MULTIDIMENSIONAL QUANTUM WALKS, RECURSION, AND QUANTUM DIVIDE & CONQUER

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POWER OF QUANTUM COMPUTERS

How powerful are quantum computers?

What problems allow quantum speedups?

Polynomial? Exponential?

TOOLS FOR CLASSICAL AND QUANTUM ALGORITHMS

CLASSICAL

- Greedy algorithms
- Dynamic programming
- Linear programming
- Heuristic algorithms
- Divide and conquer

QUANTUM

- Quantum amplitude amplification
- Quantum Fourier transform
- Quantum phase estimation
- Quantum walks
- Quantum divide and conquer

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THIS WORK

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quantum speedup for a class of problems

OUTLINE

CLASSICAL MOTIVATION

ATTEMPT TO USE QUANTUM PRIMITIVES

MAIN RESULT

APPLICATION TO DSTCON

OUTLINE

Classical motivation

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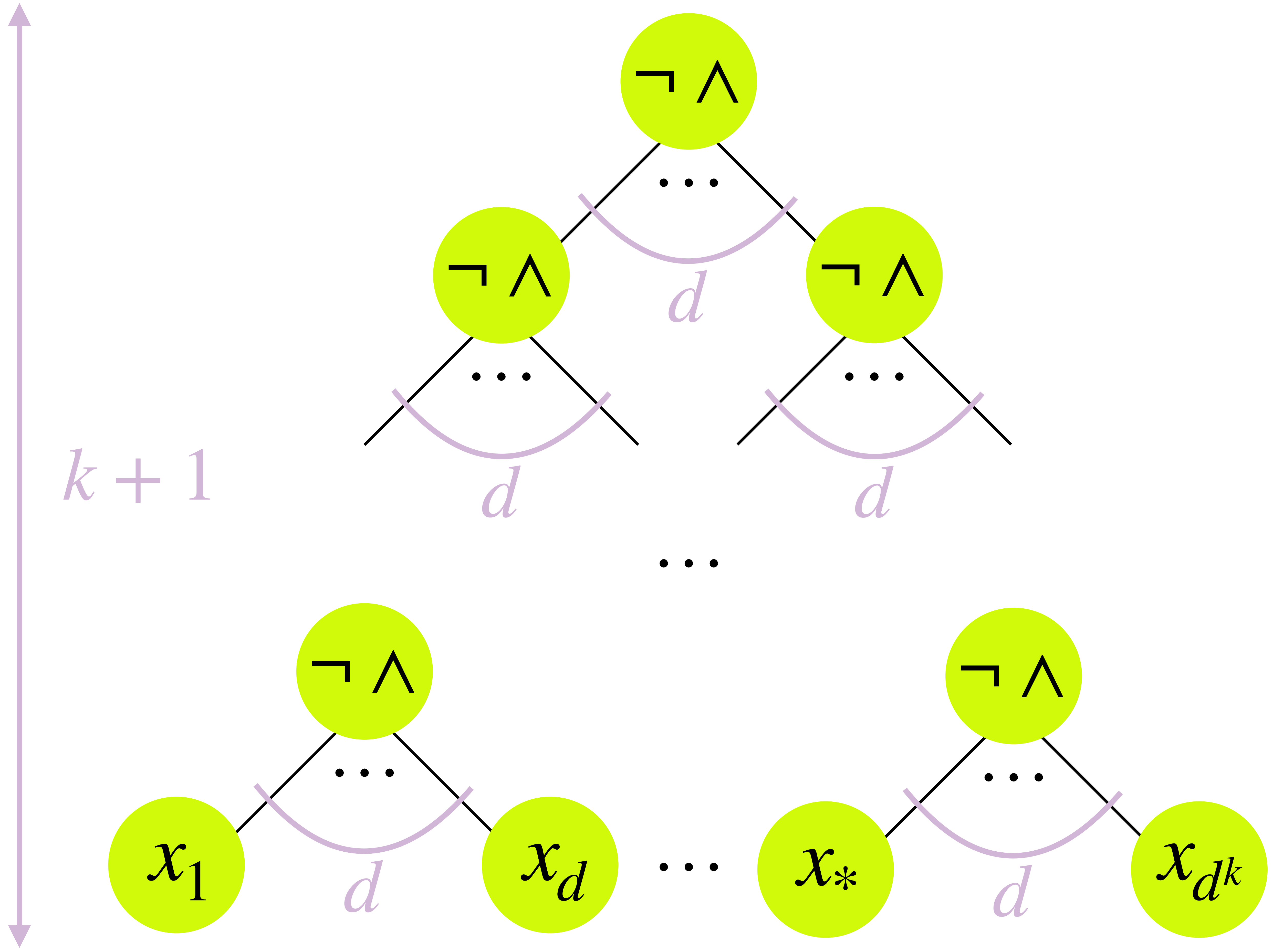
MAIN RESULT

