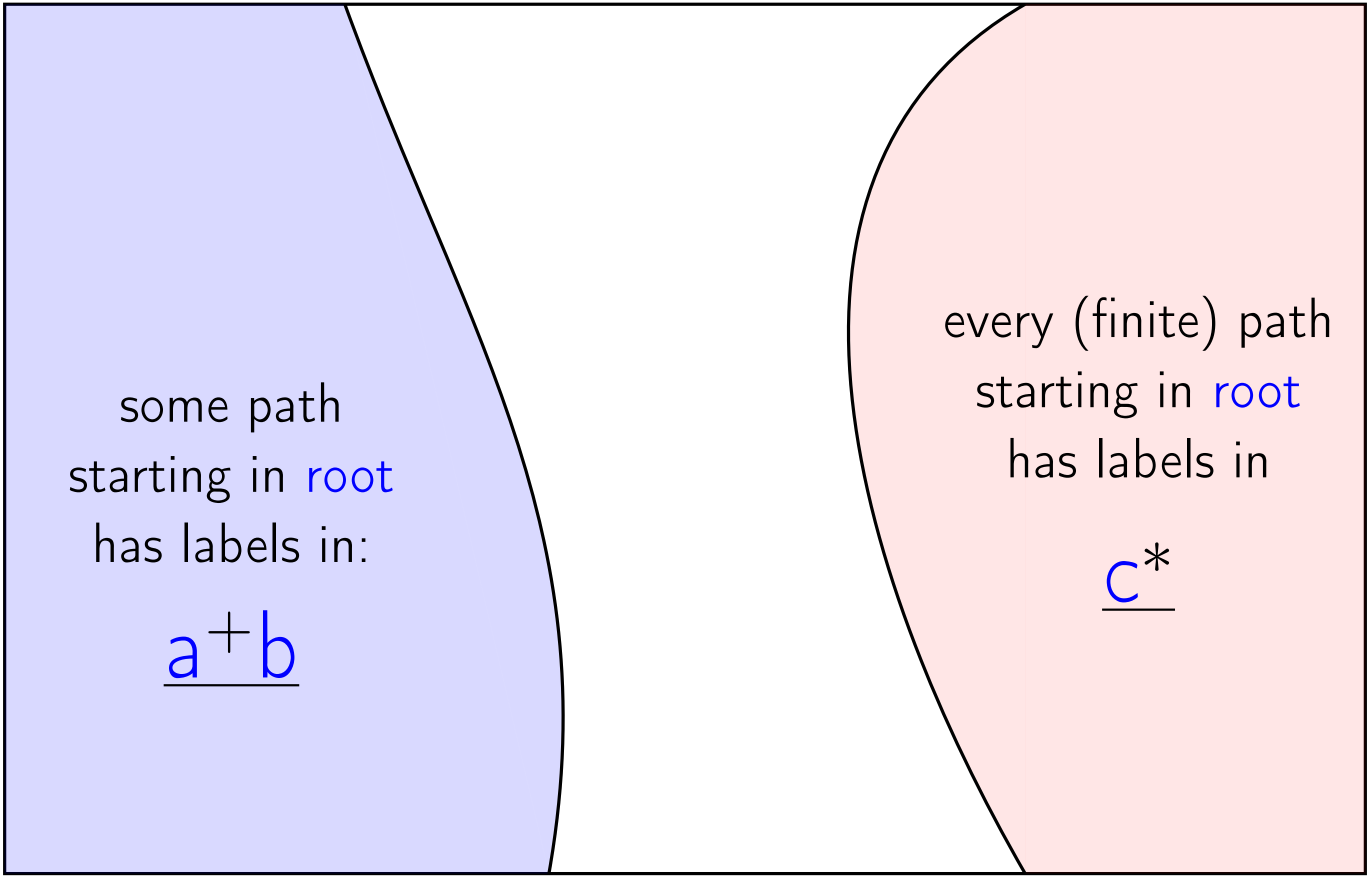

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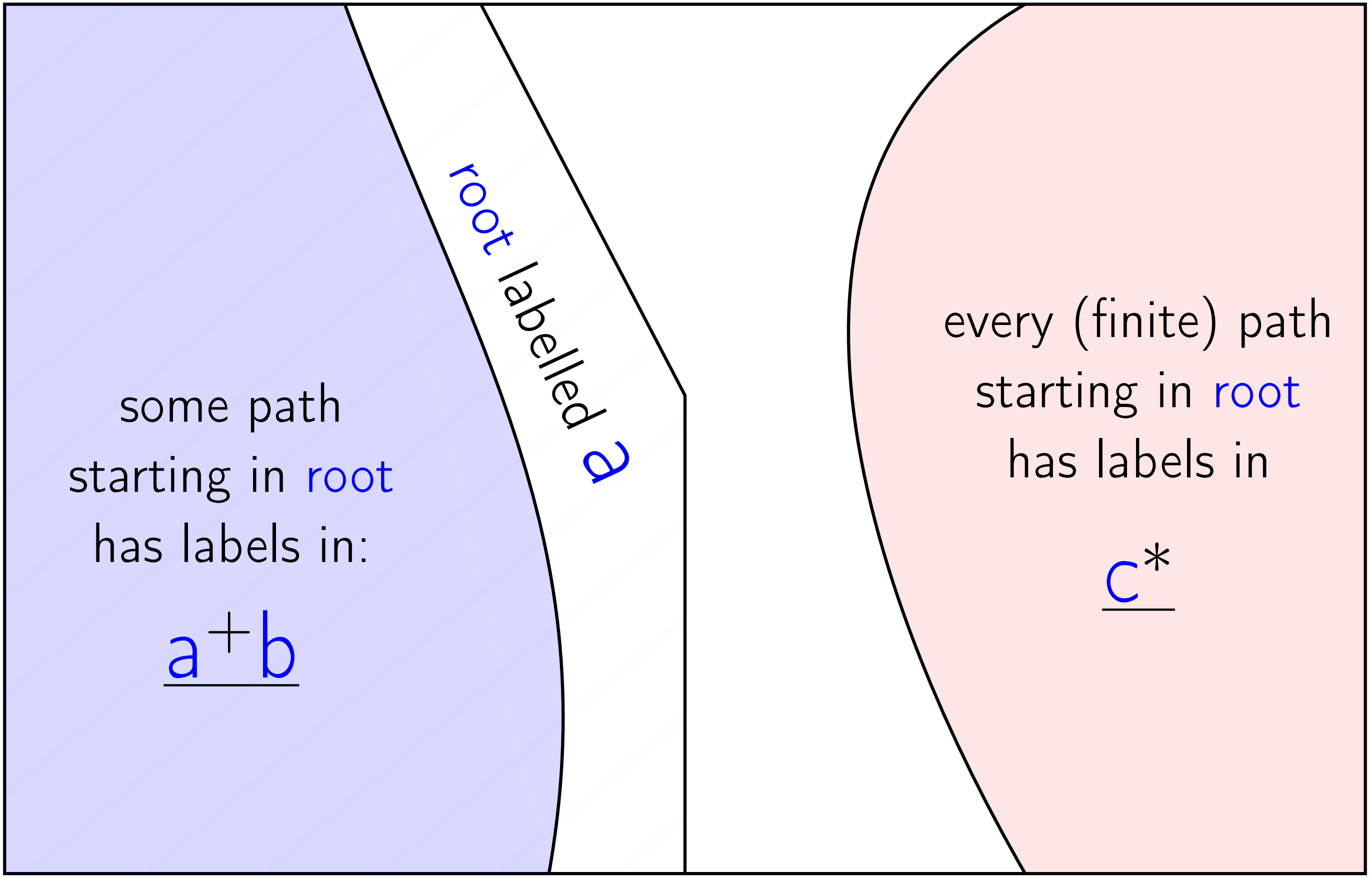

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given: $\varphi, \varphi' \in \boldsymbol{\mathcal{L}}^{+}$

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is there a separator $\psi \in \boldsymbol{\mathcal{L}}$?

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- Hence, $\mathcal{L}$-definability: "is given φ expressible in $\mathcal{L}$?"
- is a special case of $\mathcal{L}$-separability.

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$\mathcal{L}^+ = \mu$-ML = ML + **fixpoints**

The semantics of μ-ML = ML + fixpoints

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$$\mu\text{-ML}$$

$$=$$

Automata

Translations

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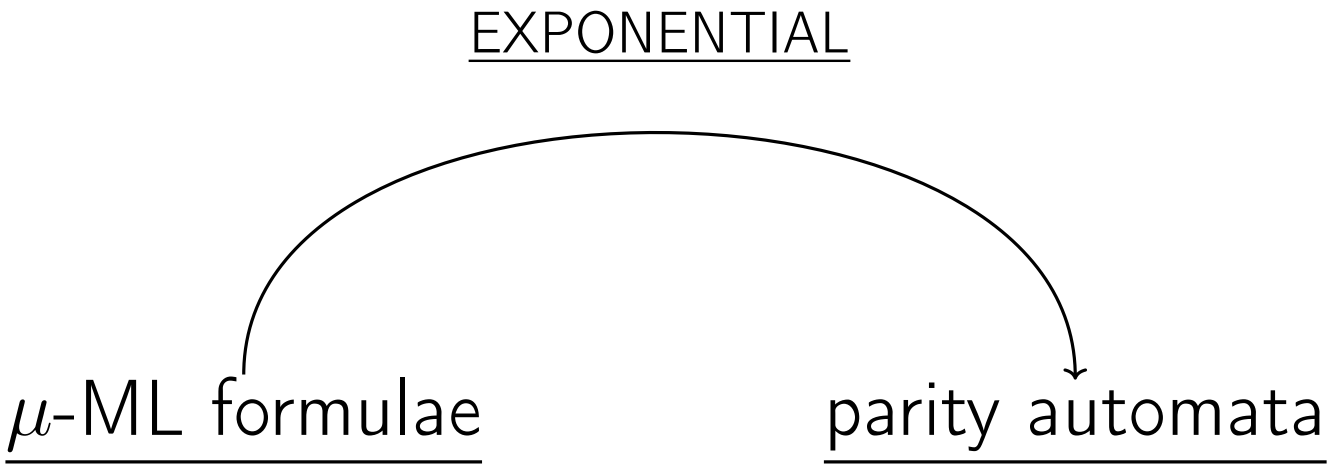
μ-ML formulae

