Monotone weak distributive laws over weakly lifted powerset monads in categories of algebras

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Monads and effects (Moggi 1989)

- Effects can be modelled using monads, e.g. (in $\mathbf{Set}$):
  - non-determinism: $X \to \mathbf{P}Y = \{\text{subsets of } Y\}$
  - probabilities: $X \to \mathbf{D}Y = \{\text{finite-support prob. distr. on } Y\}$
- In $\mathbf{Set}$, a type constructor $X \mapsto \mathrm{T}X$ is a monad when it comes with polymorphic functions

$$\begin{aligned} \texttt{mapT}\colon (X \to Y) &\to (\mathrm{T}X \to \mathrm{T}Y) \\ \texttt{joinT}\colon \mathrm{TT}X &\to \mathrm{T}X \\ \texttt{returnT}\colon X &\to \mathrm{T}X \end{aligned}$$

satisfying certain axioms, so that T-effectful morphisms

$$X \rightarrowtriangle_{\mathrm{T}} Y \equiv X \to \mathrm{T}Y$$

form a category $\mathbf{Eff}(\mathrm{T})$

Composing effects via weak distributive laws (WDLs)

- What about combining effects? In general, given monads T and S, there could be a monad structure on a retract of ST.
- True with a *weak distributive law* (Beck 1969)(Garner 2019), i.e. a polymorphic

  $$\mathtt{swapTS}\colon \mathrm{TS}X \to \mathrm{ST}X$$

  satisfying ~~four~~ three axioms.
  Equivalently, a way to weakly extend $\mathtt{map}\mathrm{T}$ to $\mathbf{Eff}(\mathrm{S})$

  $$\mathtt{map}\underline{\mathrm{T}}\colon (X \rightarrow\!\!\!\circ_{\mathrm{S}} Y) \rightsquigarrow (\mathrm{T}X \rightarrow\!\!\!\circ_{\mathrm{S}} \mathrm{T}Y)$$

  that makes T into a semi-monad $\underline{\mathrm{T}}$ on $\mathbf{Eff}(\mathrm{S})$.
- WDLs give rise to generic constructions: generalized determinization, up-to techniques
- There is no distributive law $\mathbf{DP} \Rightarrow \mathbf{PD}$ (Varacca and Winskel 2006) nor $\mathbf{PP} \Rightarrow \mathbf{PP}$ (Klin and Salamanca 2018)!

Examples of weak distributive laws

- (Garner 2019) In $\mathbf{Set}$, a WDL $\mathbf{PP} \Rightarrow \mathbf{PP}$ s.t. $(\mathbf{P} \bullet \mathbf{P})X = \{\text{sets of subsets of } X \text{ closed under non-empty unions}\}$
- (Goy and Petrişan 2020) In $\mathbf{Set}$, there is a WDL $\mathbf{DP} \Rightarrow \mathbf{PD}$ s.t. $(\mathbf{P} \bullet \mathbf{D})X = \{\text{convex subsets of prob. distr. on } X\}$
- (Garner 2019) General construction of a WDL $\mathrm{T}\mathbf{P} \Rightarrow \mathbf{P}\mathrm{T}$ on $\mathbf{Set}$:
  - $\mathbf{Eff}(\mathbf{P}) \cong \mathbf{Rel} \twoheadleftarrow \mathbf{Span}(\mathbf{Set})$

$\mathbf{Eff}(\mathbf{P})$	$f\colon X \to \mathbf{P}Y$
$\mathbf{Rel}$	$R \subseteq X \times Y$
$\mathbf{Span}(\mathbf{Set})$	$X \xleftarrow[x \leftarrow\!\shortmid (x,y)]{} R \xrightarrow[(x,y) \mapsto y]{} Y$

  - Extend $\mathtt{map}\mathrm{T}$ to $\mathbf{Eff}(\mathbf{P})$ by setting

$$\mathtt{map}\underline{\mathrm{T}}\left(X \xleftarrow{g} Z \xrightarrow{h} Y\right) = \mathrm{T}X \xleftarrow{\mathtt{map}\mathrm{T}(g)} \mathrm{T}Z \xrightarrow{\mathtt{map}\mathrm{T}(h)} \mathrm{T}Y$$

  - It's the only way to have $\mathtt{map}\mathrm{T}$ preserve the order between relations! We say the WDL is *monotone*.

Monotone weak distributive laws over **weakly lifted powerset monads in categories of algebras**

Algebras for a monad

- If T is a monad, a T-algebra is a pair $(A, a\colon \mathrm{T}A \to A)$ satisfying some axioms.
- T-algebras form a category, written $\mathbf{Alg}(\mathrm{T})$.
- Examples:
  - $\mathbf{Alg}(\mathbf{P}) \cong \mathbf{JSL}$: $\mathbf{P}$-algebras = complete join-semilattices
  - $\mathbf{Alg}(\mathbf{D}) \cong \mathbf{Conv}$: $\mathbf{D}$-algebras = barycentric algebras / convex spaces
  - $\mathbf{Alg}(\beta) \cong \mathbf{KHaus}$ (Manes 1969): β-algebras[i] = compact Hausdorff spaces

[i] β is the ultrafilter monad

Weak liftings

A WDL $\mathrm{TS} \Rightarrow \mathrm{ST}$ is equivalently given by

- $\mathtt{swapTS}\colon \mathrm{TS}X \to \mathrm{ST}X$
- a weak extension of $\mathtt{map}\mathrm{T}$ to $\mathbf{Eff}(\mathrm{S})$
- a *weak lifting* of S to $\mathbf{Alg}(\mathrm{T})$, i.e. a monad $\overline{\mathrm{S}}$ on $\mathbf{Alg}(\mathrm{T})$ s.t., up to a retraction, $\overline{\mathrm{S}}(A, a) = (\mathrm{S}A, \ldots)$.

