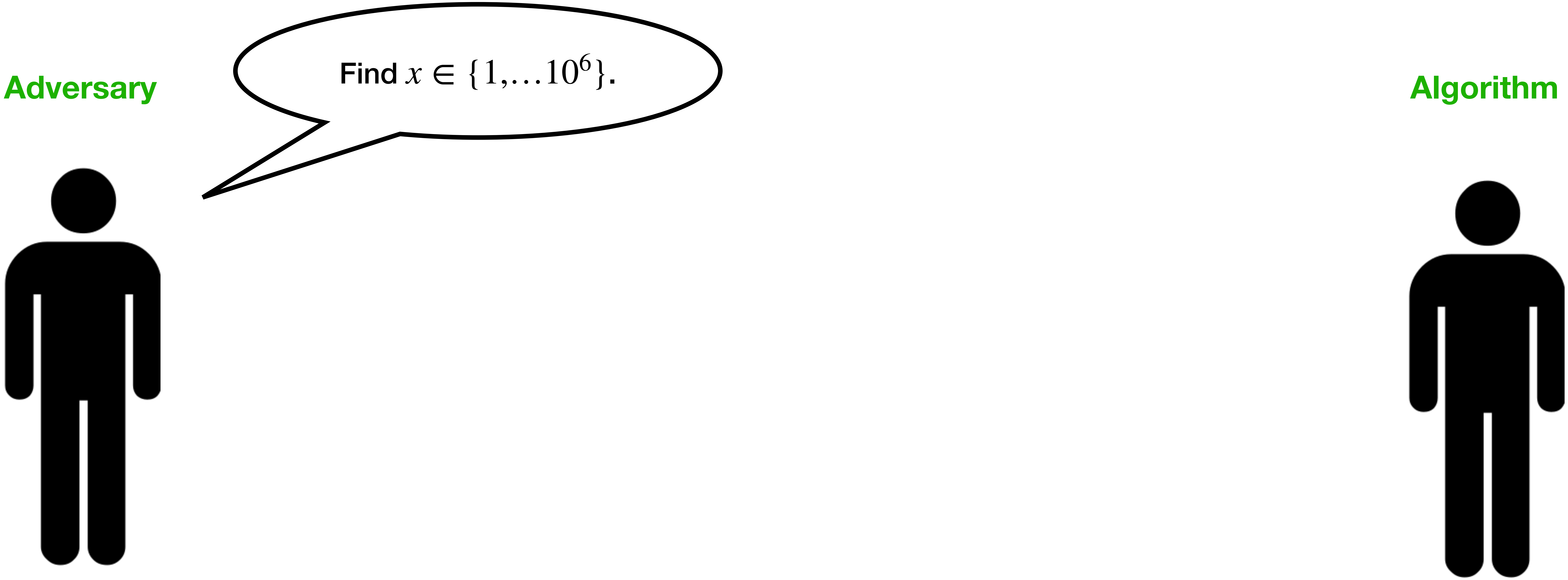
Noisy (Binary) Searching: Simple, Fast and Correct