parity automata

Translations



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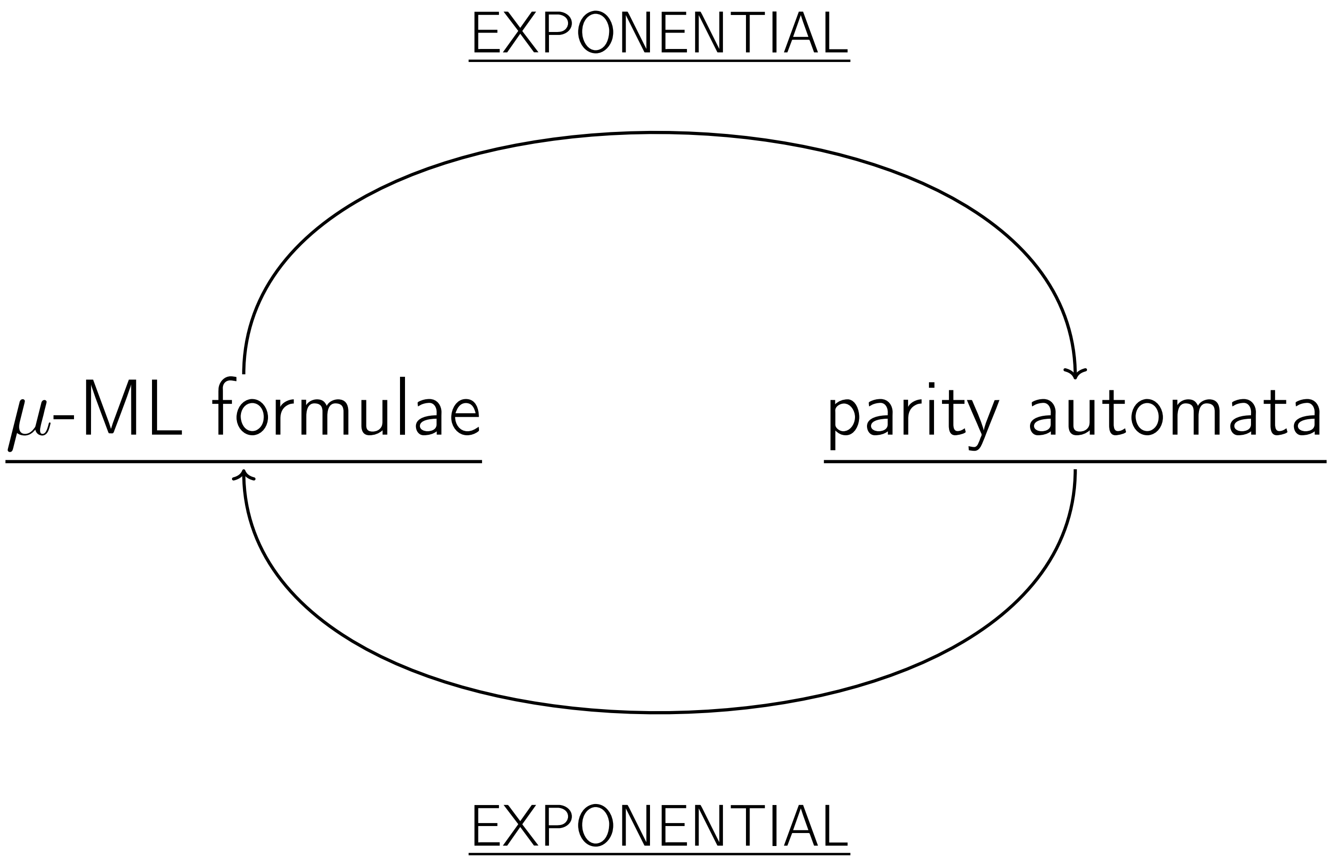

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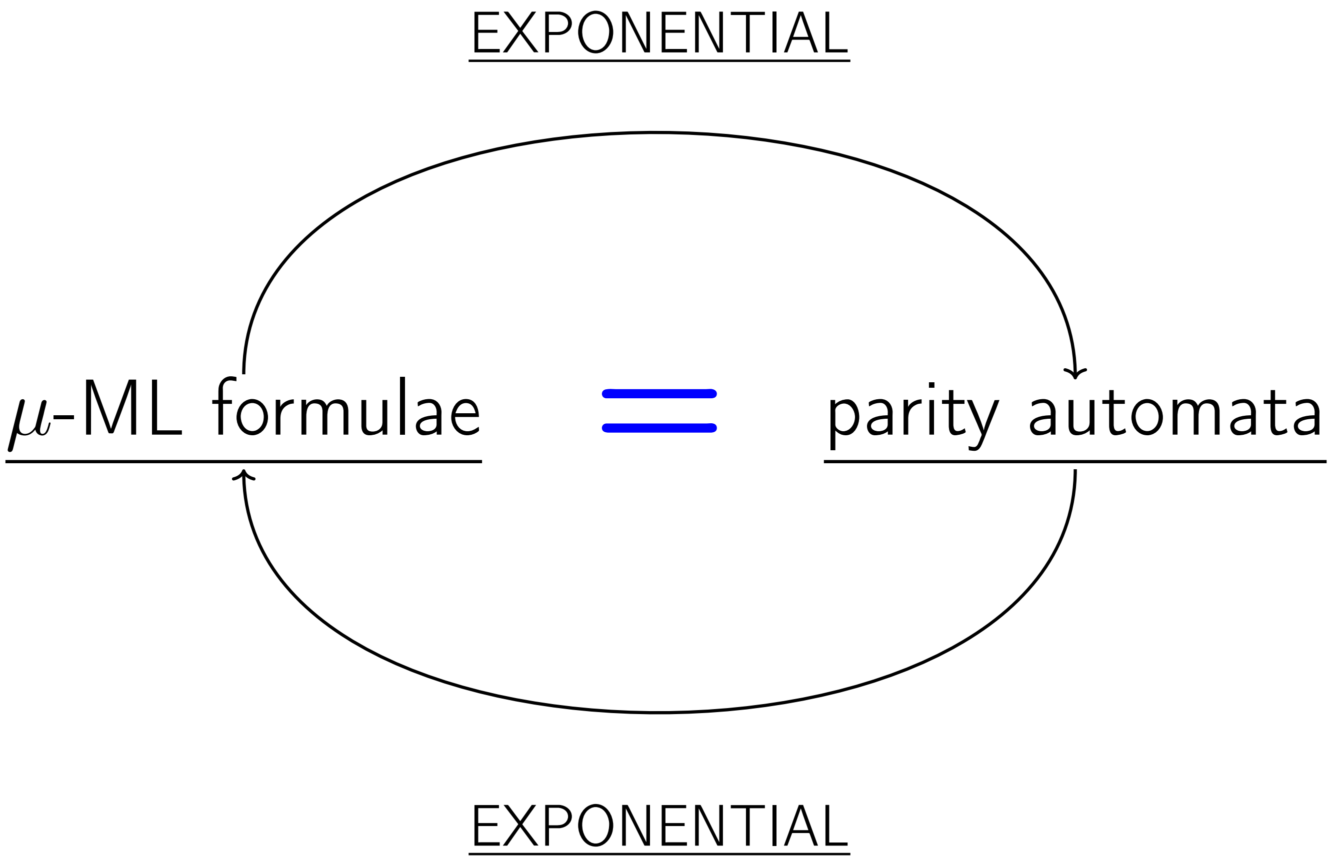

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φ entails no modal formulae!

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- in all cases trees are unordered.

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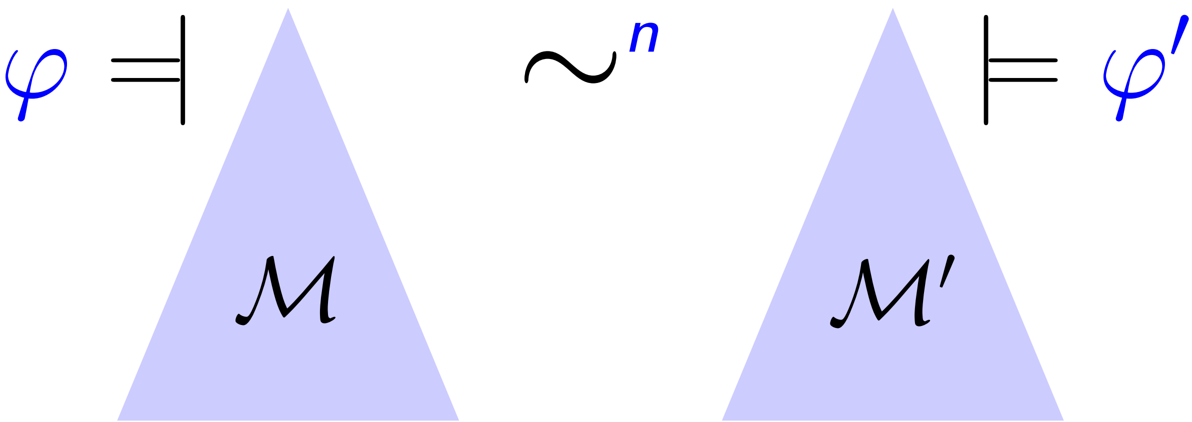