APPLICATION TO DSTCON

DIVIDE AND CONQUER APPROACH



NAND tree

DIVIDE AND CONQUER APPROACH

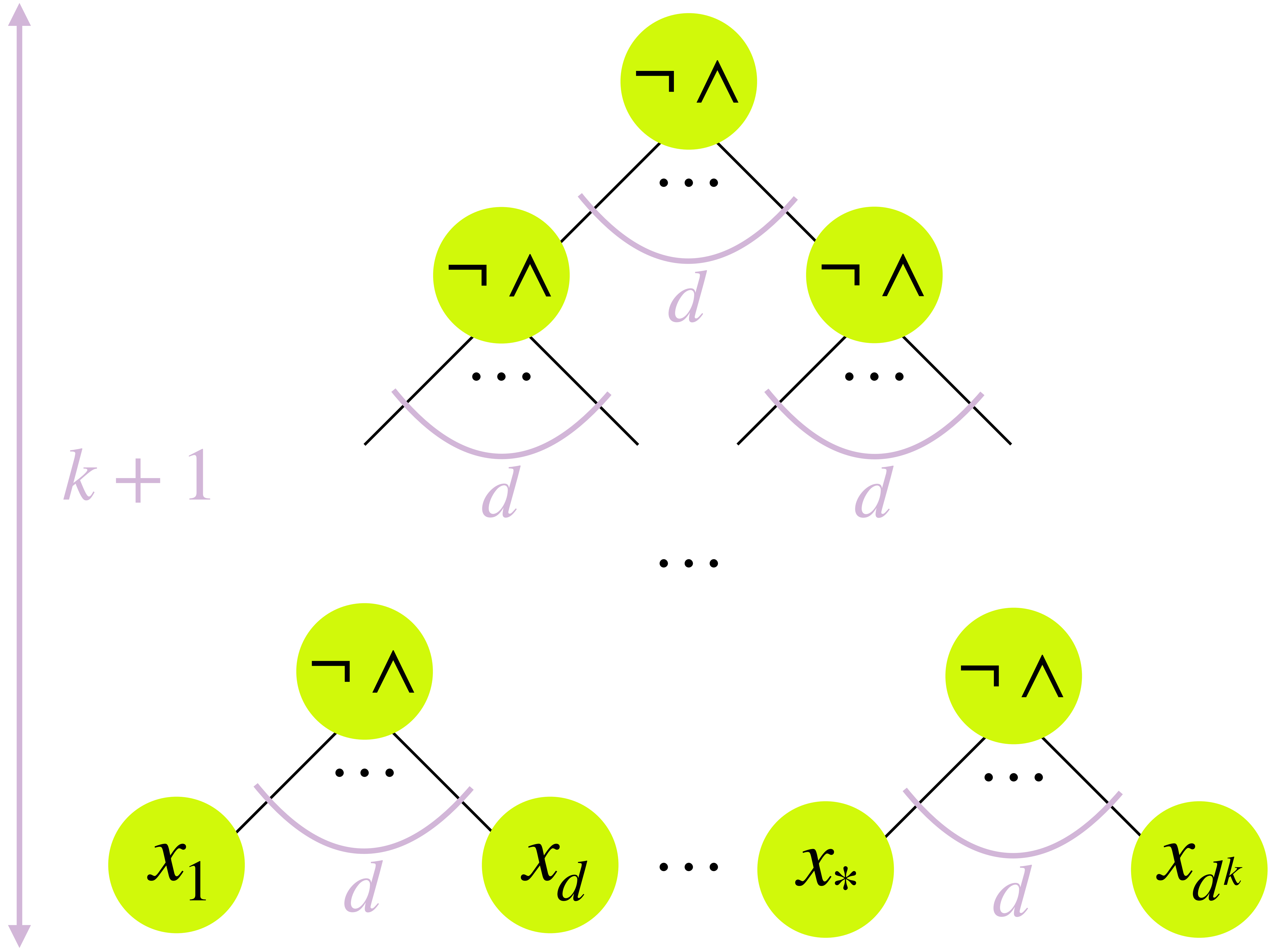
RECURSIVE RELATION

subproblems

$$T_{cl}(f_{k,d}) = dT_{cl}(f_{k-1,d}) + T_{aux}$$

subproblem complexity

combining



NAND tree

OUTLINE

CLASSICAL MOTIVATION

Attempt to use quantum primitives

MAIN RESULT

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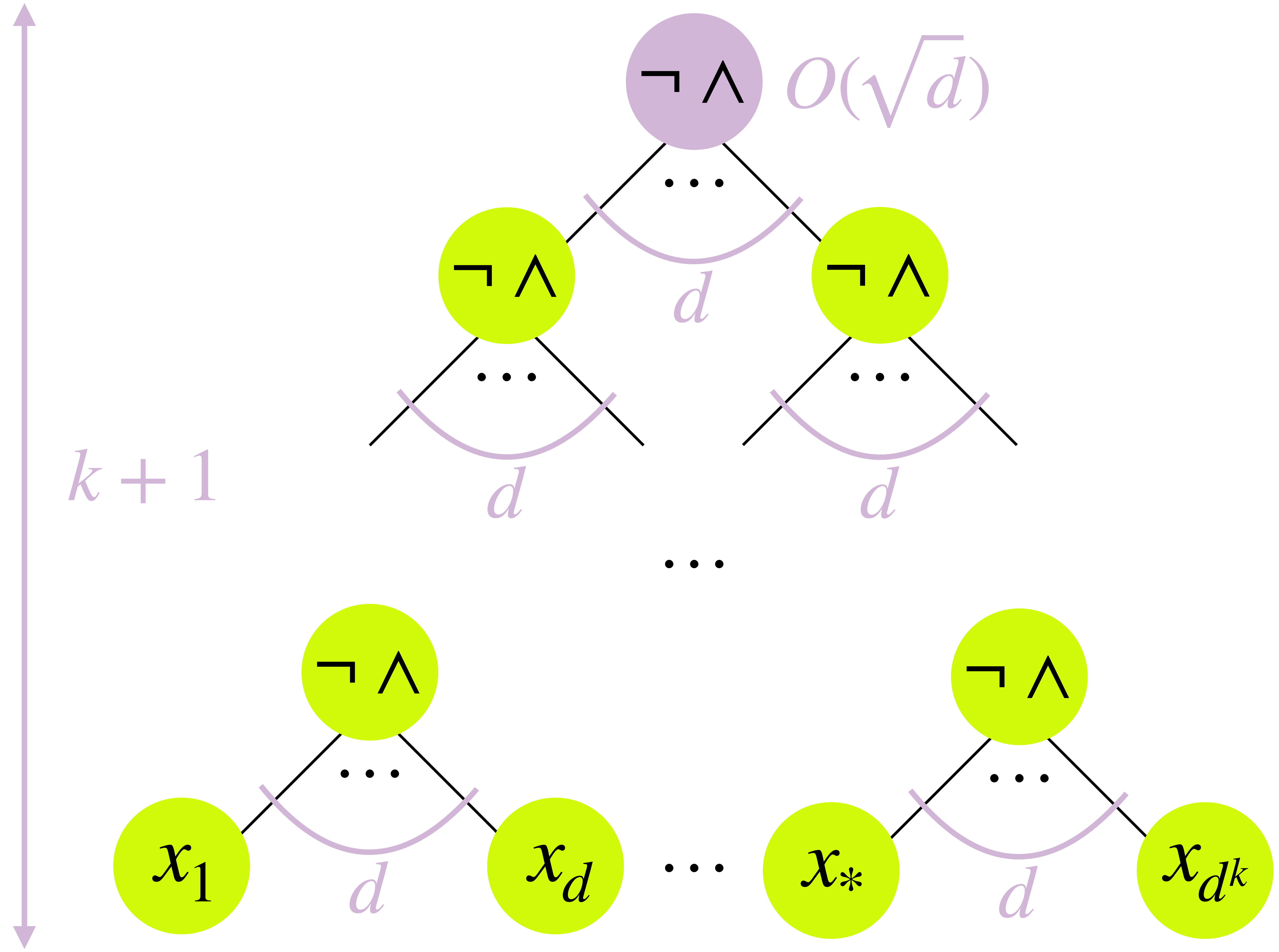
QUANTUM DIVIDE AND CONQUER ATTEMPT