Examples:

- the law $\mathbf{PP} \Rightarrow \mathbf{PP}$ yields, on $\mathbf{Alg}(\mathbf{P}) \cong \mathbf{JSL}$, the monad $\overline{\mathbf{P}}$ of subsets closed under non-empty joins
- the law $\mathbf{DP} \Rightarrow \mathbf{PD}$ yields, on $\mathbf{Alg}(\mathbf{D}) \cong \mathbf{Conv}$, the monad $\overline{\mathbf{P}}$ of convex subsets
- (Garner 2019) there is a monotone WDL $\beta\mathbf{P} \Rightarrow \mathbf{P}\beta$: it yields, on $\mathbf{Alg}(\beta) \cong \mathbf{KHaus}$, the *Vietoris monad* $\mathbf{V}$ of closed subsets

Monotone WDLs over $\overline{\mathbf{P}}$ in $\mathbf{Alg}(\mathrm{T})$

Characterization of monotone WDLs in $\mathbf{Alg}(\mathrm{T})$

There are powerset-like monads in $\mathbf{Alg}(\mathbf{P}) \cong \mathbf{JSL}$, $\mathbf{Alg}(\mathbf{D}) \cong \mathbf{Conv}$, $\mathbf{Alg}(\beta) \cong \mathbf{KHaus}$ (and more). Do they weakly distribute over themselves, just like $\mathbf{P}$ does in $\mathbf{Set}$?

- Given a monotone WDL $\mathrm{T}\mathbf{P} \Rightarrow \mathbf{P}\mathrm{T}$ in $\mathbf{Set}$, can the setting for <u>monotone</u> WDLs over $\mathbf{P}$ be lifted to $\mathbf{Alg}(\mathrm{T})$?
  - $\mathbf{Alg}(\mathrm{T})$ is a *regular category*: it has a category of relations $\mathbf{Rel}(\mathbf{Alg}(\mathrm{T})) \twoheadleftarrow \mathbf{Span}(\mathbf{Alg}(\mathrm{T}))$.
  - $\mathbf{Eff}(\overline{\mathbf{P}}) \hookrightarrow \mathbf{Rel}(\mathbf{Alg}(\mathrm{T}))$. $\overline{\mathbf{P}}$-effectful morphism correspond to spans $X \xleftarrow{f} \cdot \xrightarrow{g} Y$ where f is *decomposable*.
  - If $\mathtt{map}\mathrm{S}$ extends to $\mathbf{Rel}$, making $\underline{\mathrm{S}}$ into a monad/semi-monad, the same is true of $\mathtt{map}\overline{\mathrm{S}}$ and $\mathbf{Rel}(\mathbf{Alg}(\mathrm{T}))$.
  - $\overline{\mathrm{S}}$ has a monotone WDL over $\overline{\mathbf{P}}$ iff $\mathtt{map}\overline{\mathrm{S}}$ preserves decomposable morphisms.

Case study: compact Hausdorff spaces

From the monotone WDL $\beta\mathbf{P} \Rightarrow \mathbf{P}\beta$, we automatically retrieve:

- $\mathbf{Alg}(\beta) \cong \mathbf{KHaus}$ has a category of relations $\mathbf{Rel}(\mathbf{KHaus})$ (closed relations)
- $\mathbf{Eff}(\mathbf{V}) \hookrightarrow \mathbf{Rel}(\mathbf{KHaus})$
- decomposable continuous functions are the open ones
- $\mathtt{map}\mathbf{V}$ preserves open functions (easy): there is a monotone WDL $\mathbf{V}\mathbf{V} \Rightarrow \mathbf{V}\mathbf{V}$

Using similar techniques for the *Radon monad* $\mathbf{R}$:

- $\mathtt{map}\mathbf{R}$ does not preserve open functions: there is no monotone WDL $\mathbf{R}\mathbf{V} \Rightarrow \mathbf{V}\mathbf{R}$ (new!)
- $\mathtt{map}\mathbf{R}$ preserves open surjections: there is a monotone WDL $\mathbf{R}\mathbf{V}_* \Rightarrow \mathbf{V}_*\mathbf{R}$ (new!)

Conclusion: no-go theorems for monotone WDLs

$\mathbf{PP} \Rightarrow \mathbf{PP}$ and $\mathbf{VV} \Rightarrow \mathbf{VV}$ look the same... but monotone WDLs over $\overline{\mathbf{P}}$ are quite rare otherwise:

	KHaus		**JSL**	**Conv**	**Mon**				**CMon**		
	$\mathbf{V}$	$\mathbf{R}$	$\overline{\mathbf{P}}$	$\overline{\mathbf{P}}$	$\overline{\mathbf{M}}$	$\overline{\mathbf{D}}$	$\overline{\mathbf{P}}$	$\overline{\mathbf{M}_S}$	$\overline{\mathbf{M}}$	$\overline{\mathbf{D}}$	$\overline{\mathbf{P}}$
$\overline{\mathbf{P}}$	✓	✗	✗	✗	✗	✗	✗	✗	✗	✗	✗
$\overline{\mathbf{P}_*}$	✓	✓	✗	✗	✗	✗	✗	✗	✗	✗	✗

- What's next?
  - extending this framework: Pos-regular categories, other monads of relations
  - no-go theorems for (all) WDLs
  - seeing this in the setting of monoidal topology

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