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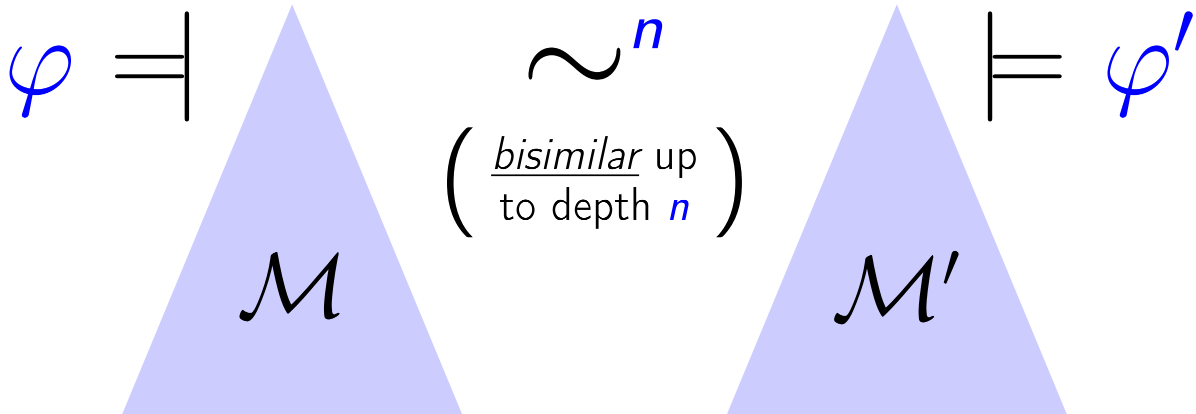

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Turing machine T with

2^n memory cells

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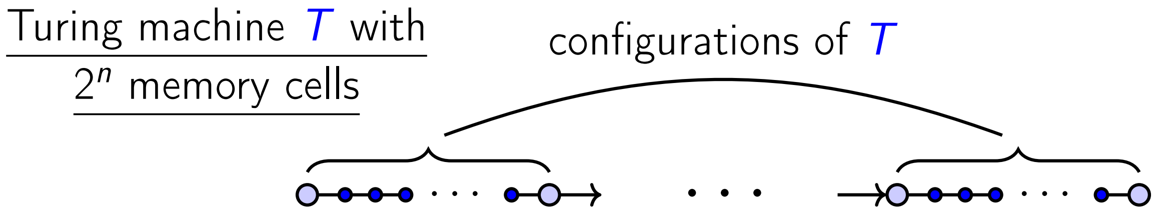

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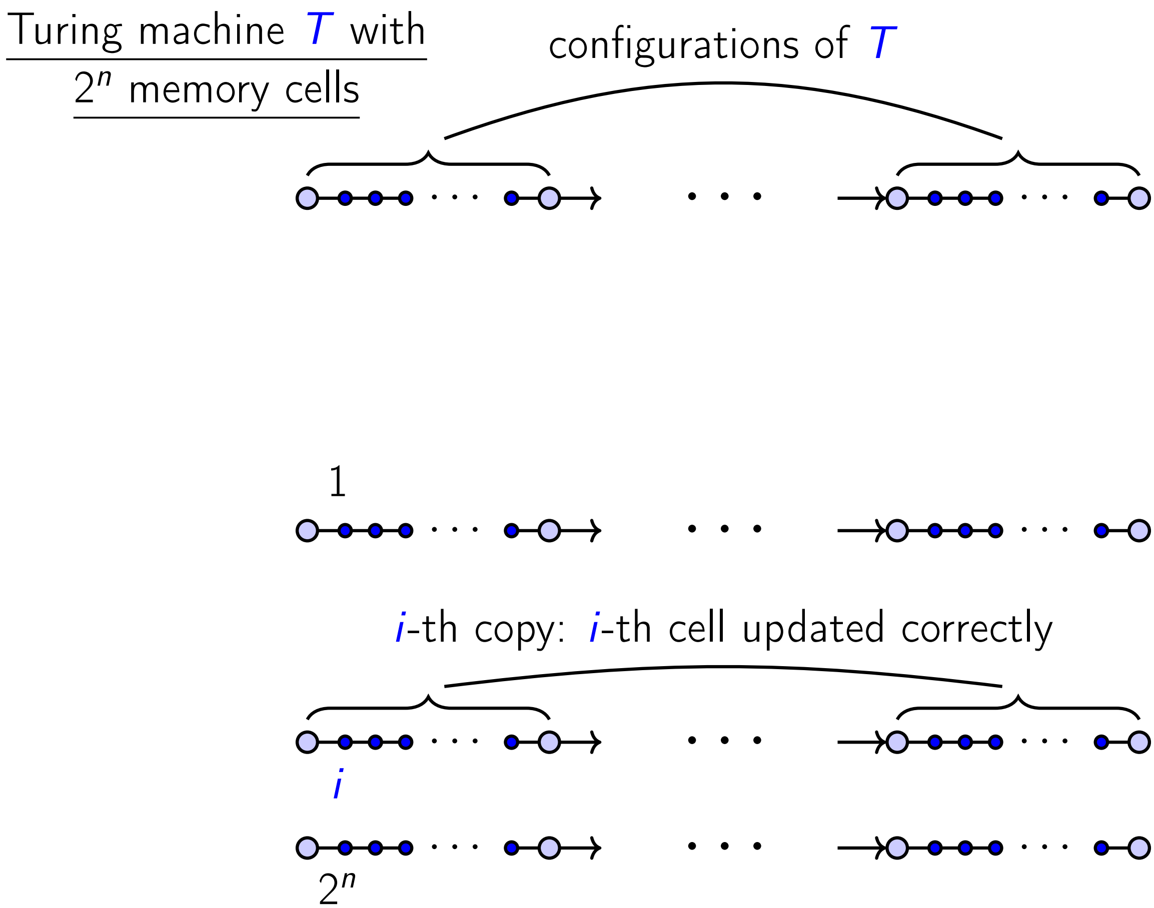

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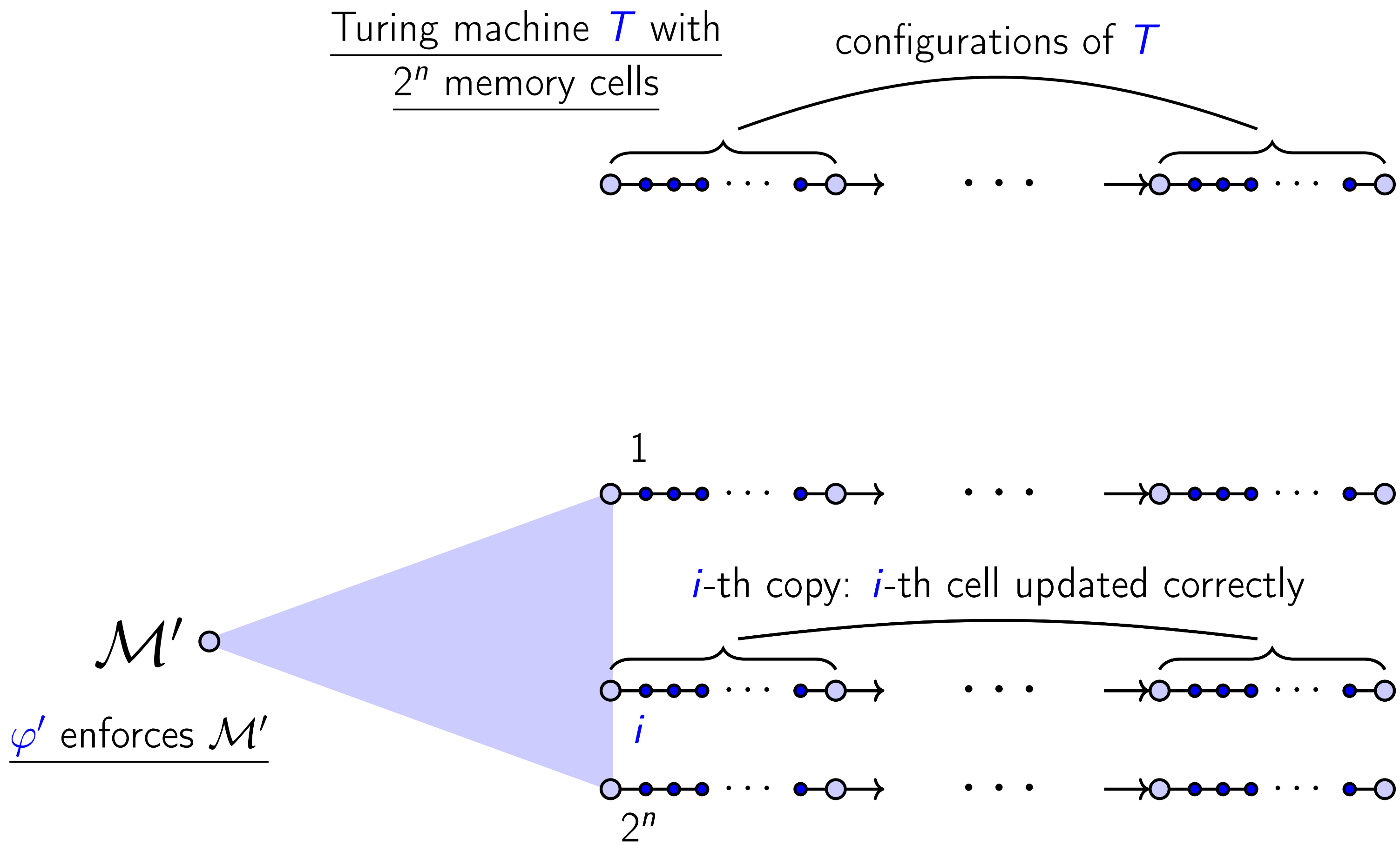