- Grover's algorithm computes $\neg \wedge$ in $O(\sqrt{d})$ queries



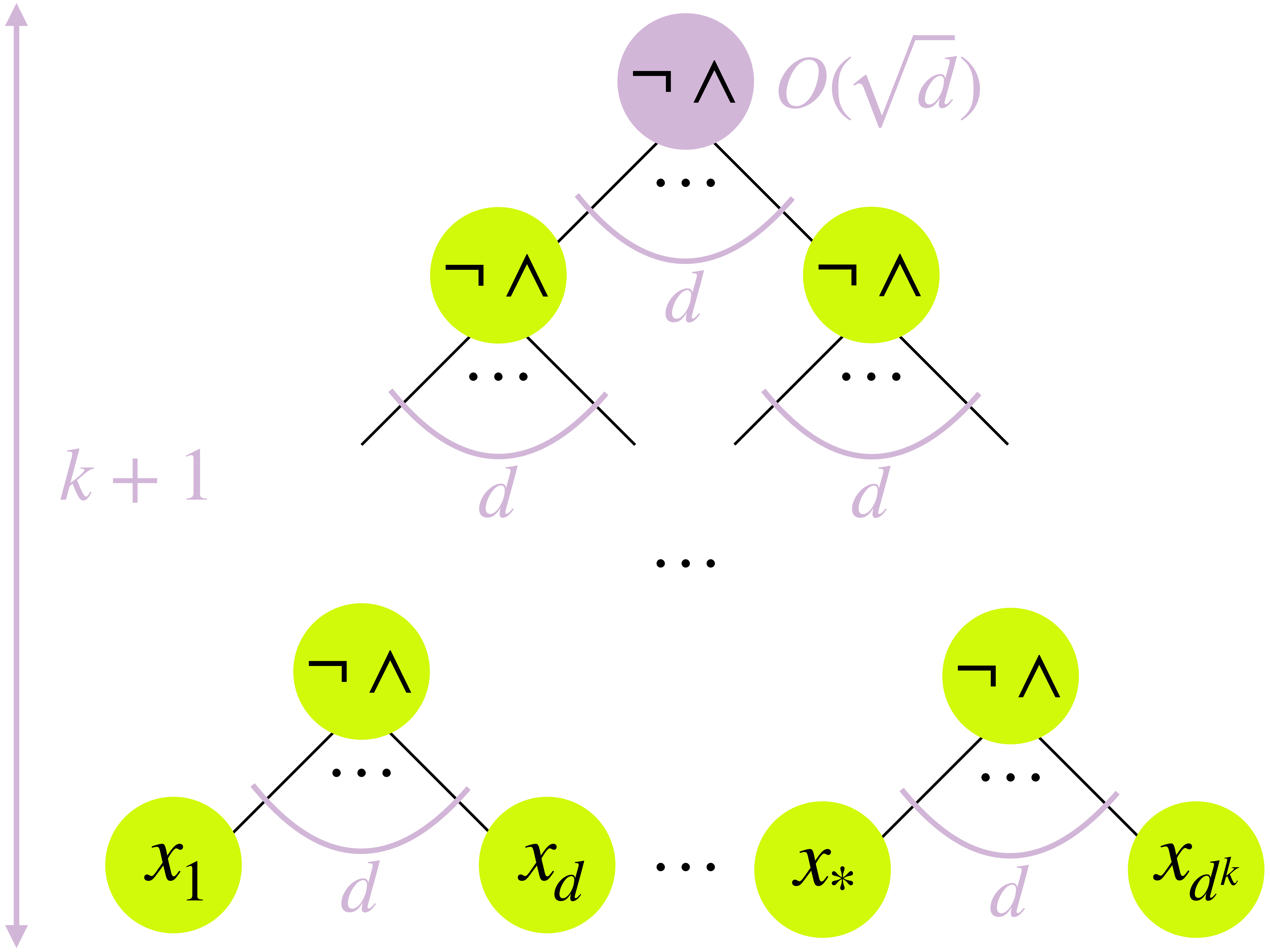
NAND tree

QUANTUM DIVIDE AND CONQUER ATTEMPT

- Grover's algorithm computes $\neg\wedge$ in $O(\sqrt{d})$ queries

if Grover's alg is perfect

- $T_q(f_{k,d}) = c\sqrt{d}T_q(f_{k-1,d})$



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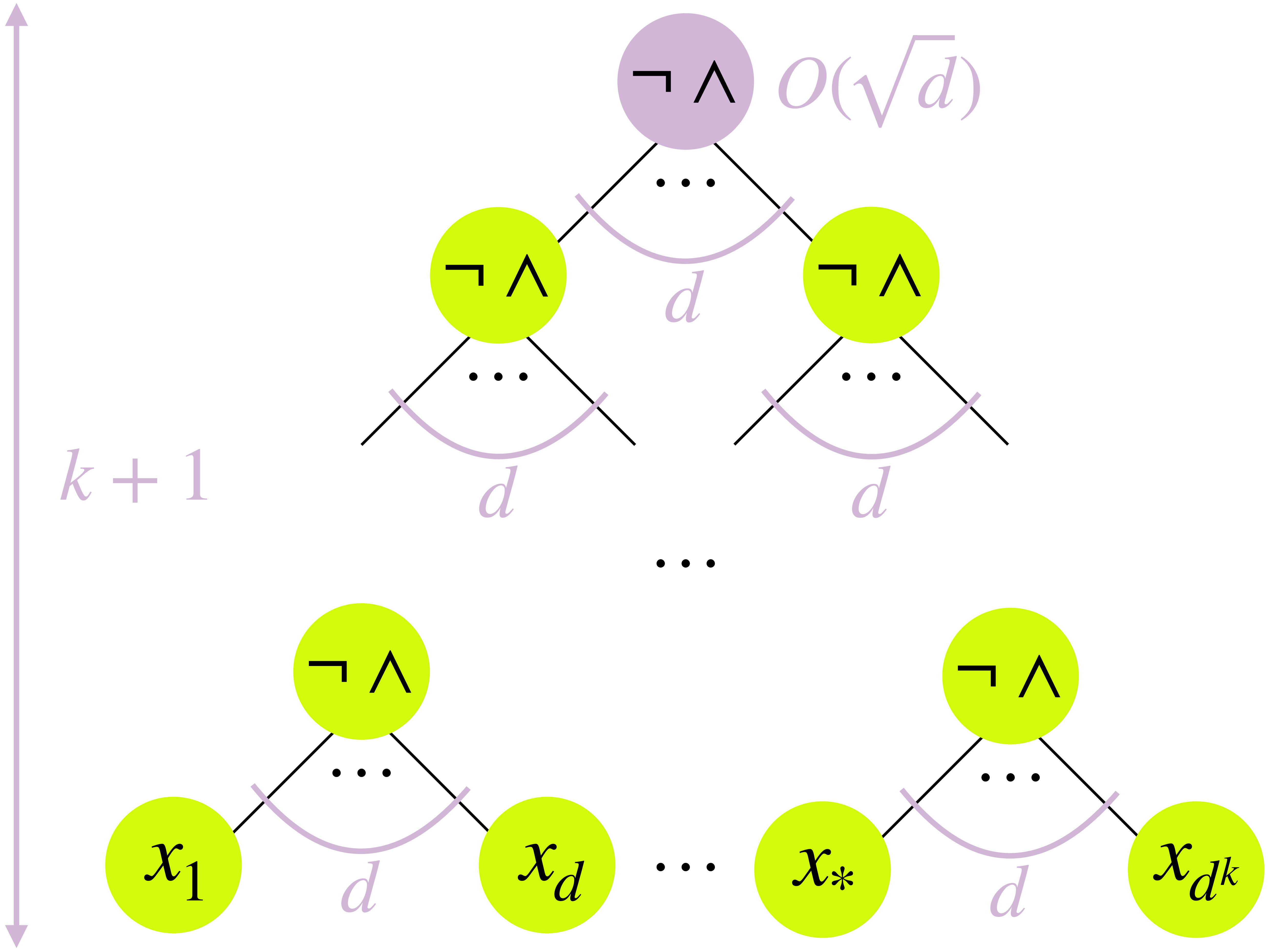
- $T_q(f_{k,d}) = c\sqrt{d}\,T_q(f_{k-1,d})$

$$= c^k(\sqrt{d})^k$$

kills the speedup



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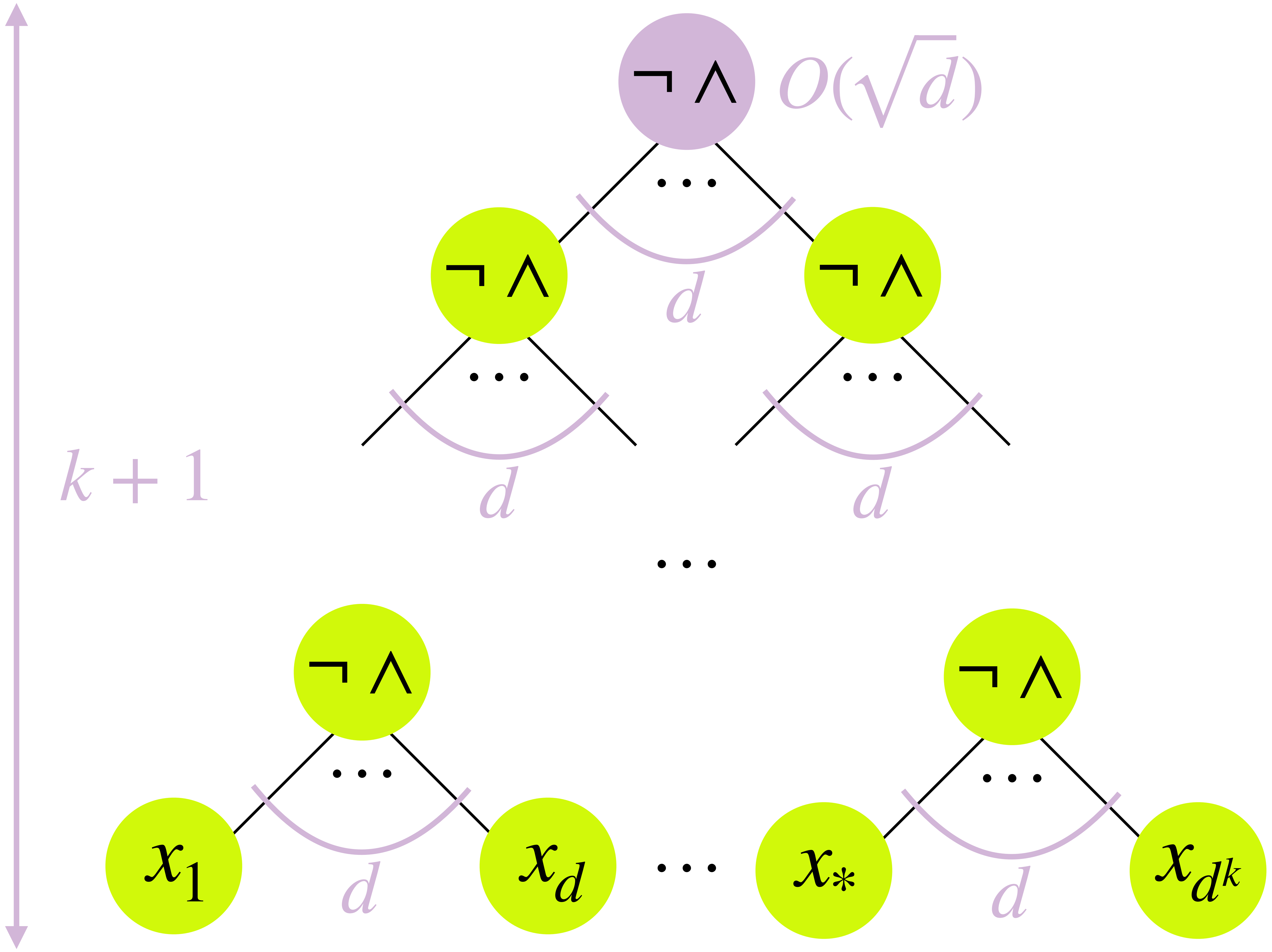
- But we know $O(\sqrt{d^k})$ [Rei11]