By Dariusz Dereniowski, Aleksander Łukasiewicz and Przemysław Uznański

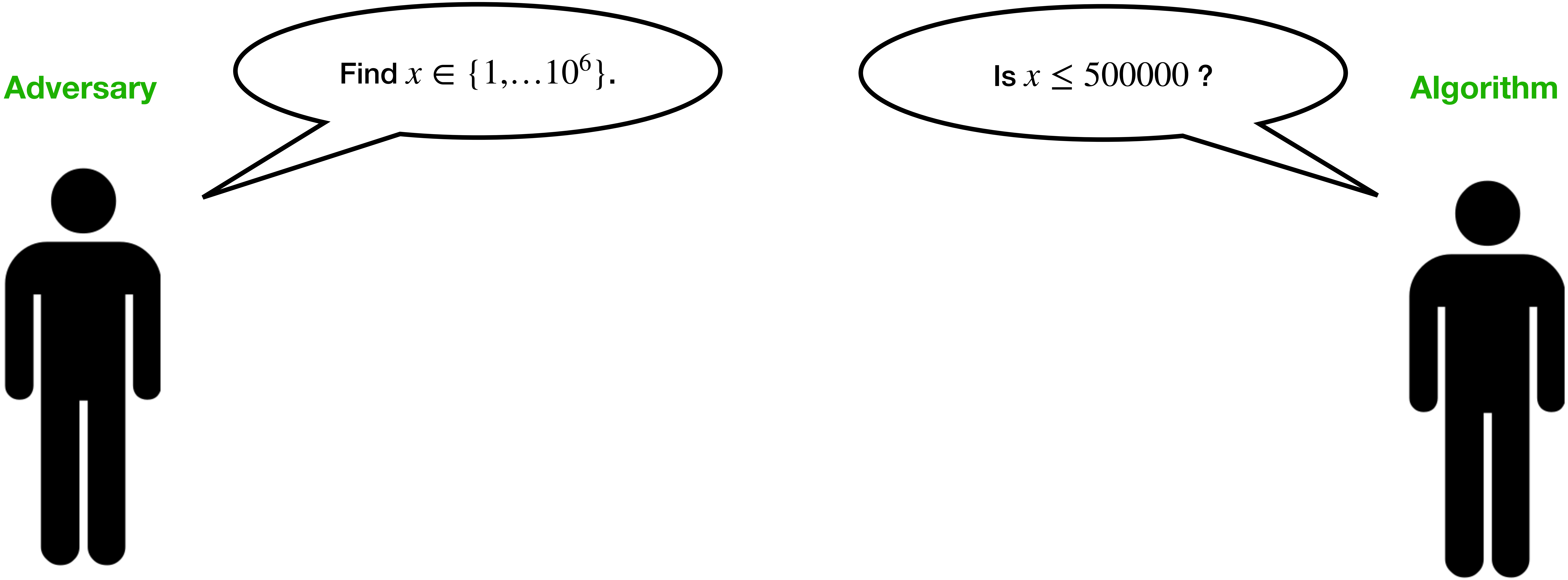
Part I

The problem, the model and all that jazz…

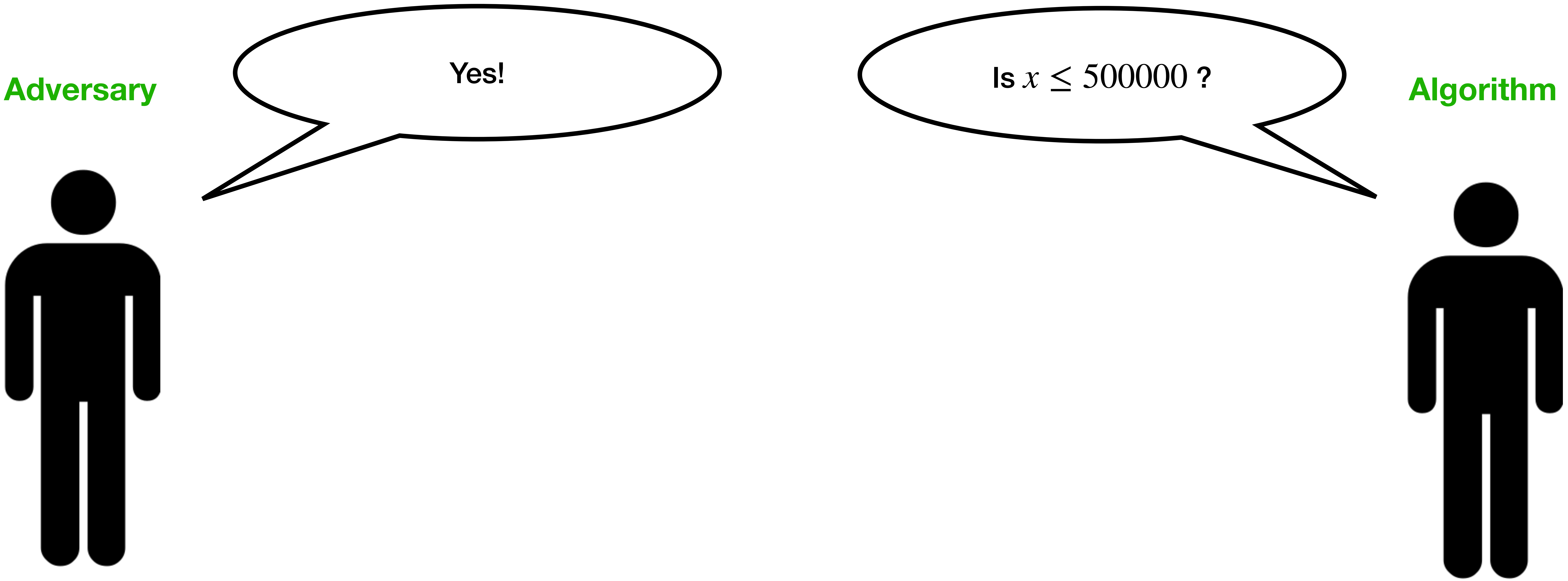
Binary search as a game



Binary search as a game



Binary search as a game



Binary search as a game

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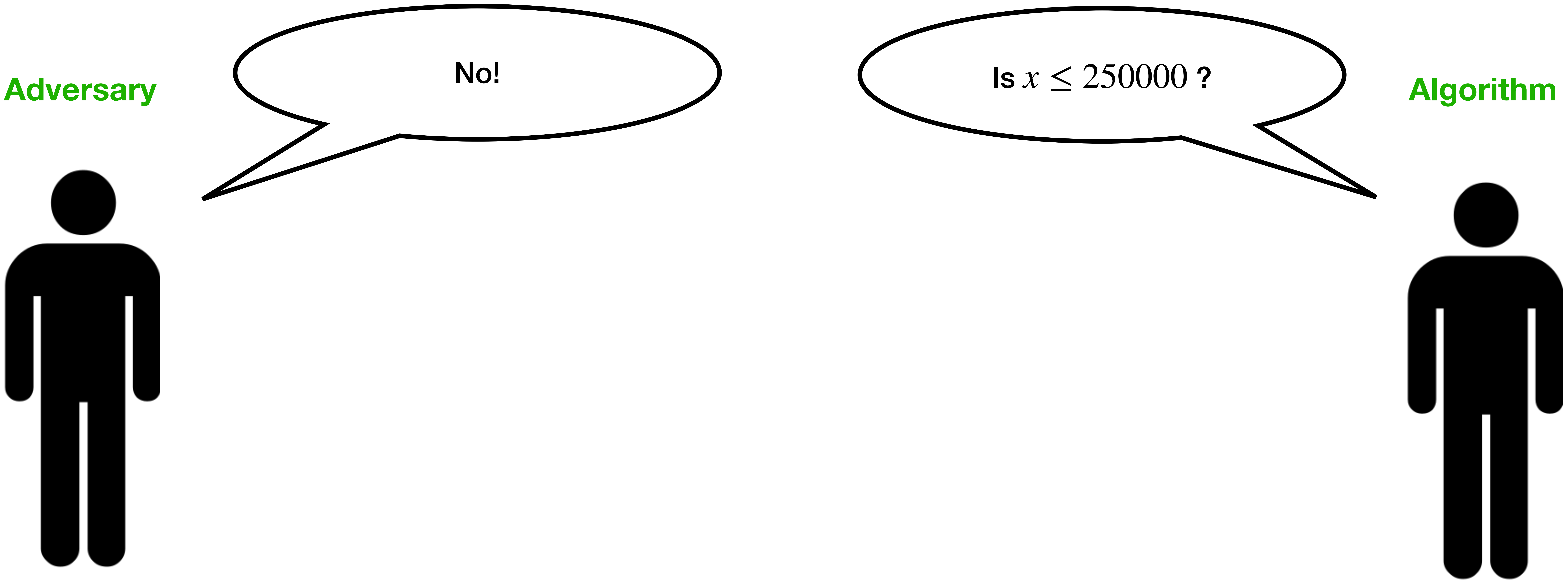
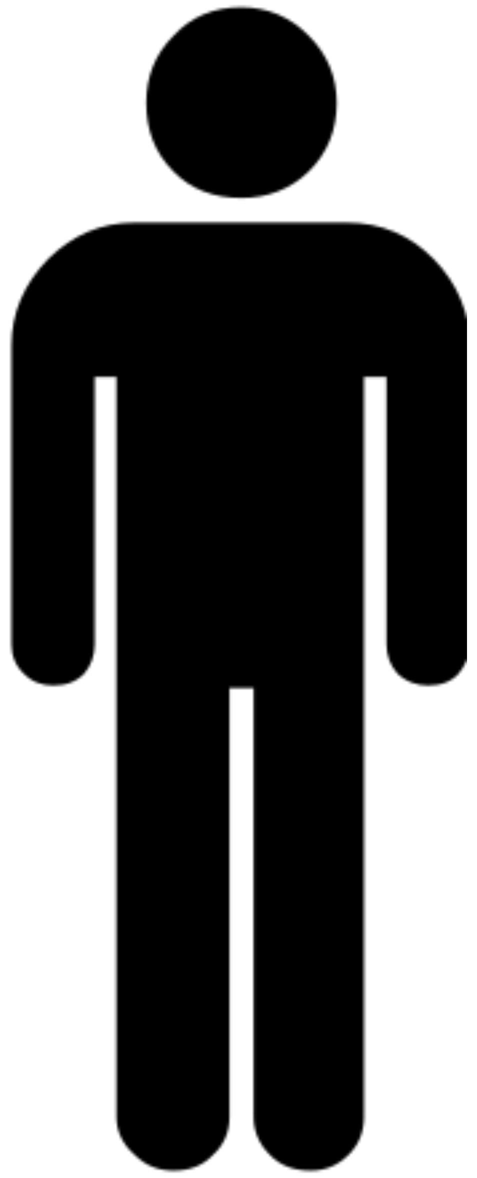