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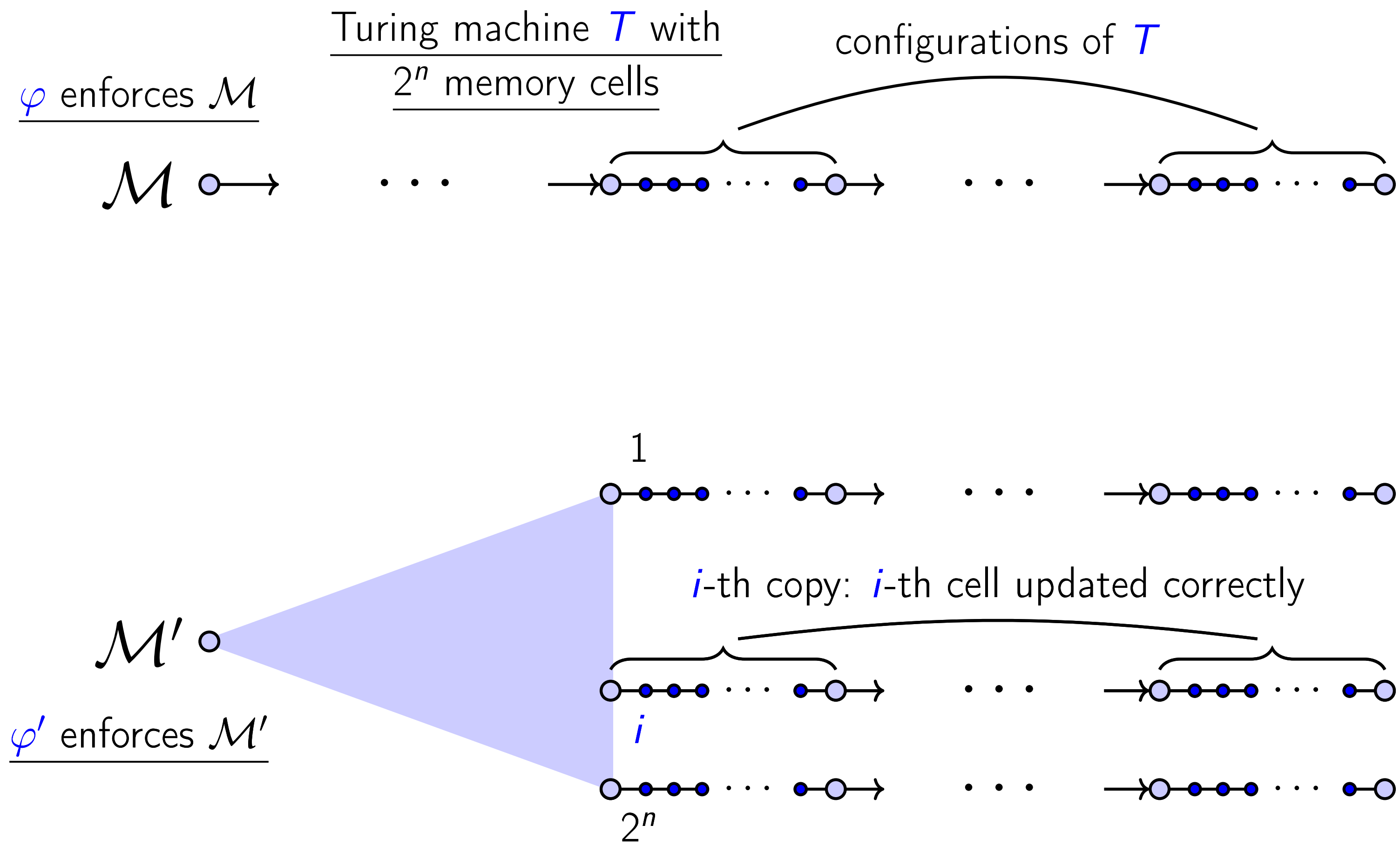

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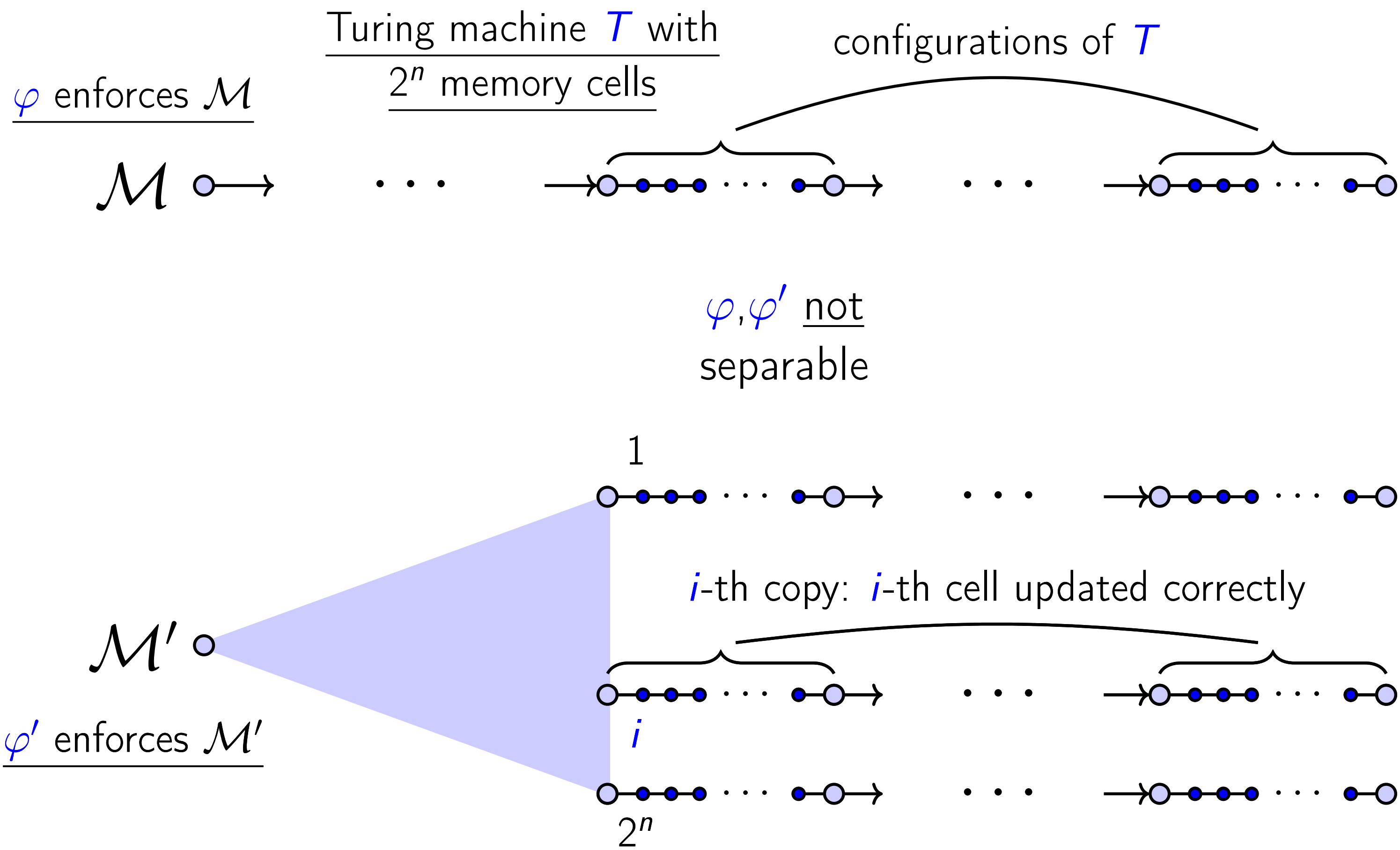

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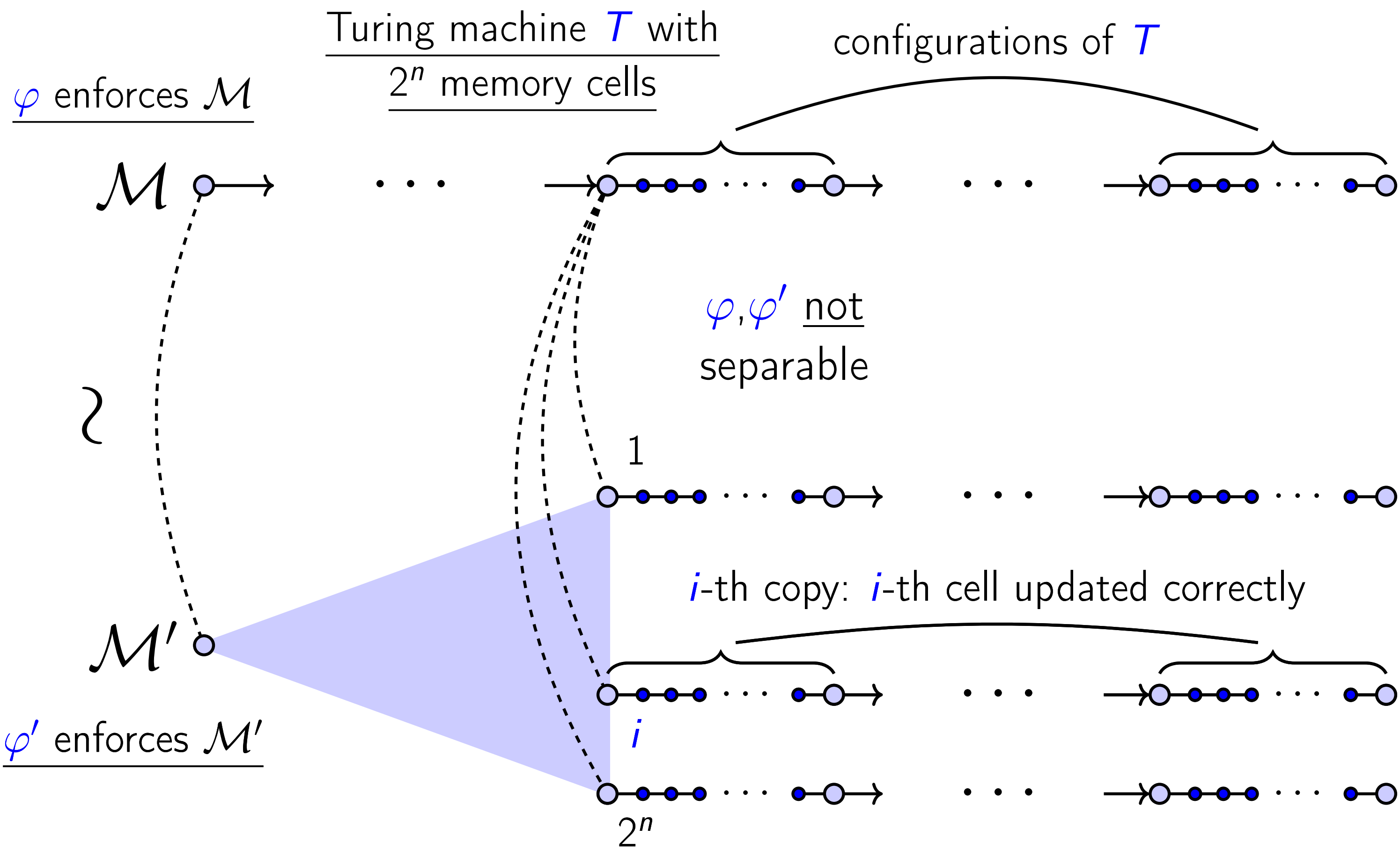