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Main result

APPLICATION TO DSTCON

QUANTUM DIVIDE AND CONQUER

THEOREM. Let $f_{l,n} = \phi(\underbrace{f_{\frac{l}{b},n}, \ldots, f_{\frac{l}{b},n}}_{a}) \vee f_{aux,l,n}$

Then the quantum time complexity of $f_{l,n}$ is $\tilde{O}(T(l,n))$, where

$$T(l,n) \leq \sqrt{a}T(l/b,n) + T^{q}_{aux}$$

and space complexity $O(S^{q}_{aux}(l,n) + \log T(l,n))$.

vs classical $T(l,n) \leq aT(l/b,n) + T^{cl}_{aux}$

QUANTUM DIVIDE AND CONQUER

$f_{l,n} : D_{l,n} \to \{0,1\}$

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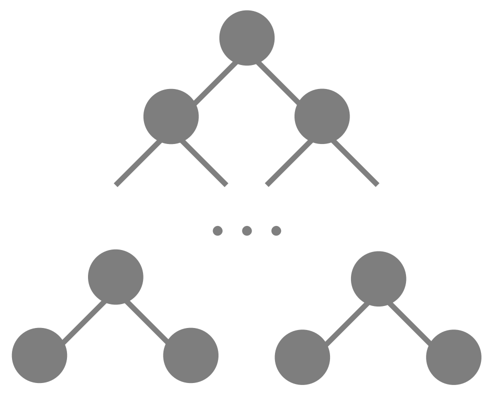

$f_{l,n} : D_{l,n} \to \{0,1\}$ symmetric Boolean formula

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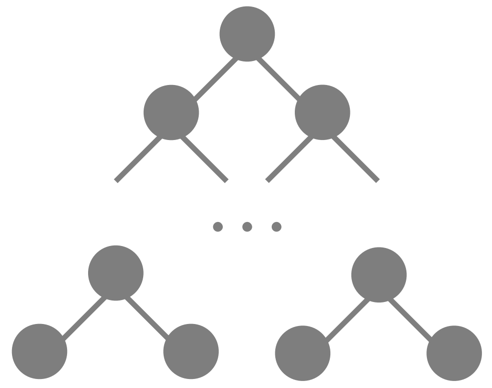

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QUERY VS TIME COMPLEXITY

- Assume oracle access to the input $x \in \{0,1\}^m$

$$|i\rangle|0\rangle \rightarrow \boxed{O_x} \rightarrow |i\rangle|x_i\rangle$$

QUERY VS TIME COMPLEXITY

can encode e.g. a function or an adjacency matrix

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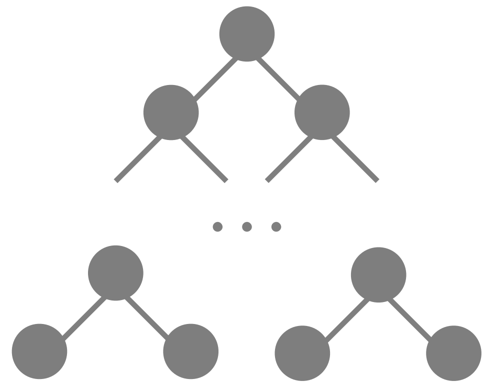
- Often the most expensive operation of the algorithm
- QUERY COMPLEXITY: only count oracle calls
- TIME COMPLEXITY: count all the operations

STRONGER

QUANTUM DIVIDE AND CONQUER



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CONTEXT

QUERY COMPLEXITY

[CKKD+22]

- Arbitrary Boolean formulas & more general functions
- Only query complexity
- Not constructive

TIME COMPLEXITY

[ABB+23]

- AND or OR
- MIN or MAX

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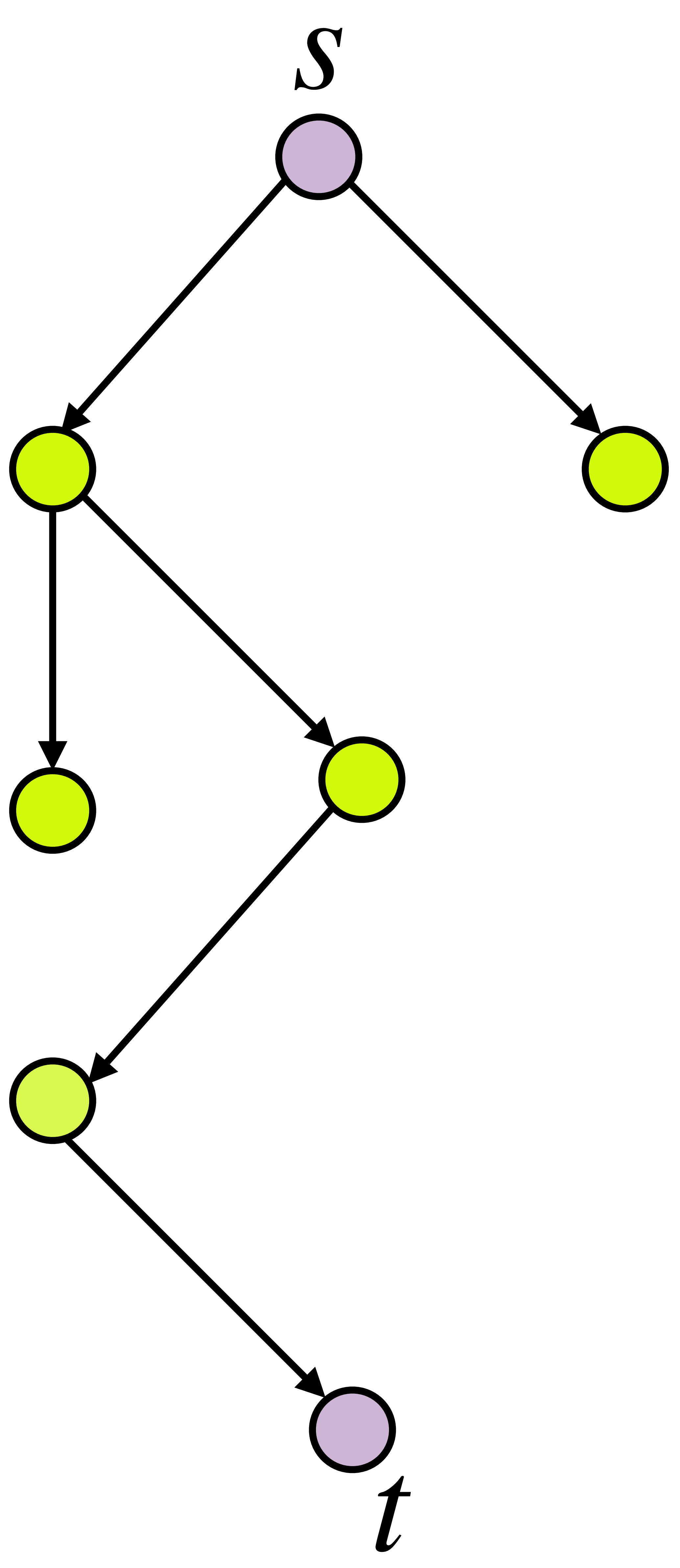
DIRECTED ST-CONNECTIVITY

- Directed graph $G = (V, E)$
- Special vertices $s, t \in V$
- Is there a path from s to t?