Binary search as a game

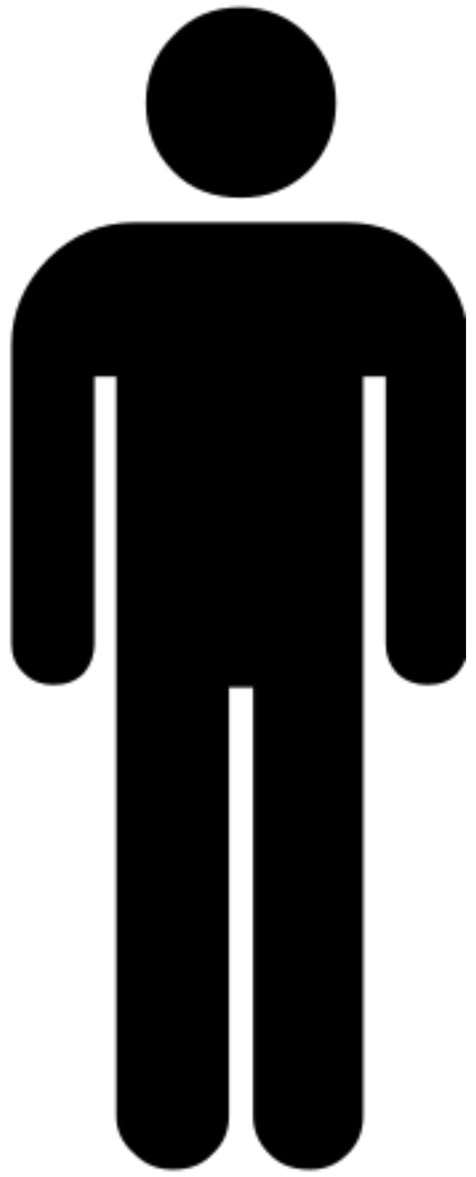
Adversary



No!

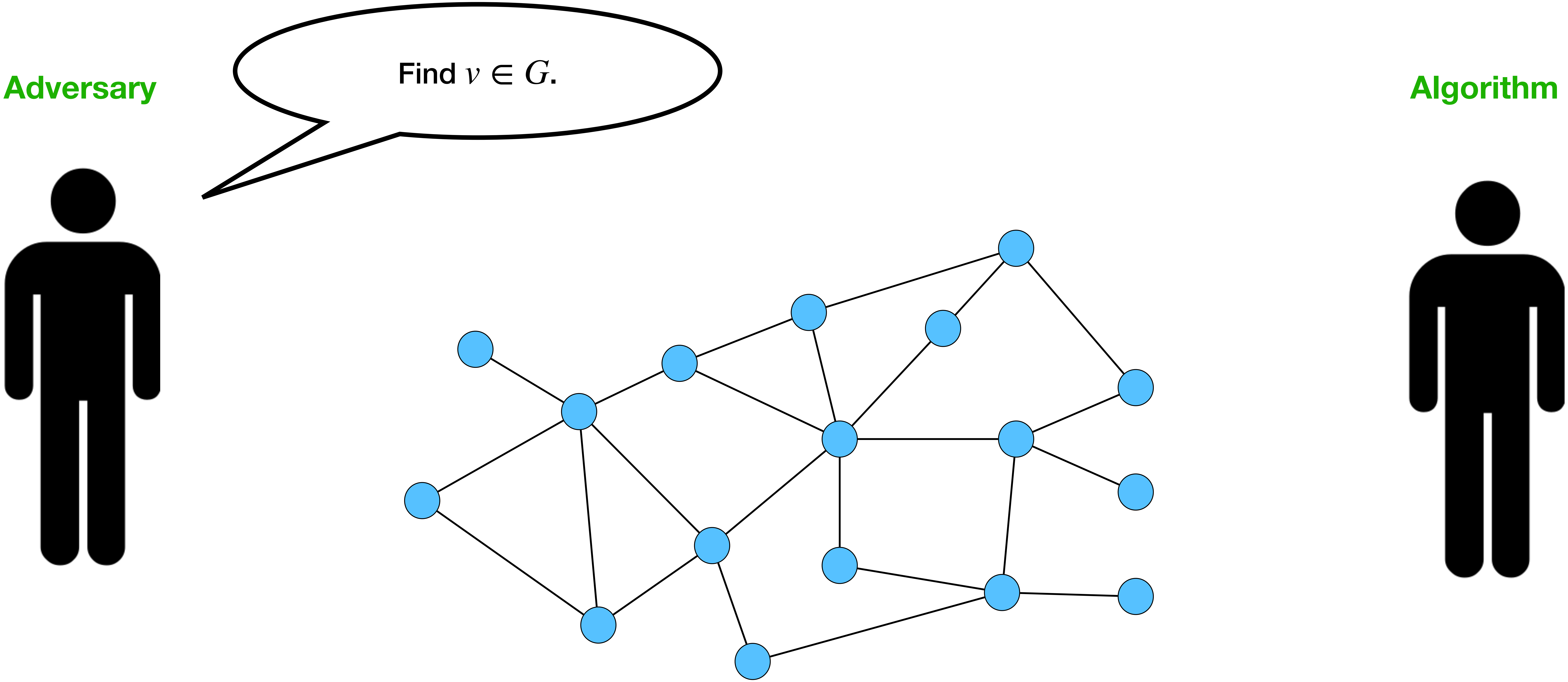
Is $x \leq 250000$?