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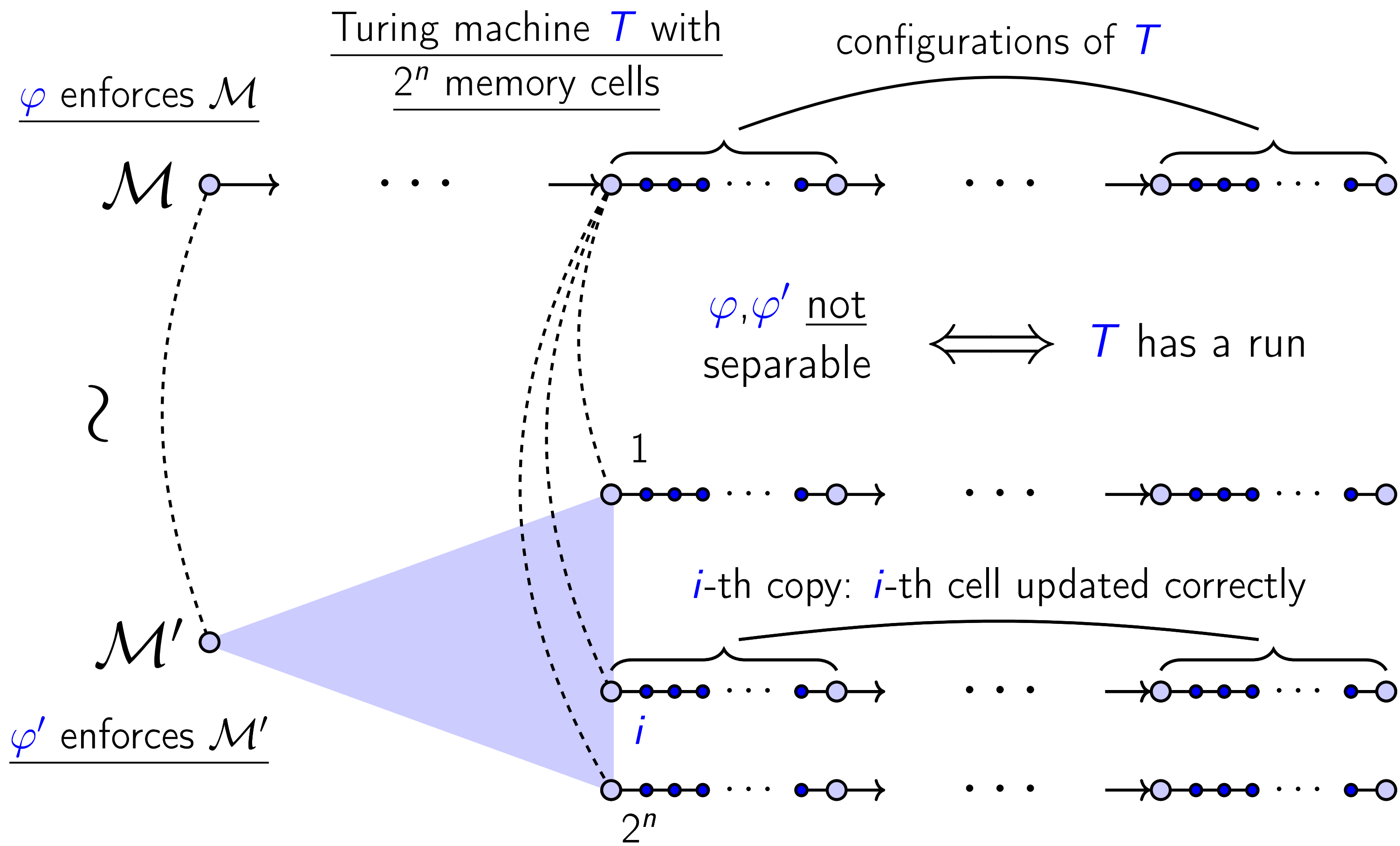

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- conclusion: modal separation is 2-ExpTime-hard over ternary trees!

Ternary case: upper bound

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- assume: tree $\mathcal{M} = (V, E)$, every node with at most ternary branching

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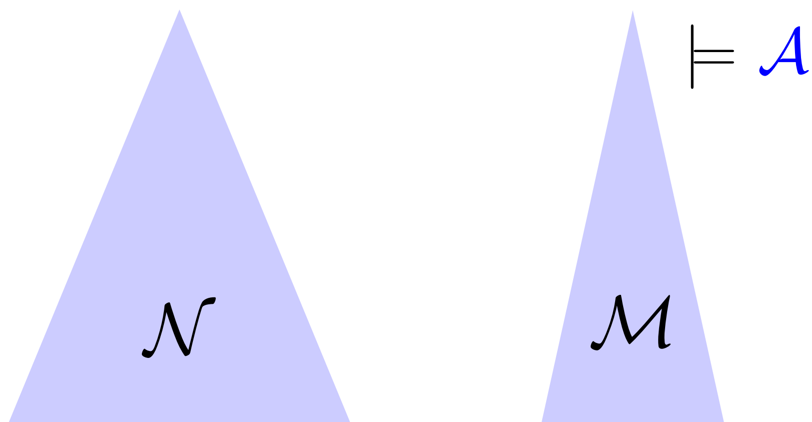

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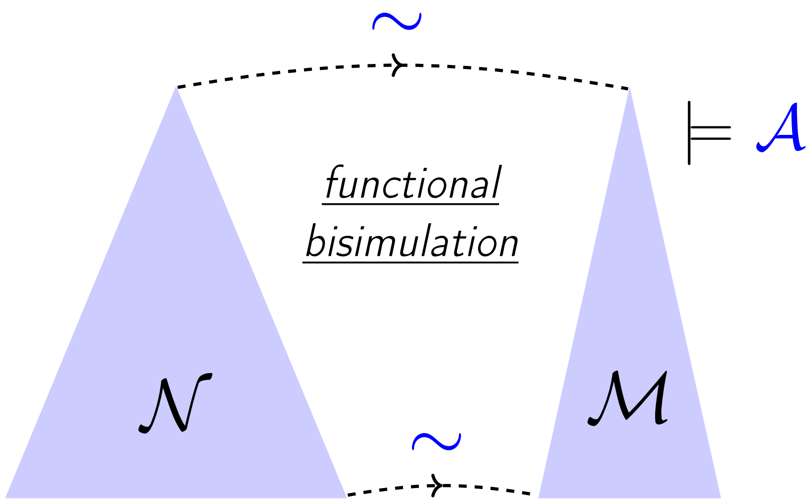

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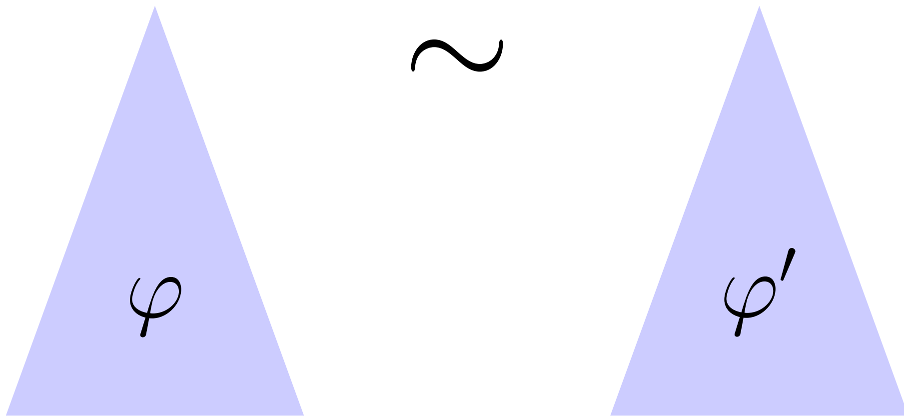