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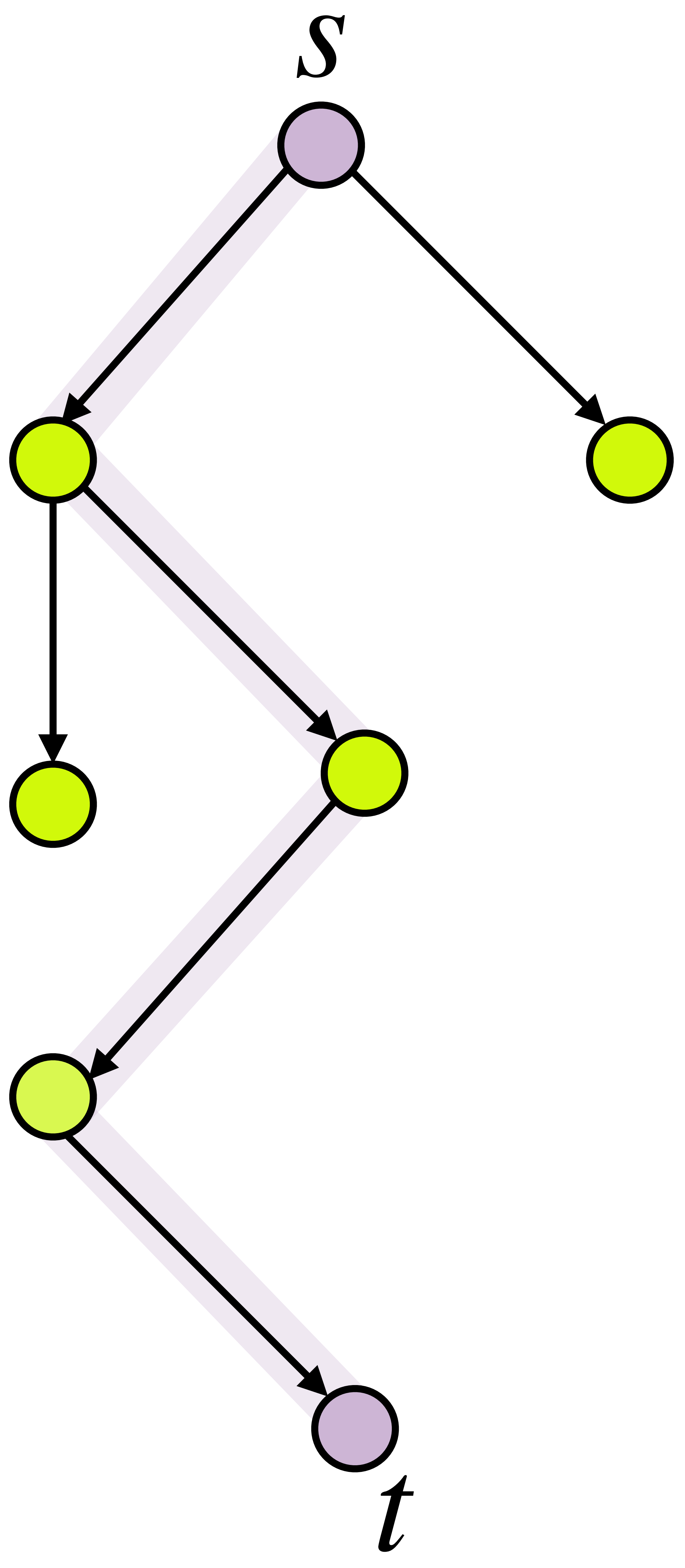

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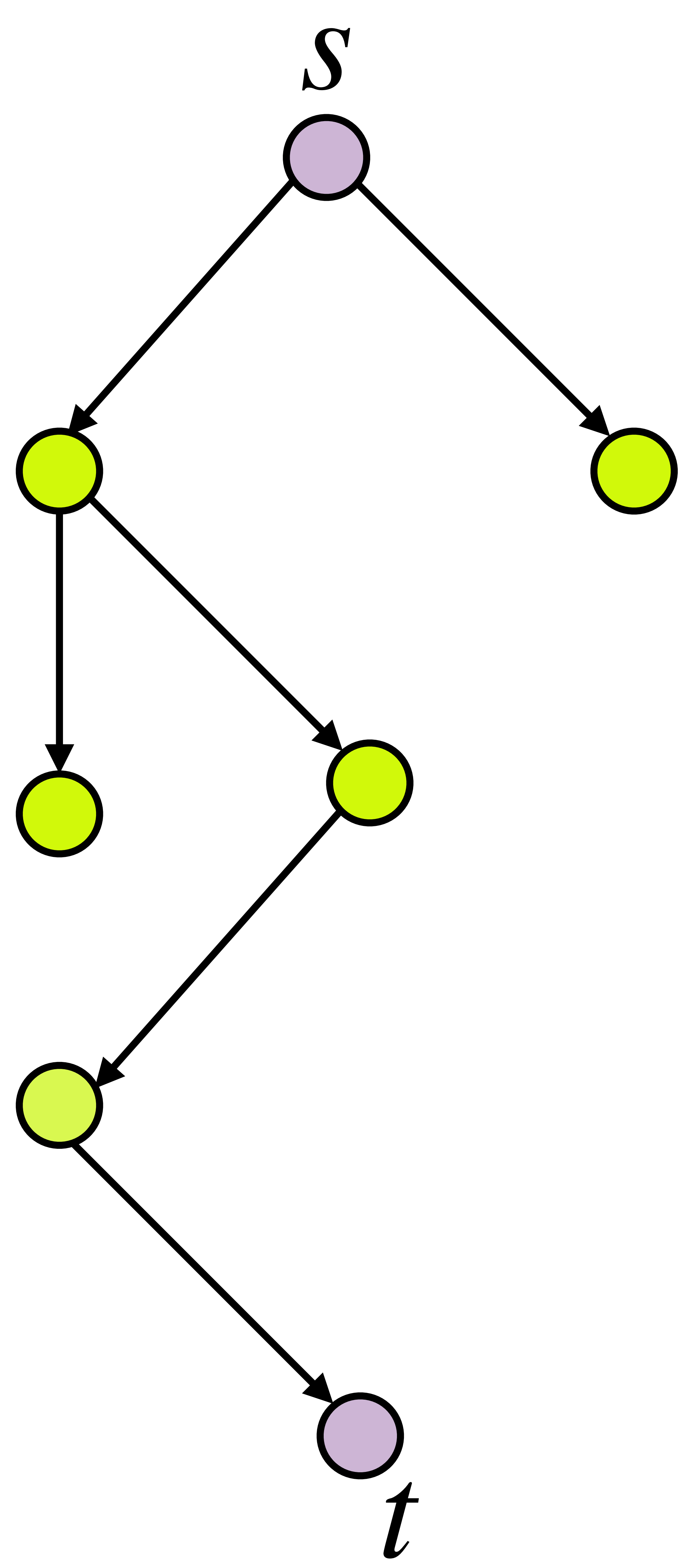

SAVITCH'S ALGORITHM [Sav70]

For $v \in V$:

Check if there is a path $s \to v$ of length $n/2$

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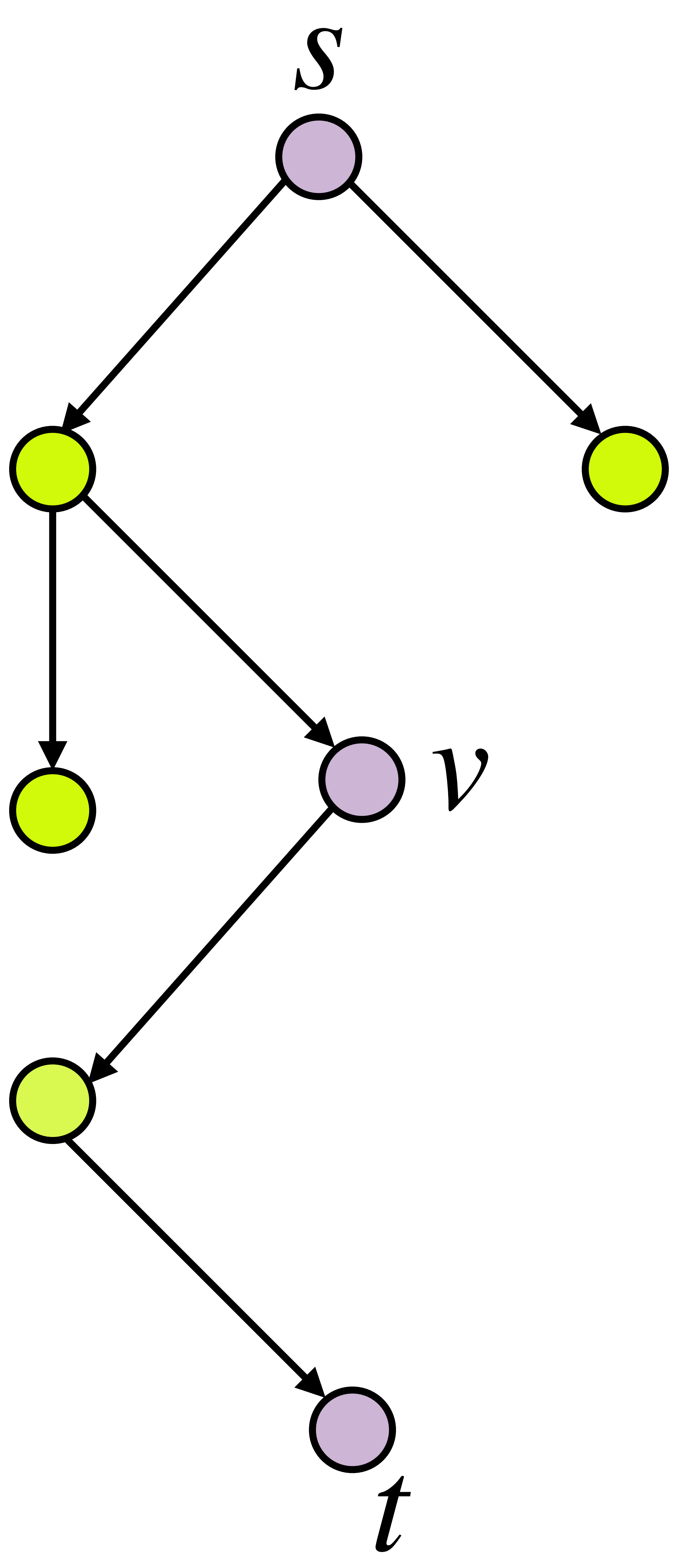

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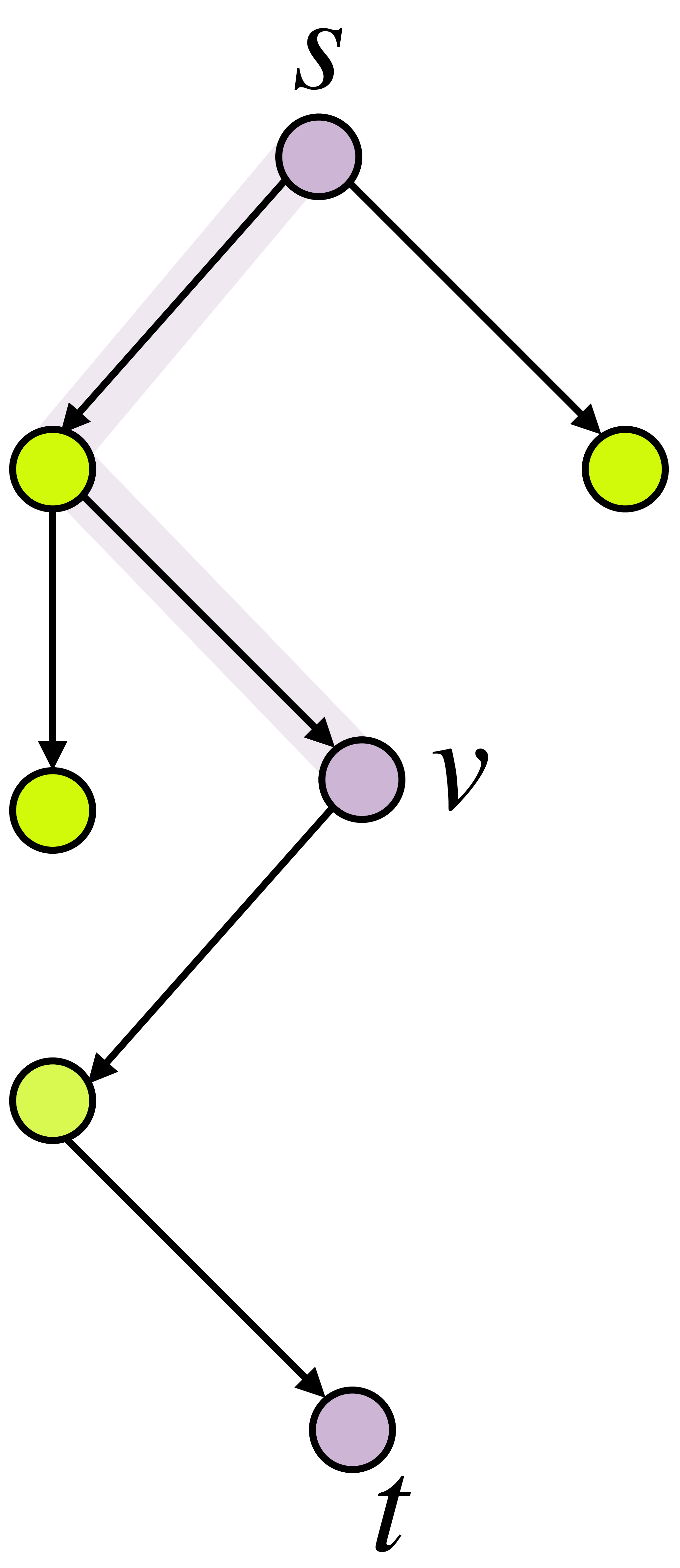

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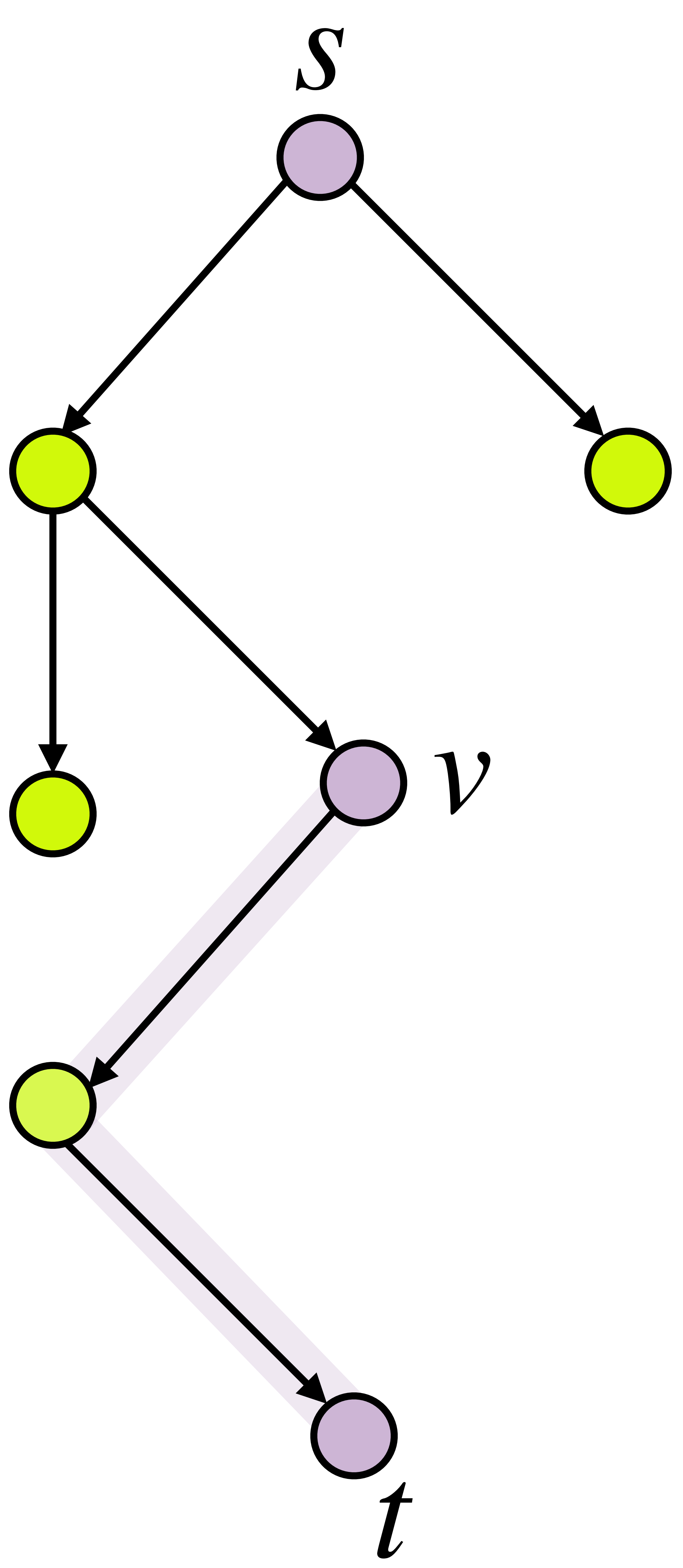