Algorithm

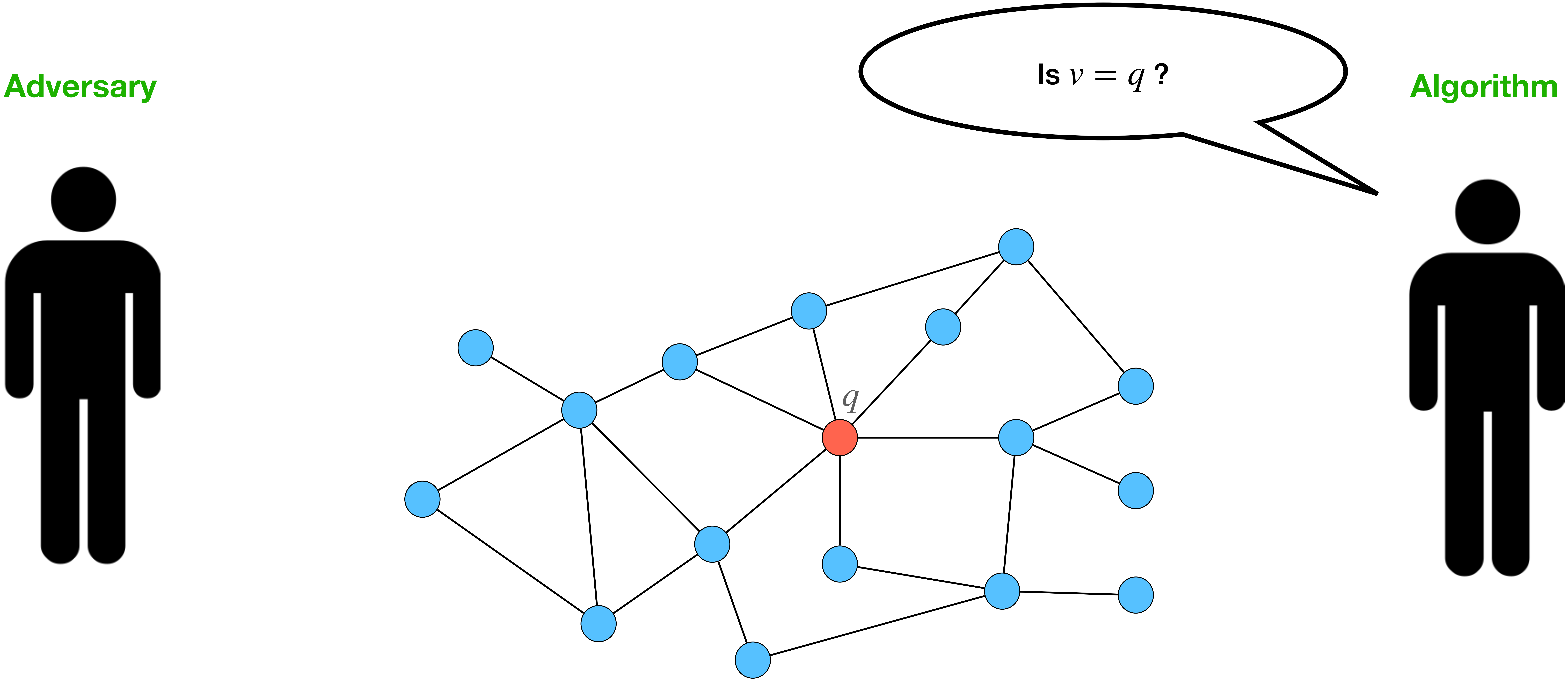


And so on, and so forth…

Game on a graph



Game on a graph



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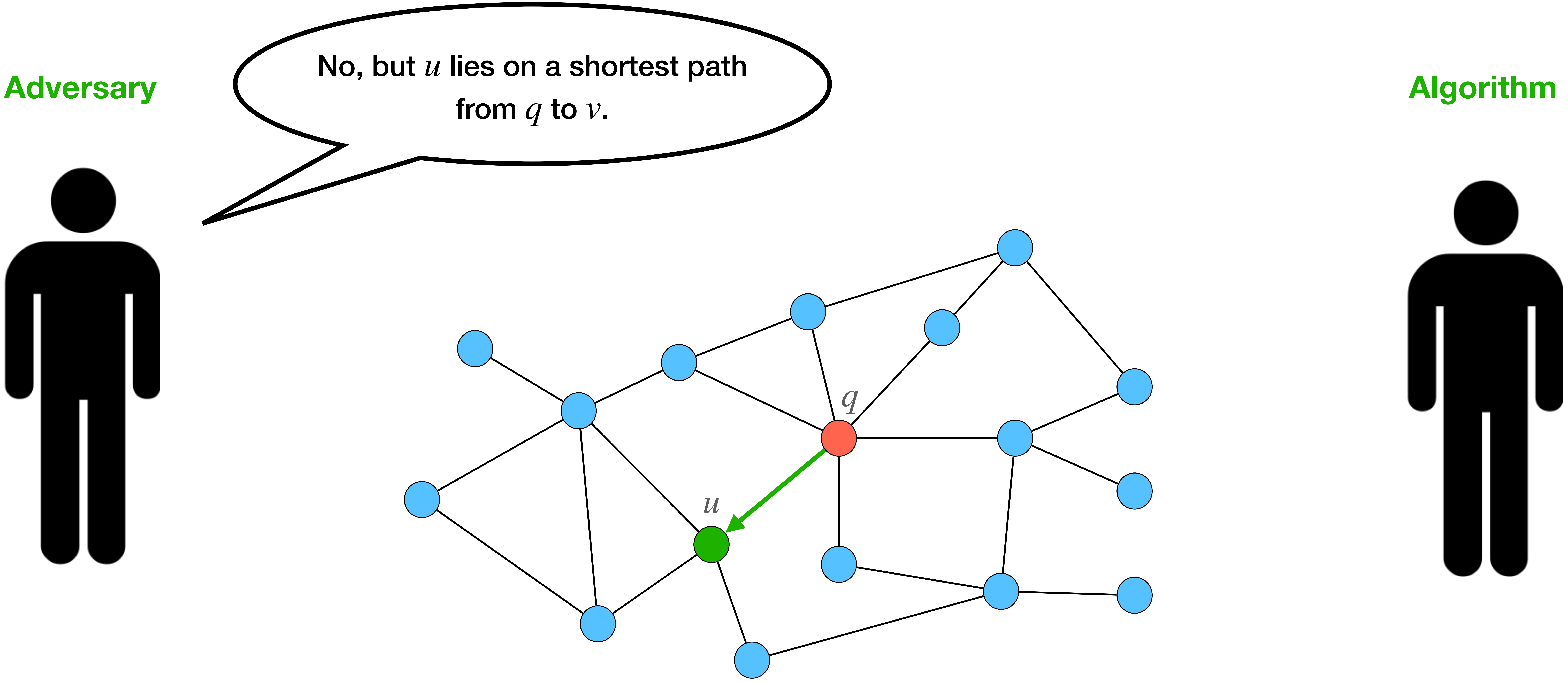
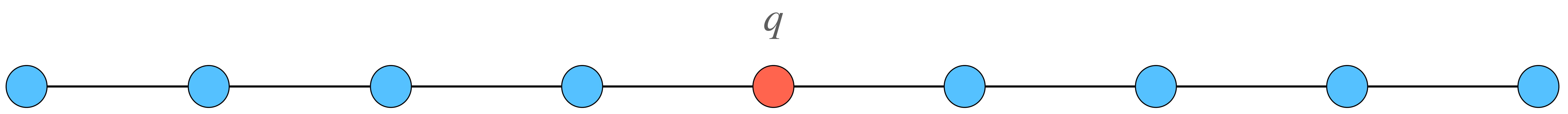
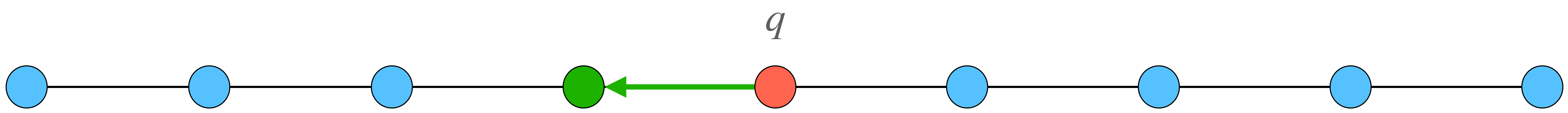


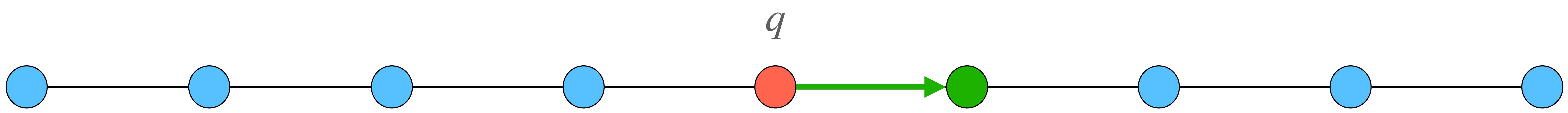
(Almost) a generalization of binary search