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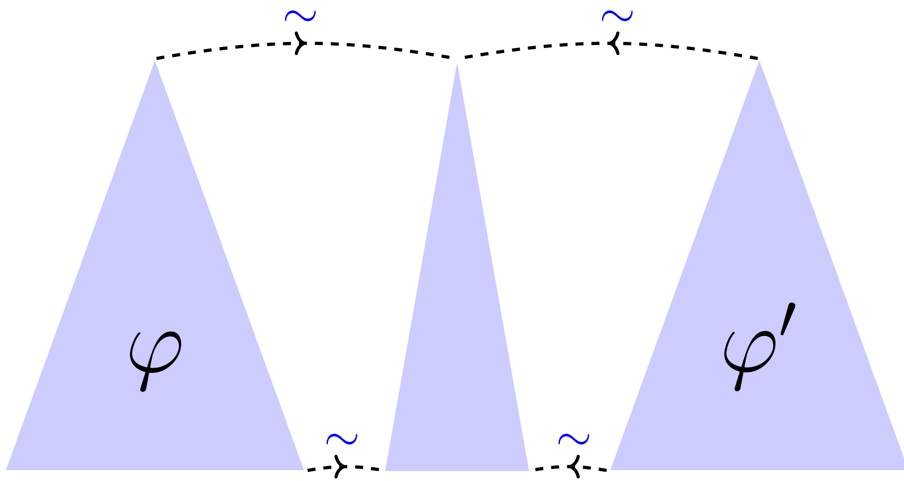

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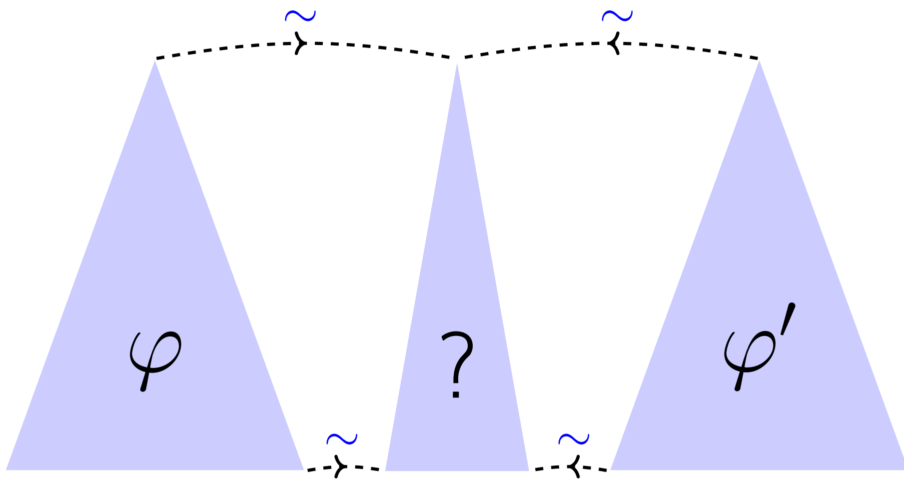

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Thank you!

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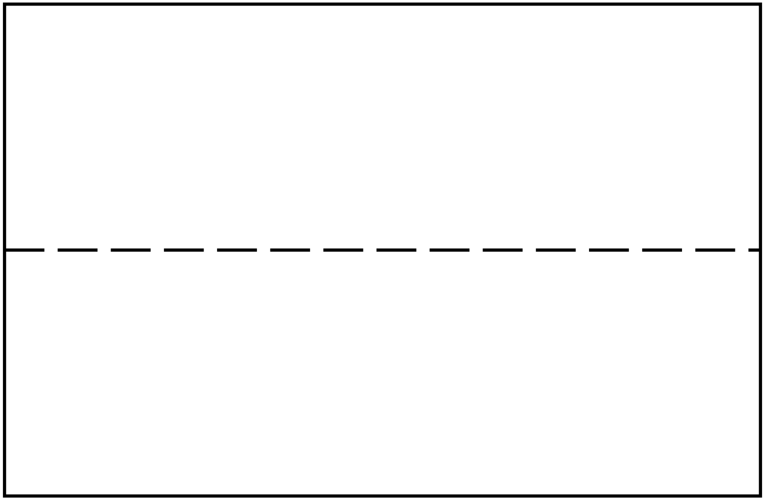

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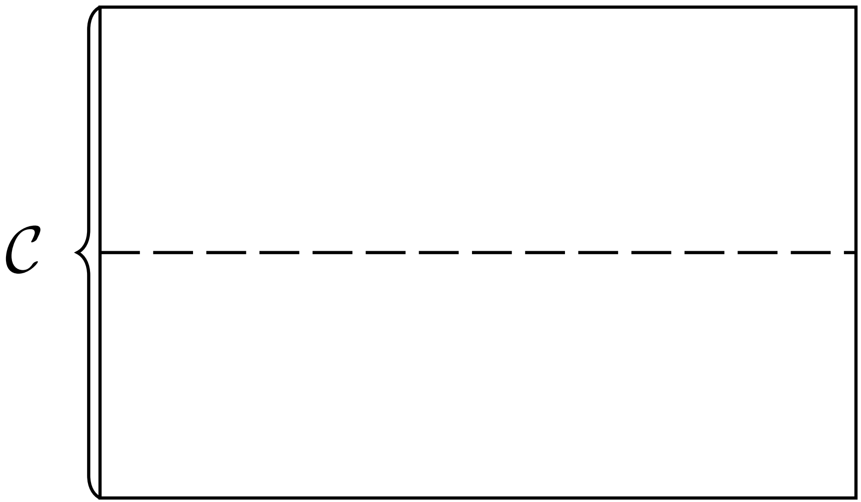

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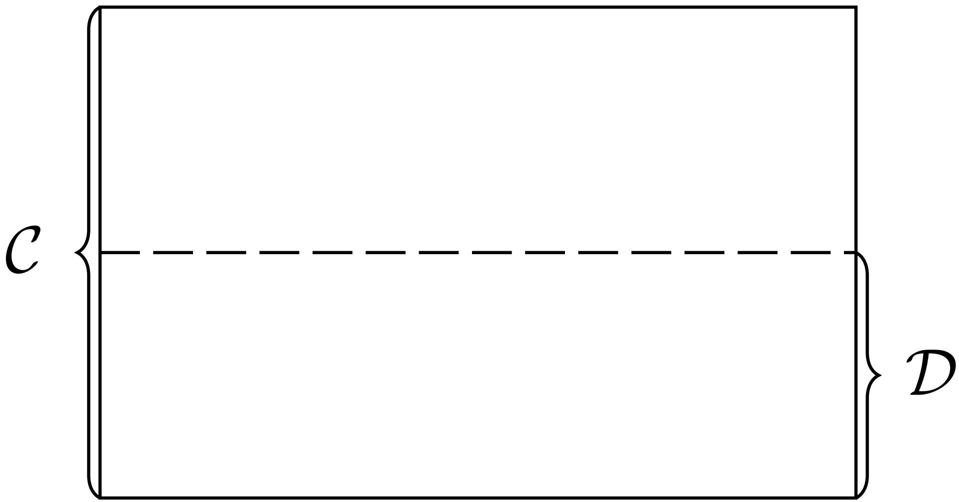

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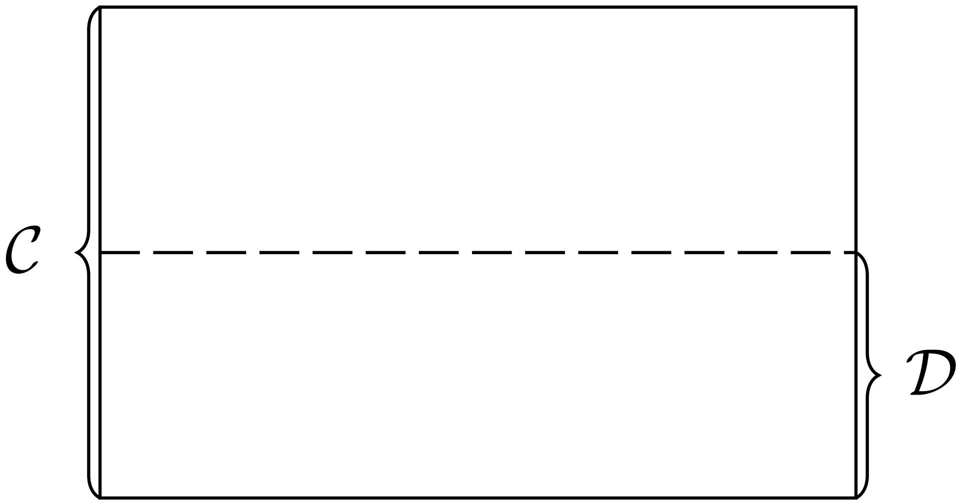
Finitely branching models



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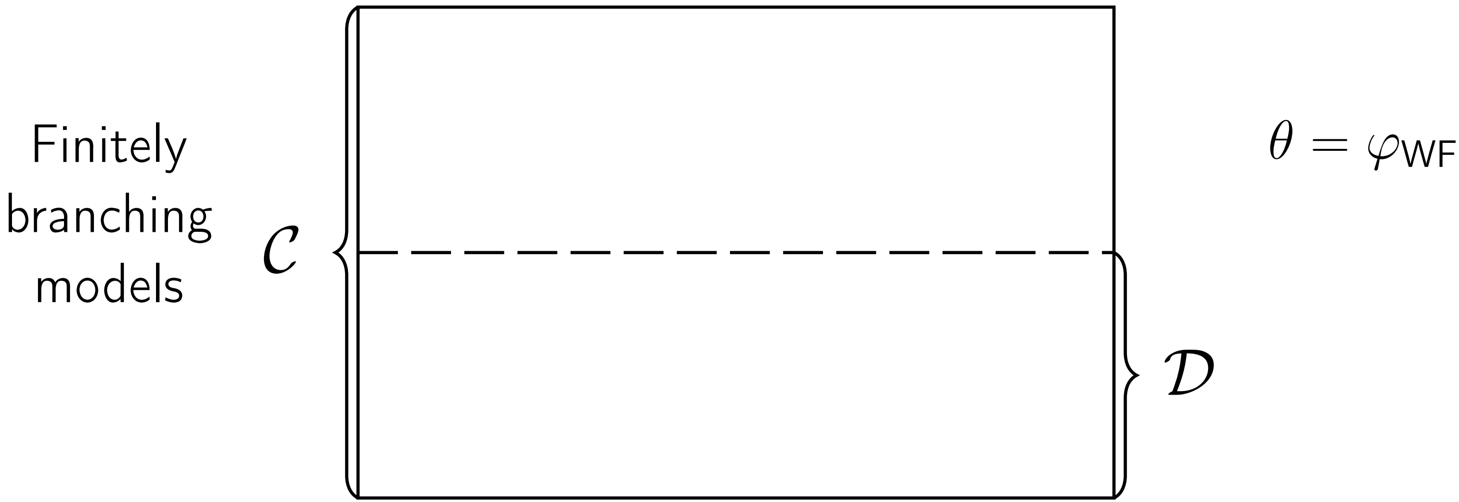