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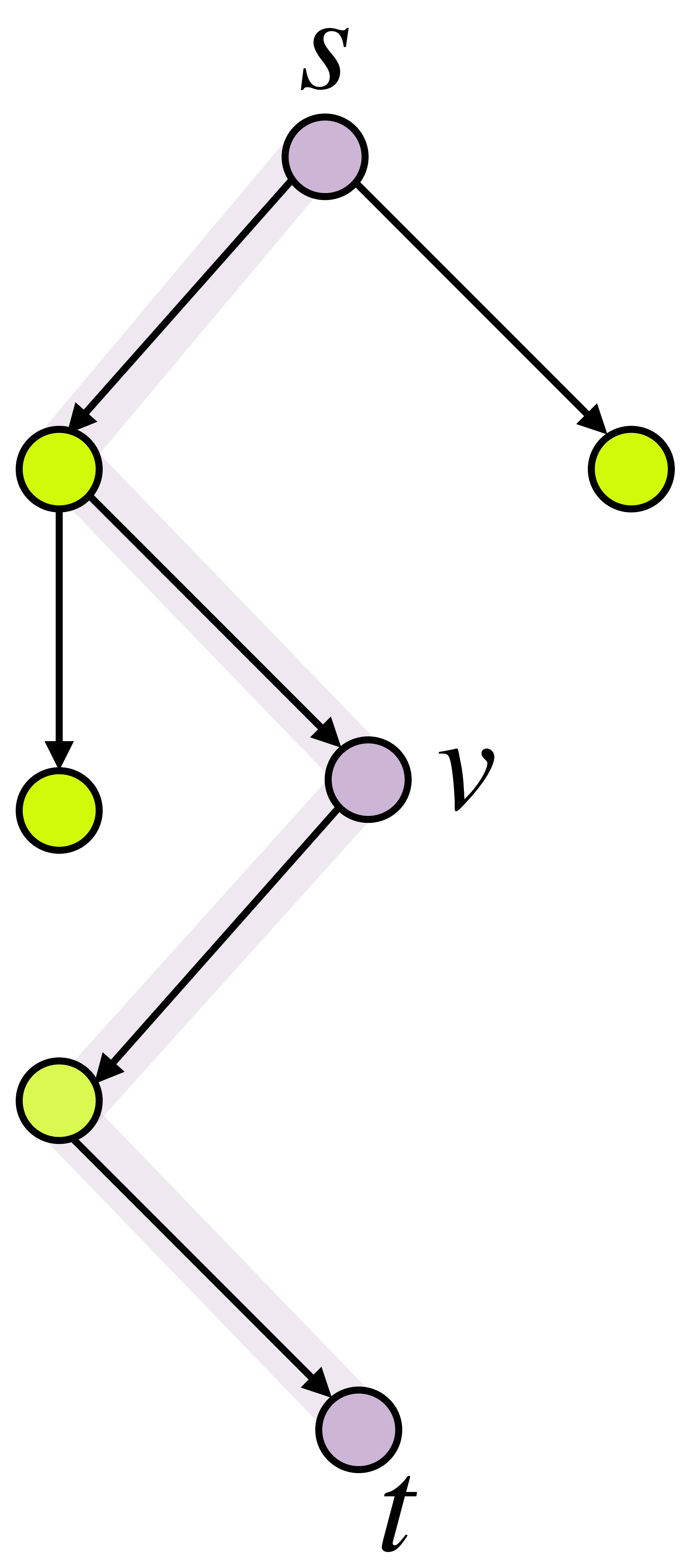

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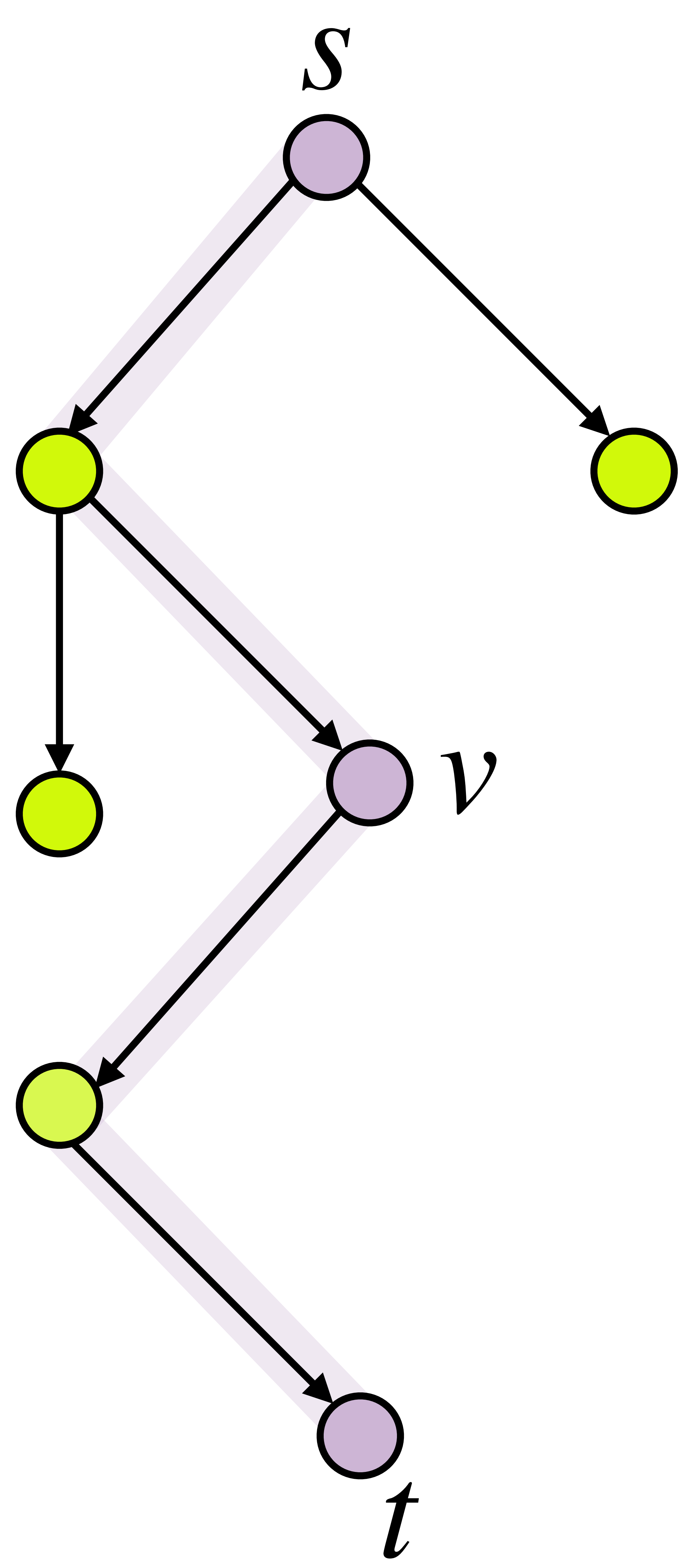