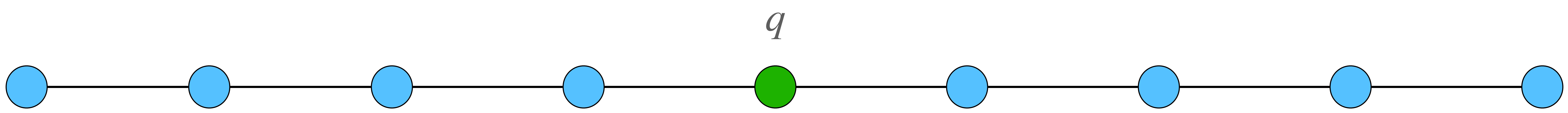
(Almost) a generalization of binary search



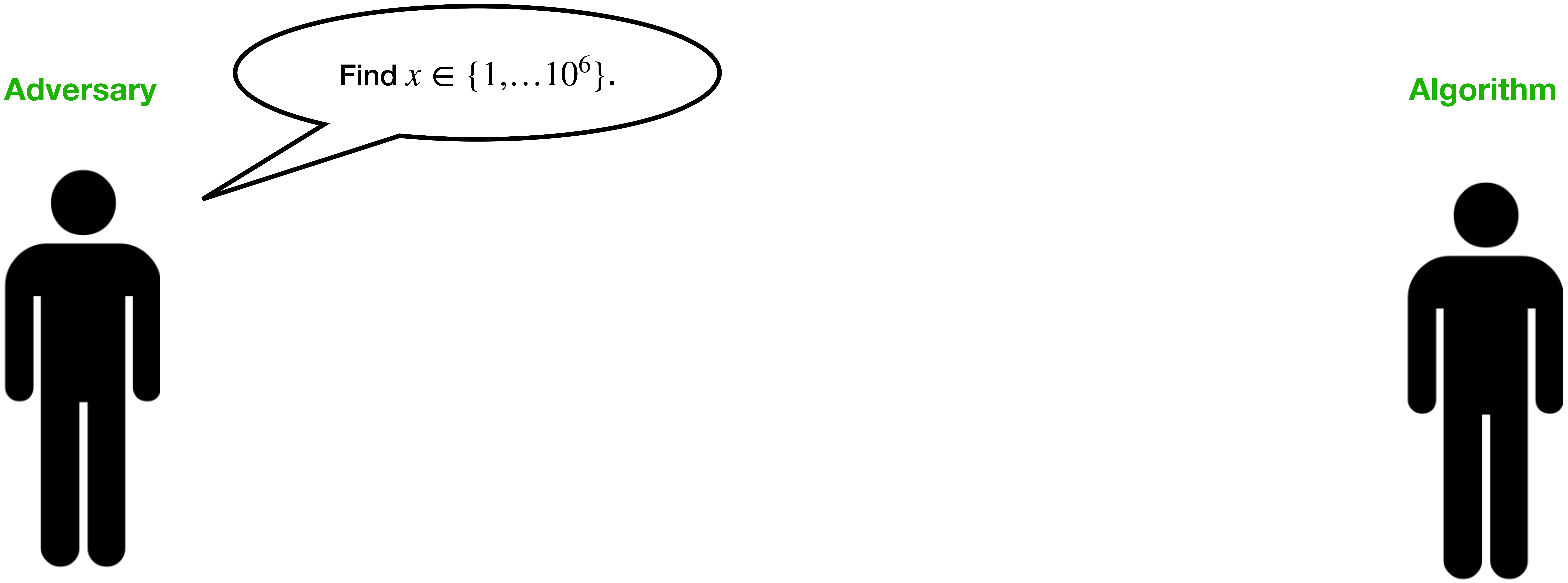
(Almost) a generalization of binary search



(Almost) a generalization of binary search



Searching with stochastic noise



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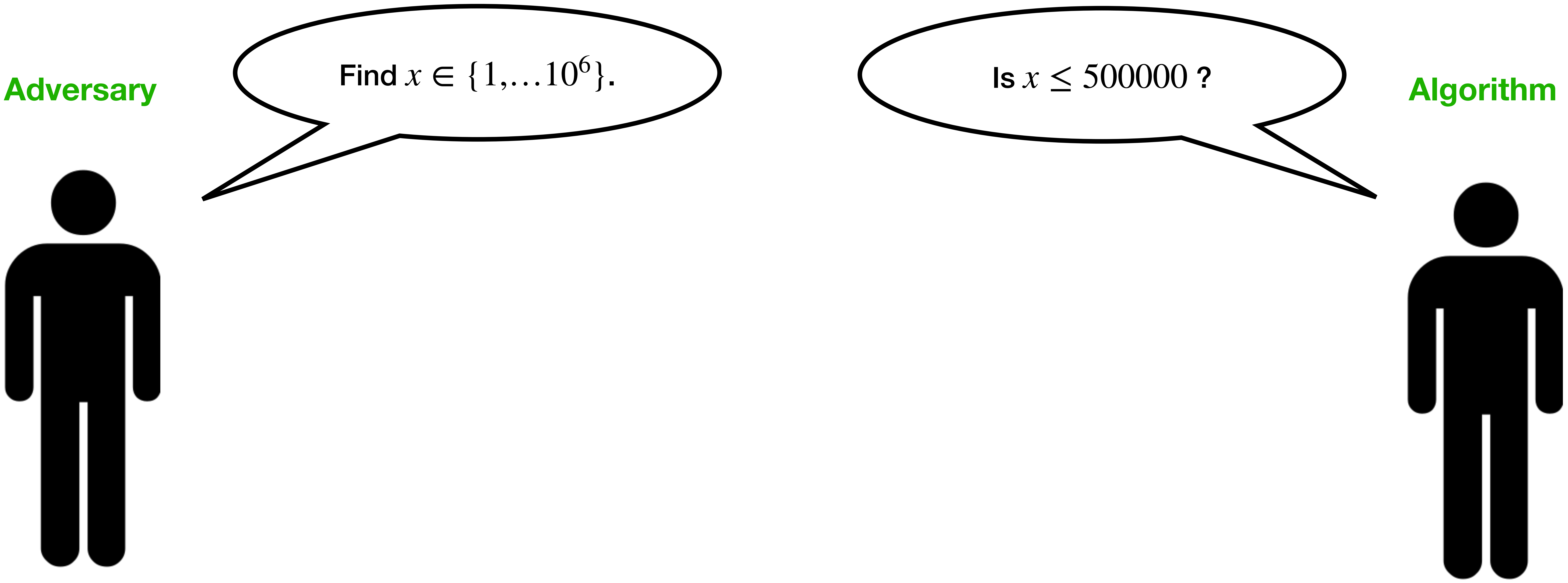