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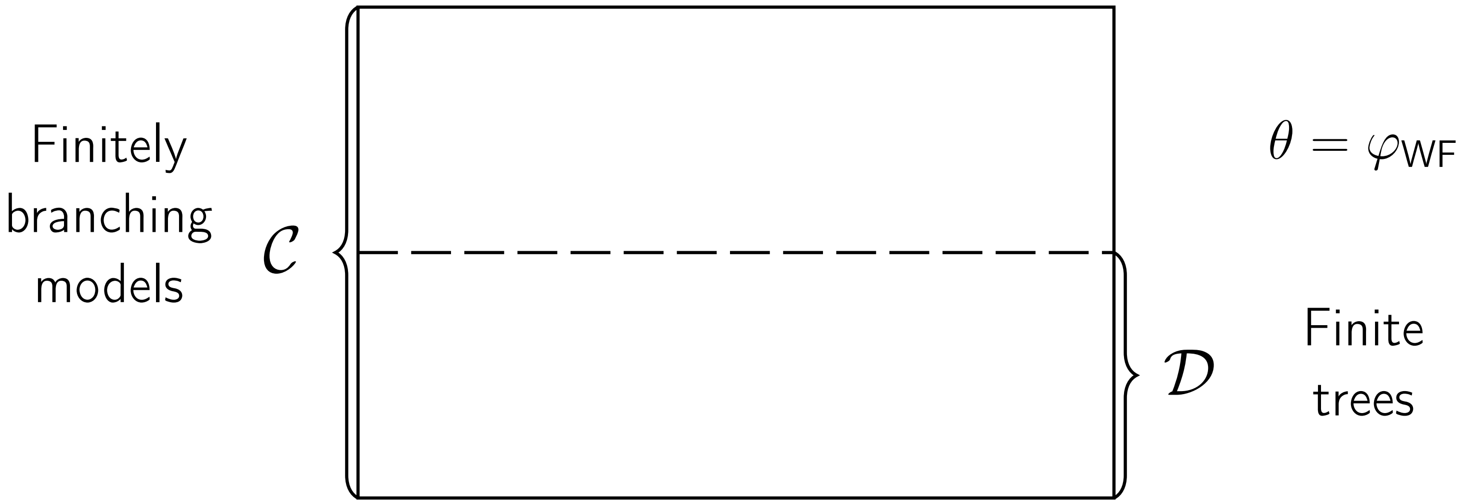

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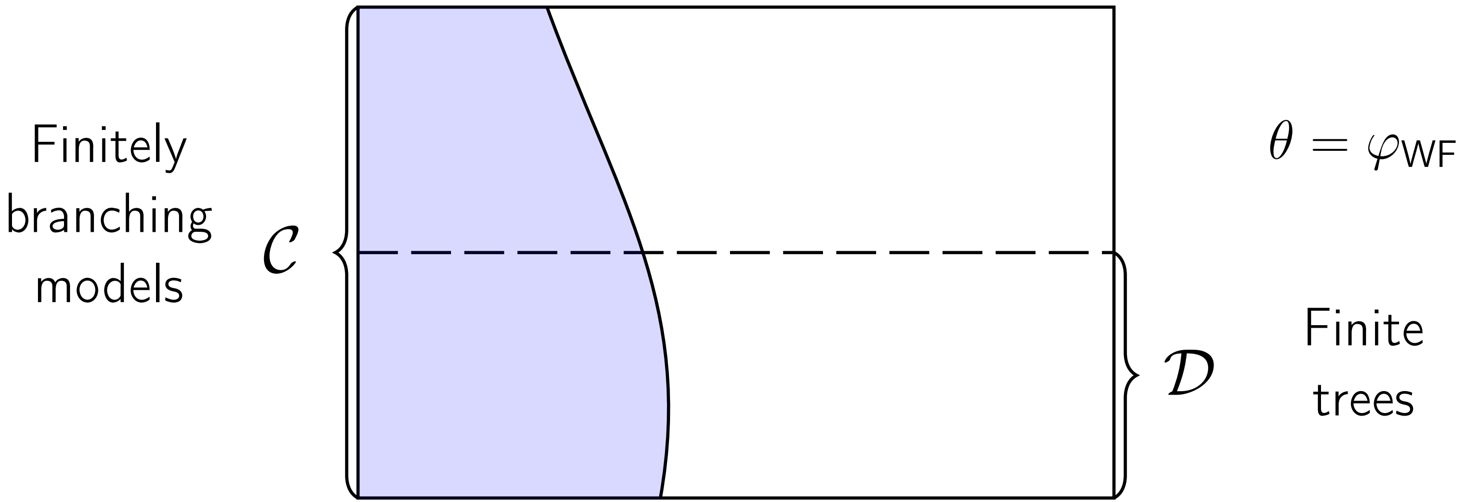

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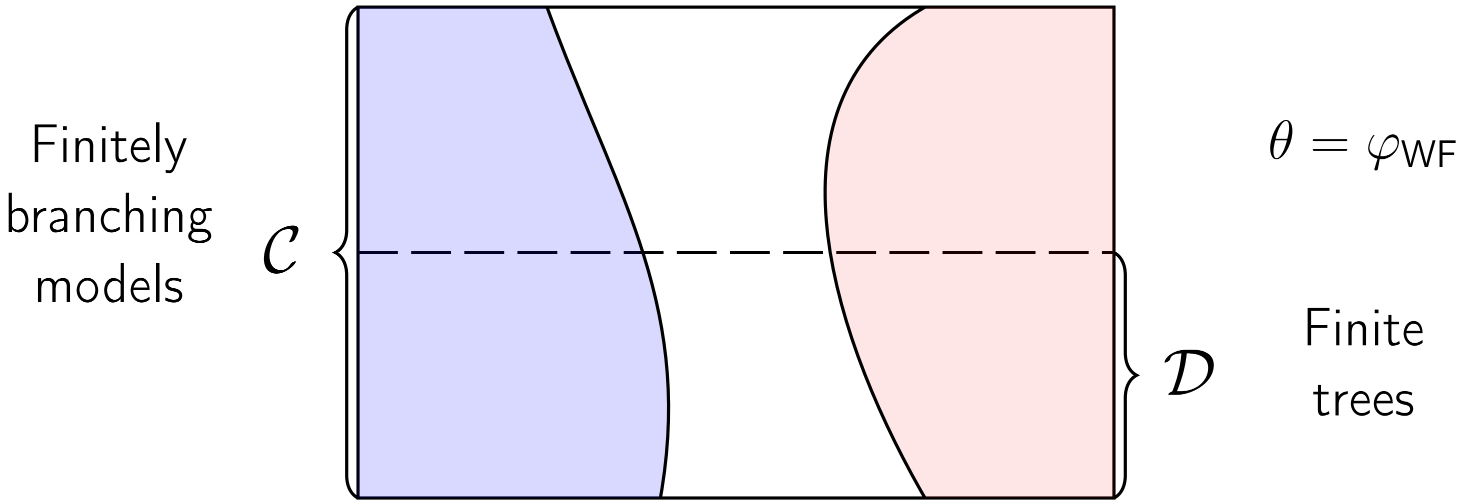

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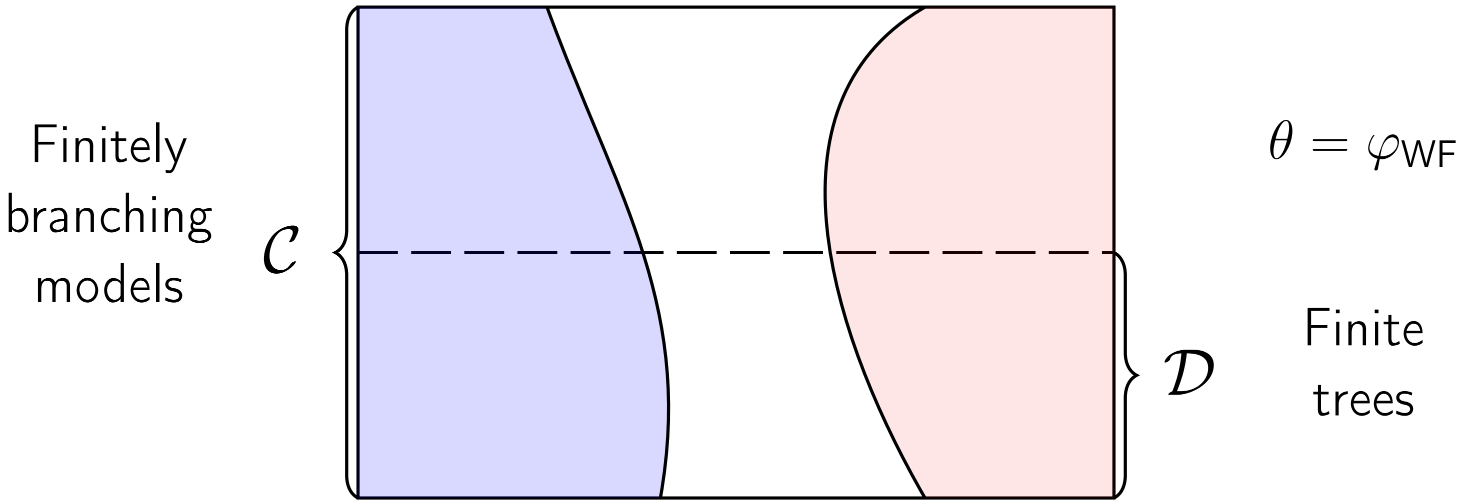