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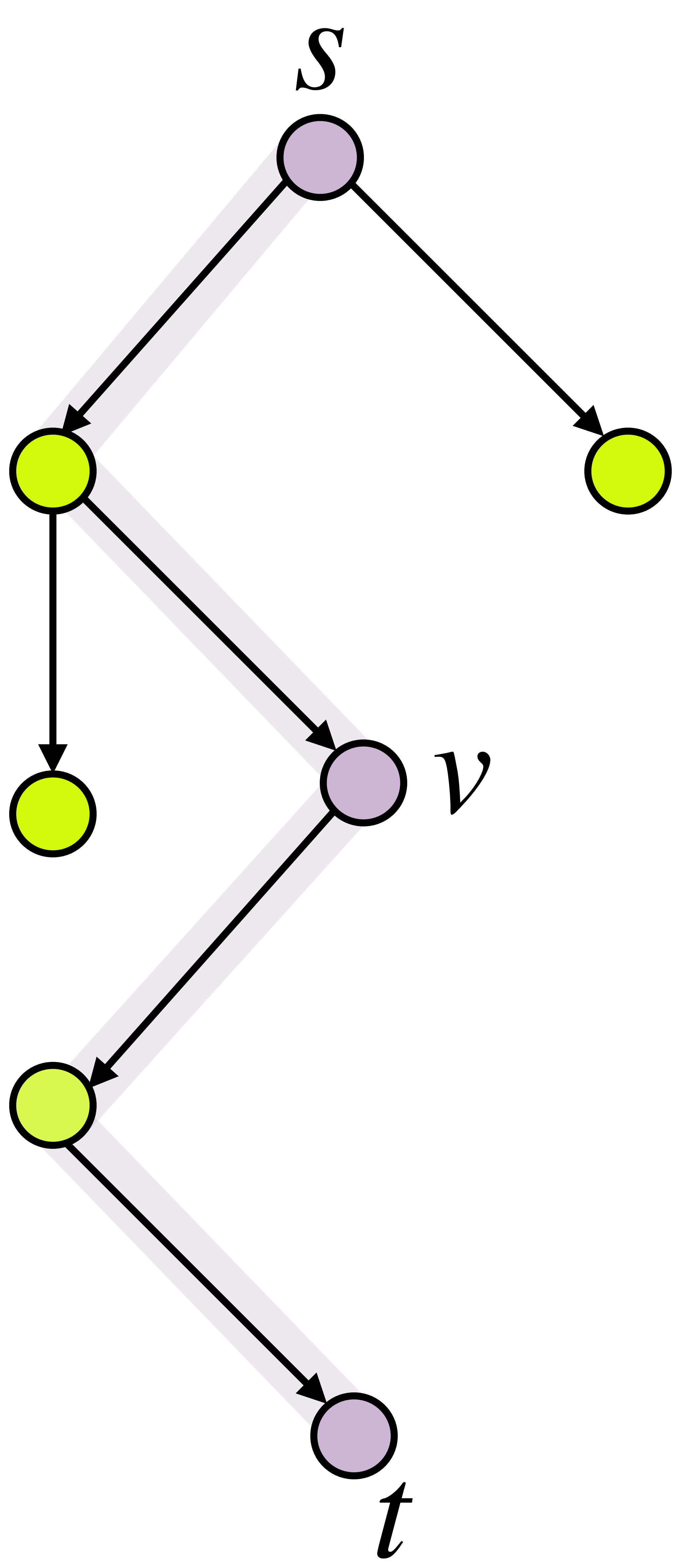
SPACE	$O\left(\log^2 n\right)$

$|V| = n$



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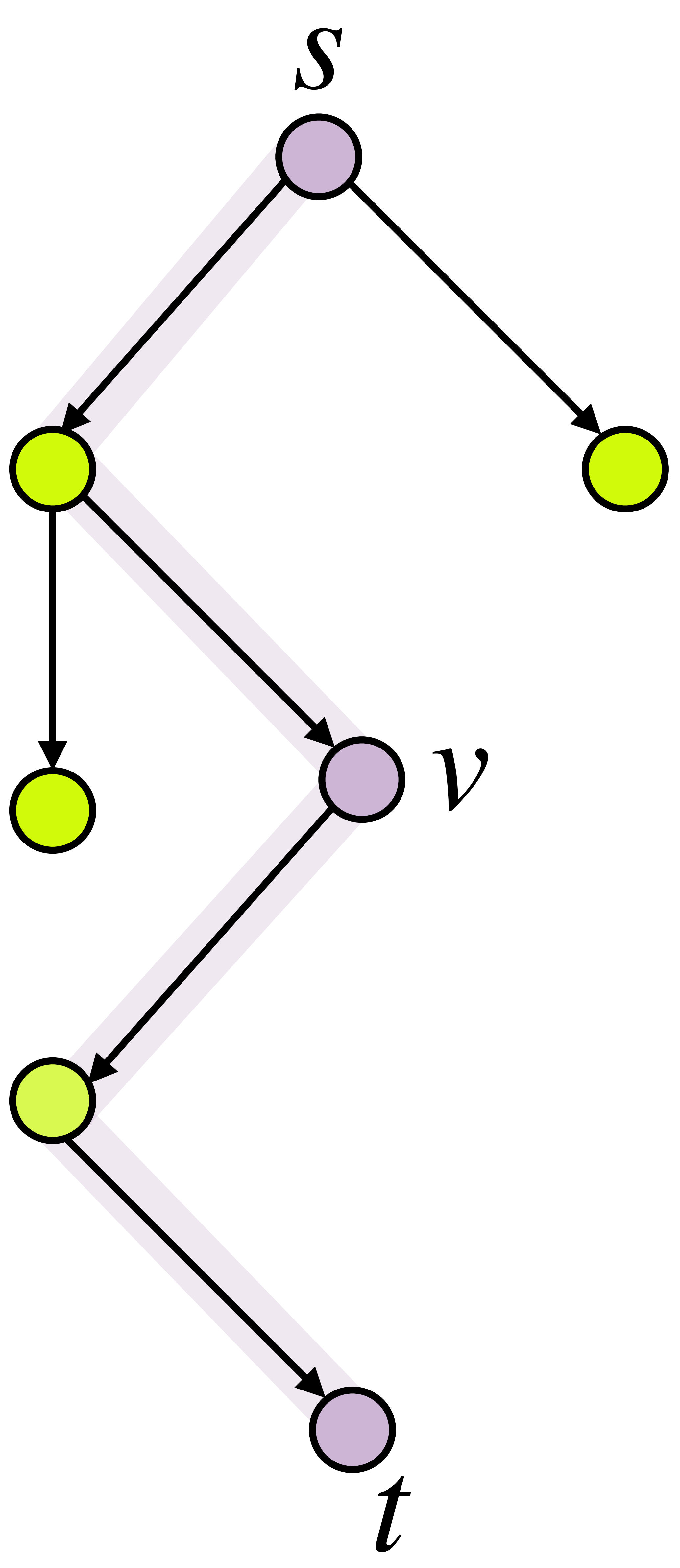
Can write down as a formula

$$Path(s,t,n) = \vee_{v \in V} \left(Path(s,v,n/2) \wedge Path(v,t,n/2)\right)$$



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	CLASSICAL	QUANTUM
TIME	$O\left((2n)^{\log n}\right)$	$\boxed{O\left(\sqrt{2n}^{\log n}\right)}$
SPACE	$O\left(\log^2 n\right)$	$O\left(\log^2 n\right)$

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- Classical divide and conquer + quantum primitives doesn't work

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- We develop a time-efficient quantum divide and conquer framework

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- Classical divide and conquer + quantum primitives doesn’t work
- We develop a time-efficient quantum divide and conquer framework
- Quadratic speedup for DSTCON in the low-space regime

THANK YOU

for your attention