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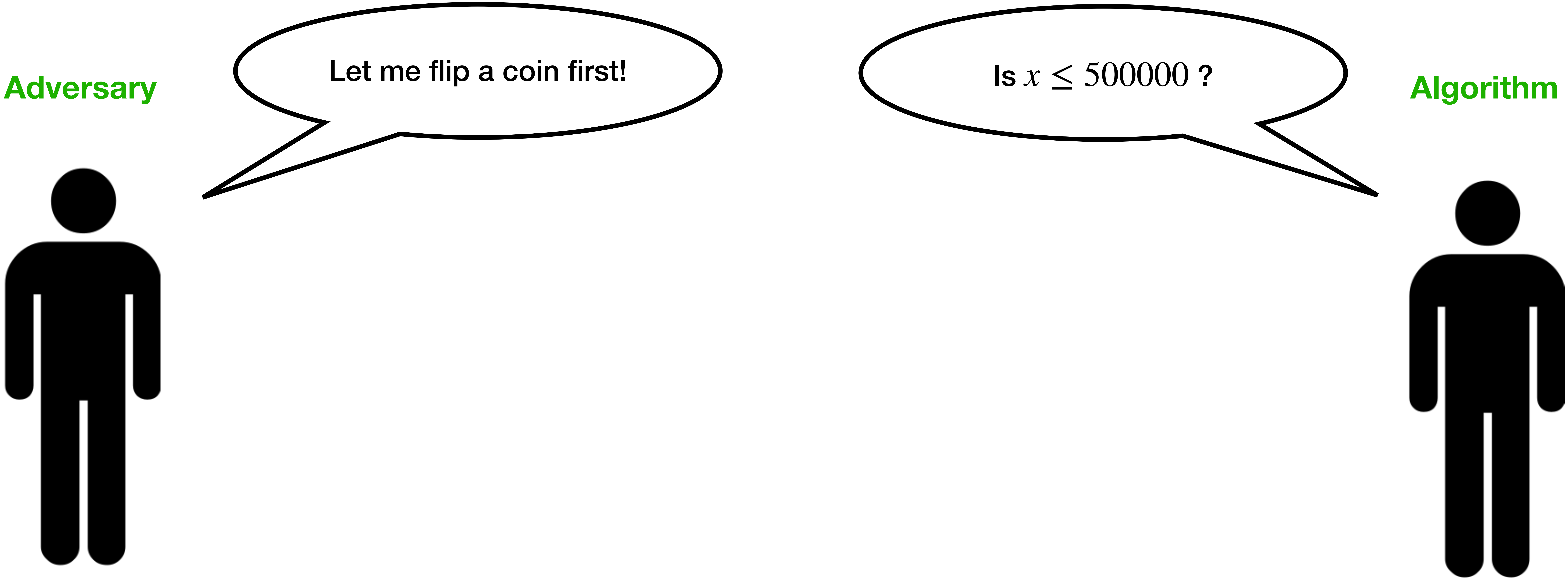
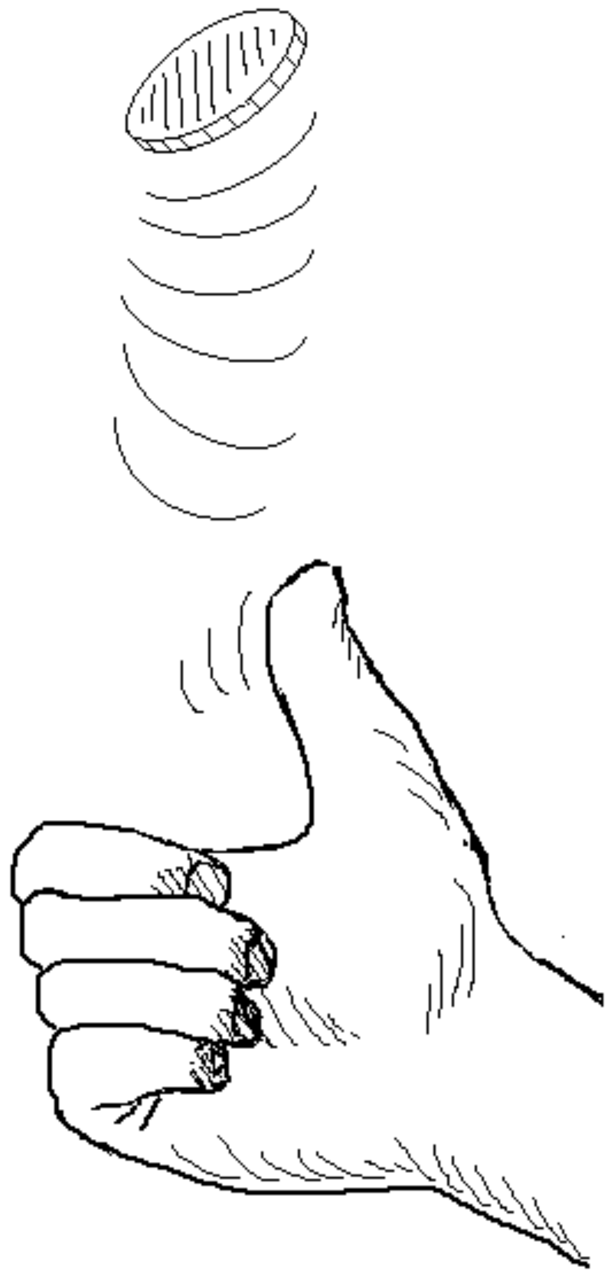
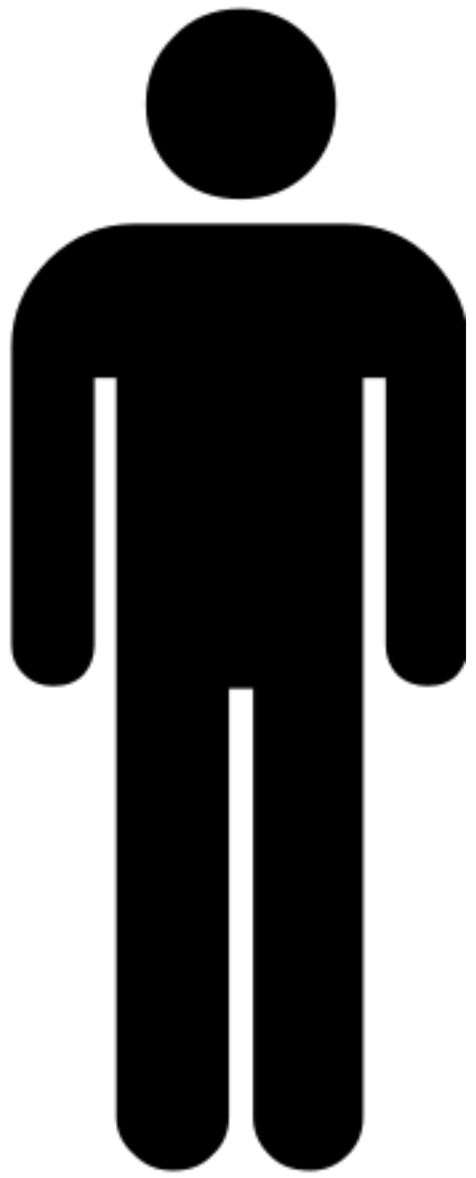