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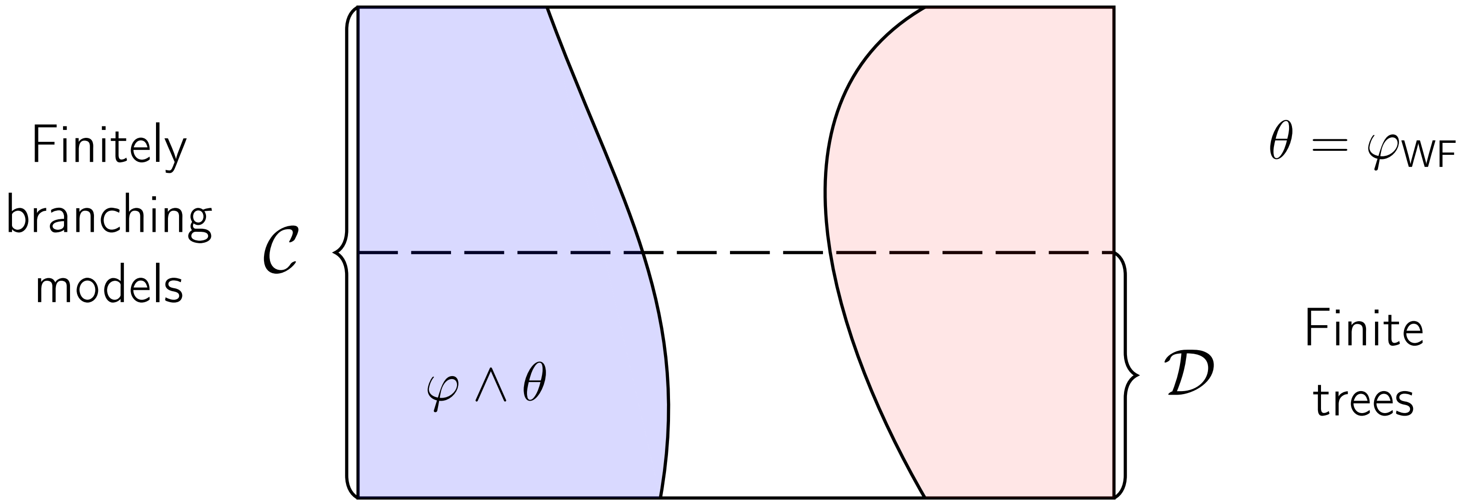

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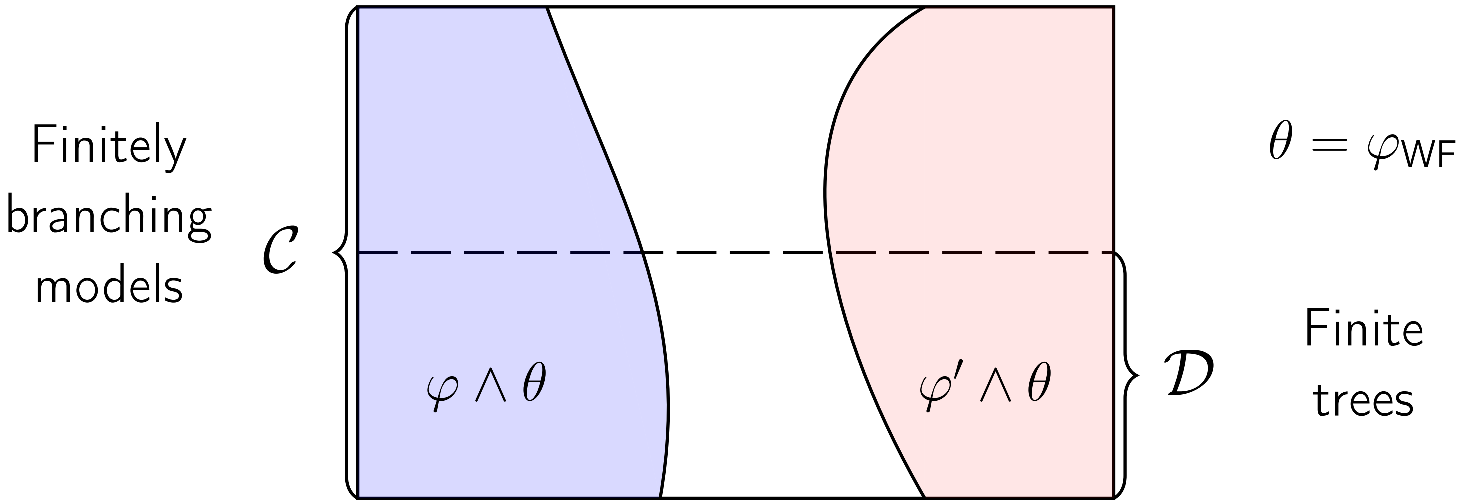

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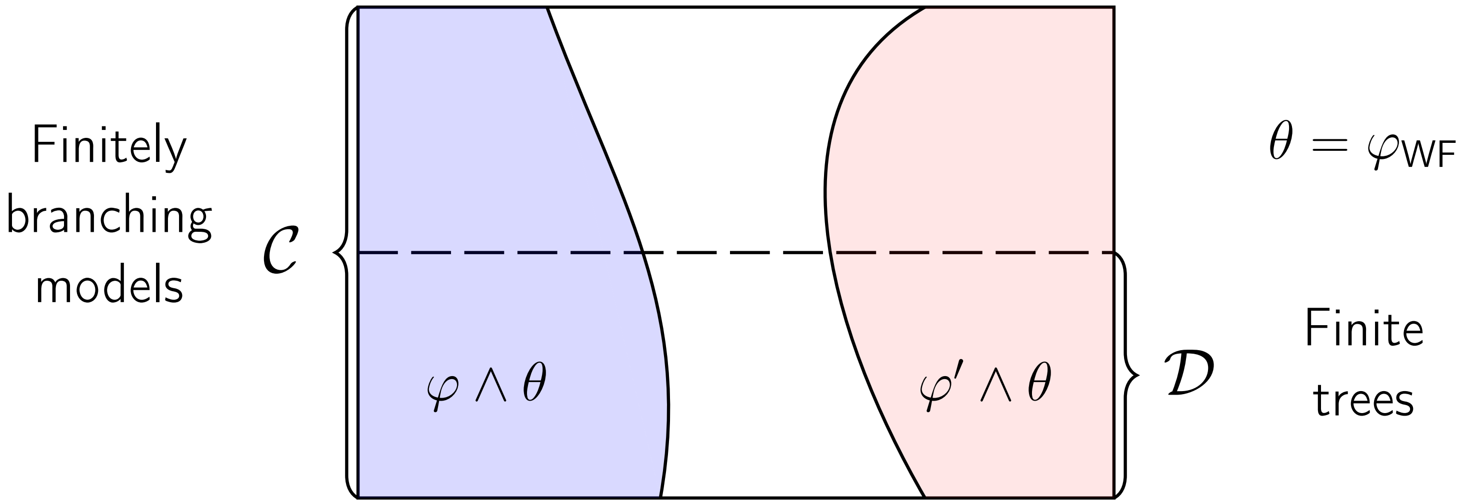

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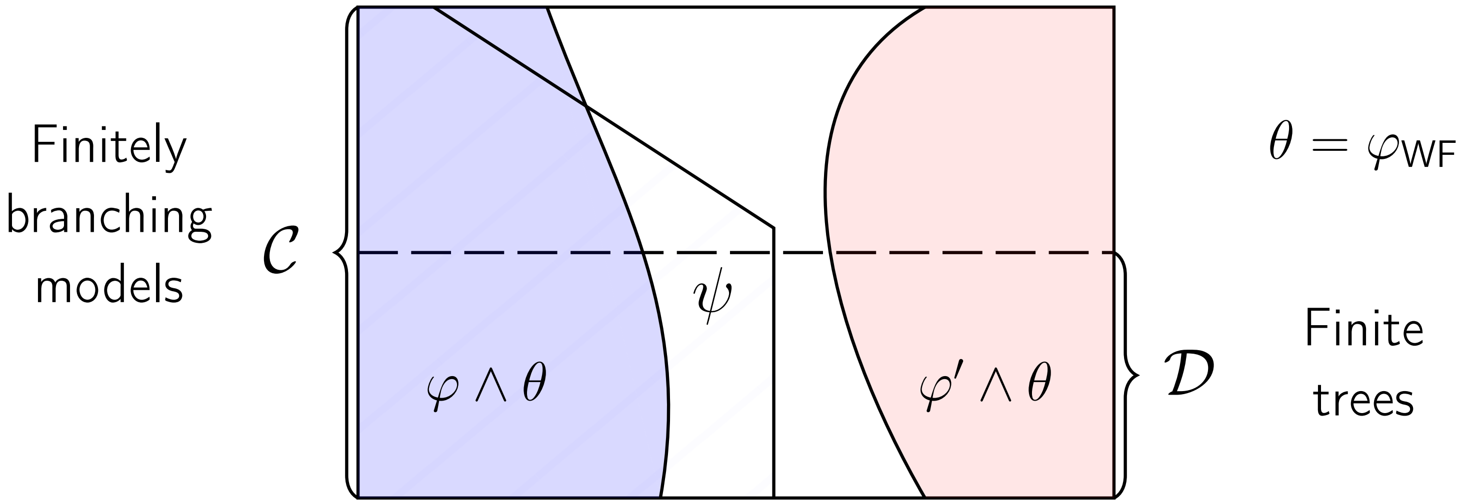

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