Searching with stochastic noise

Adversary



Algorithm

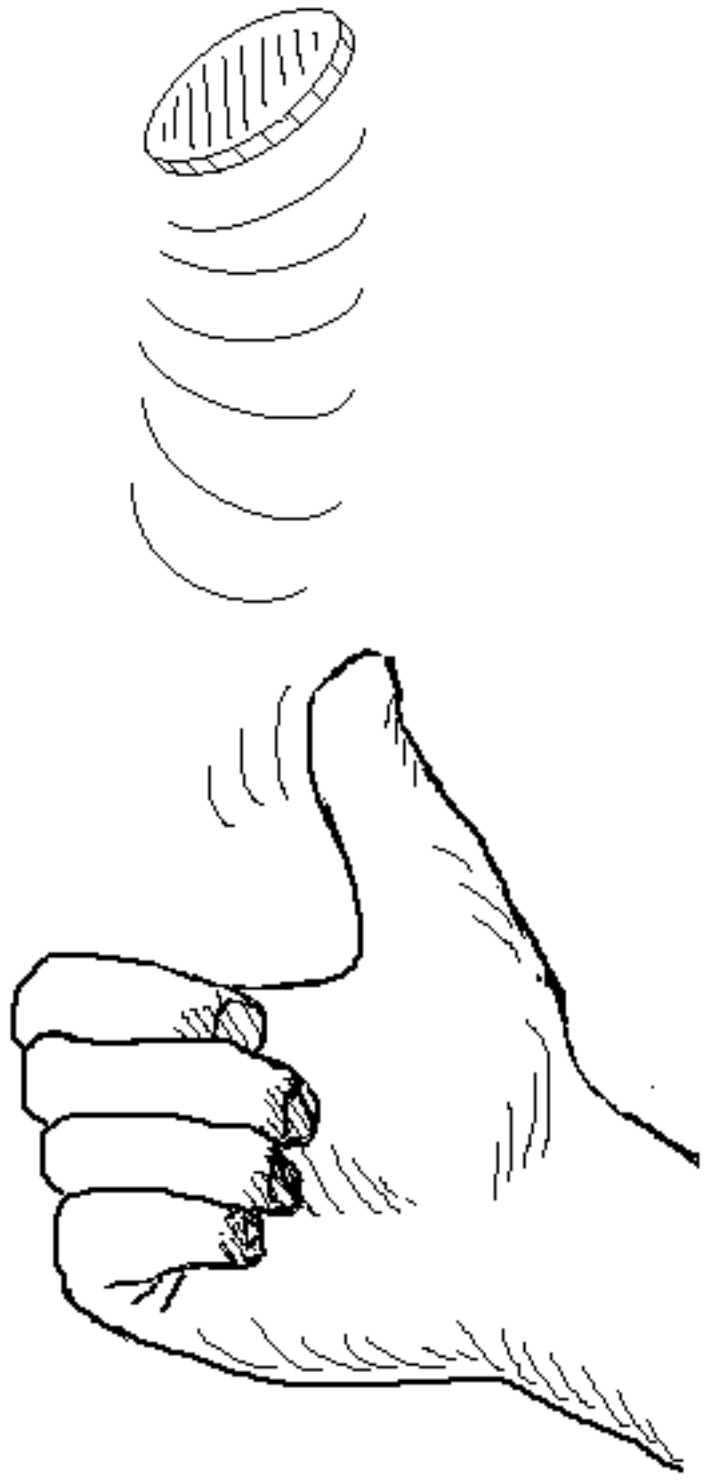


Searching with stochastic noise

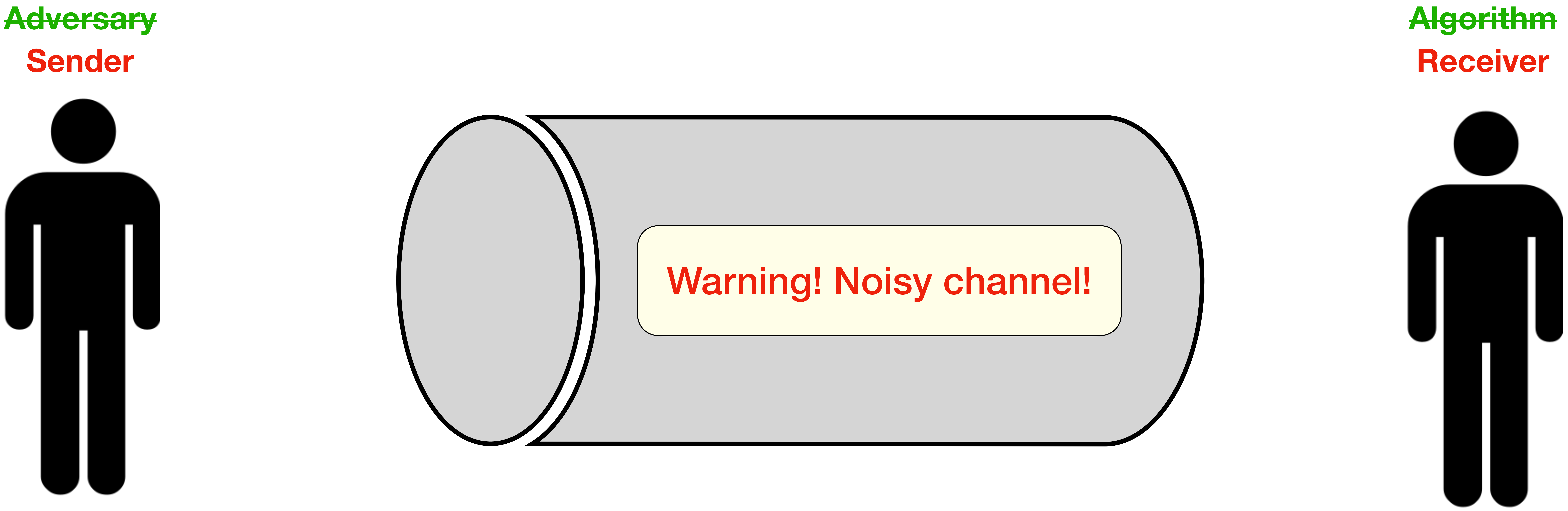
Searching with stochastic noise

Probabilistic errors

- For each query an erroneous reply independently with probability p, $0 \le p < \frac{1}{2}$.
- We want correct output with probability at least $1 - \delta$.
- We want to know the exact query complexity (we care about constants).



The tale of information theory



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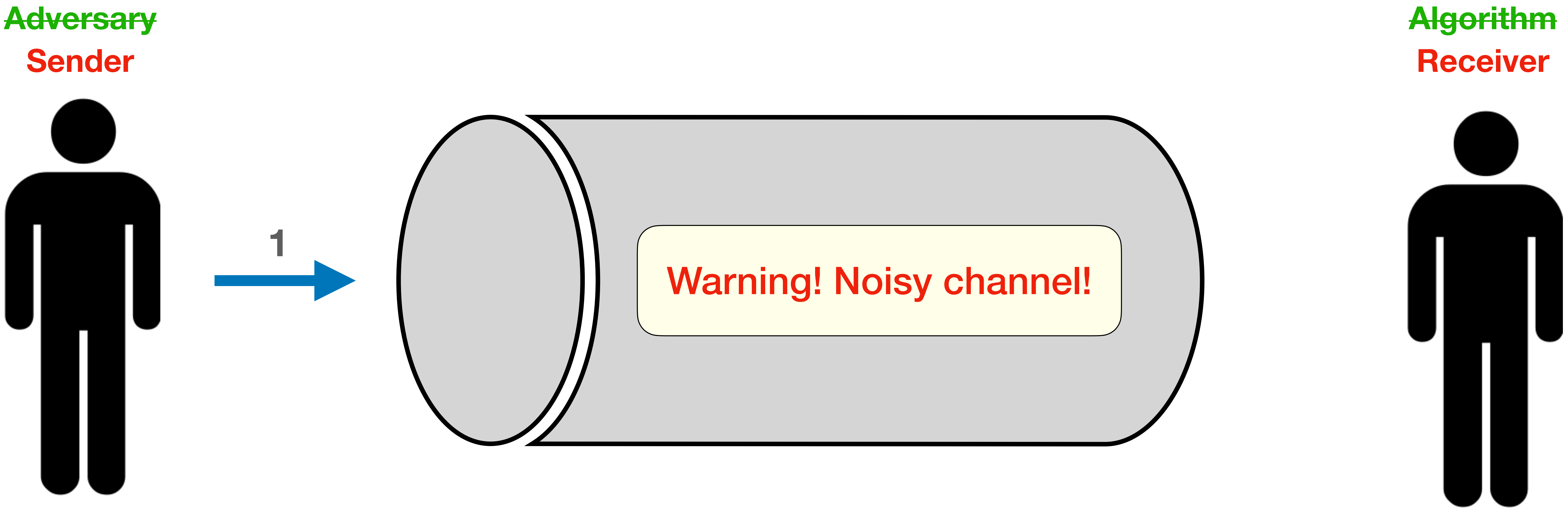

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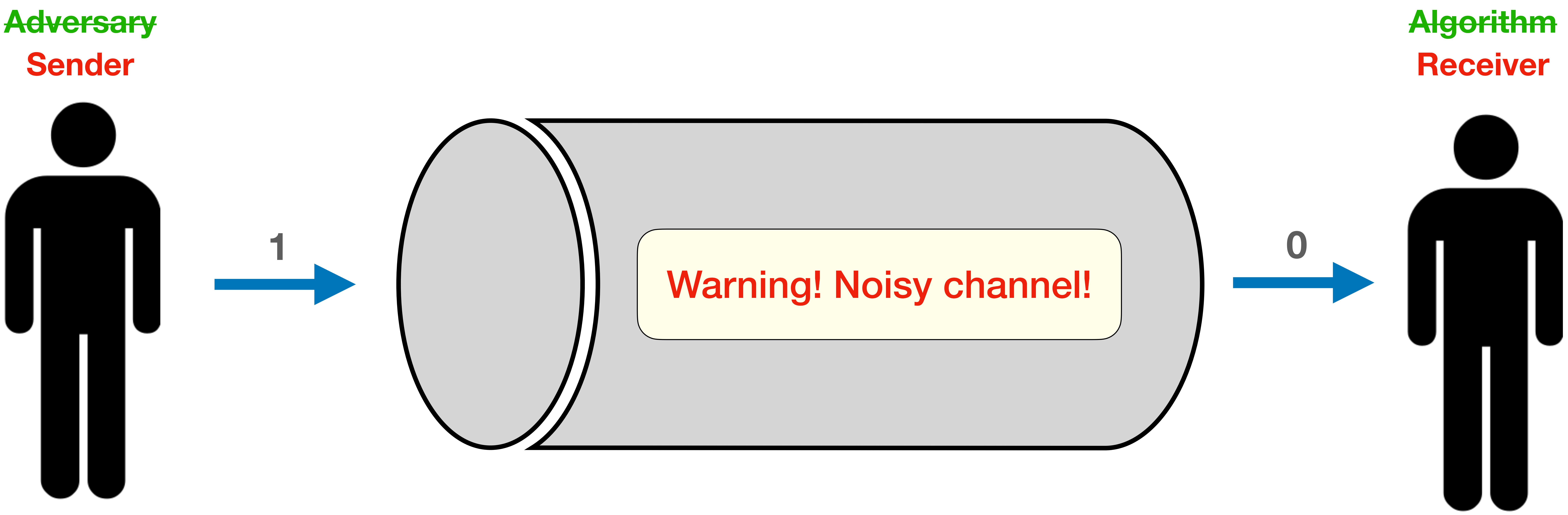

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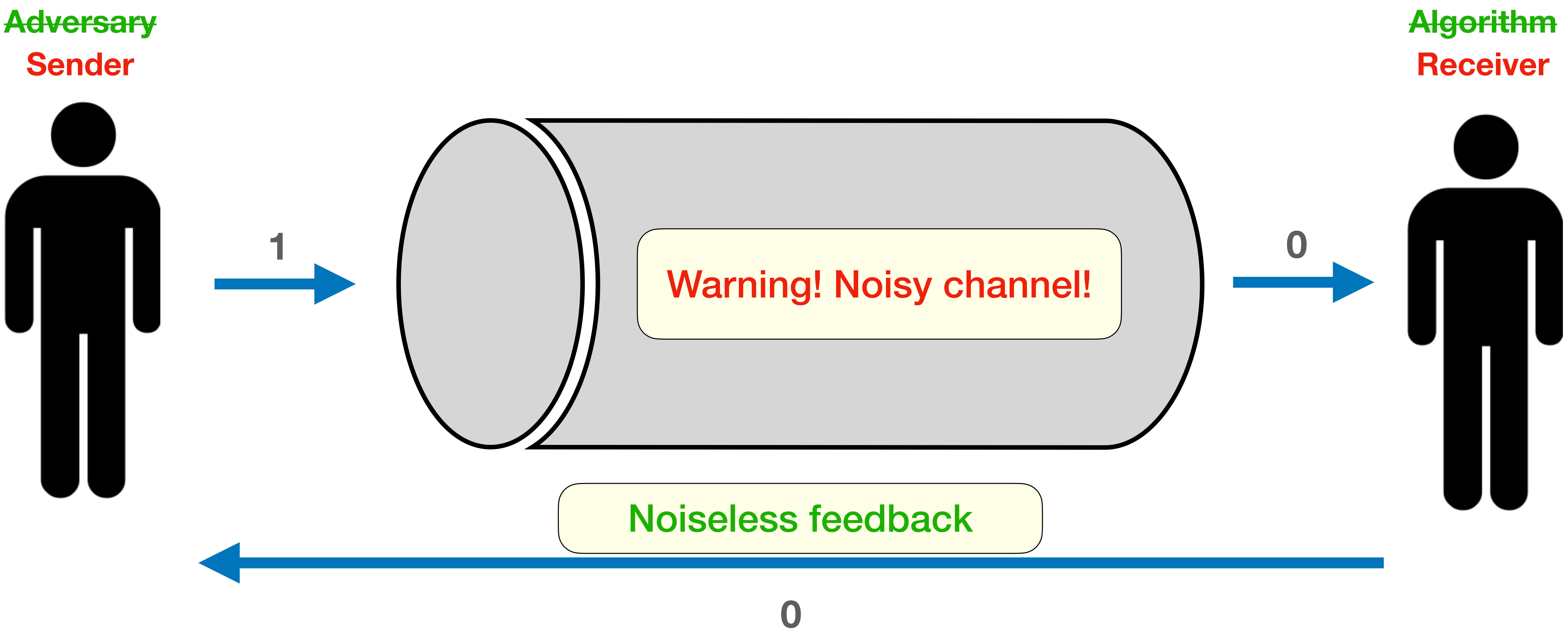


Can't beat roughly

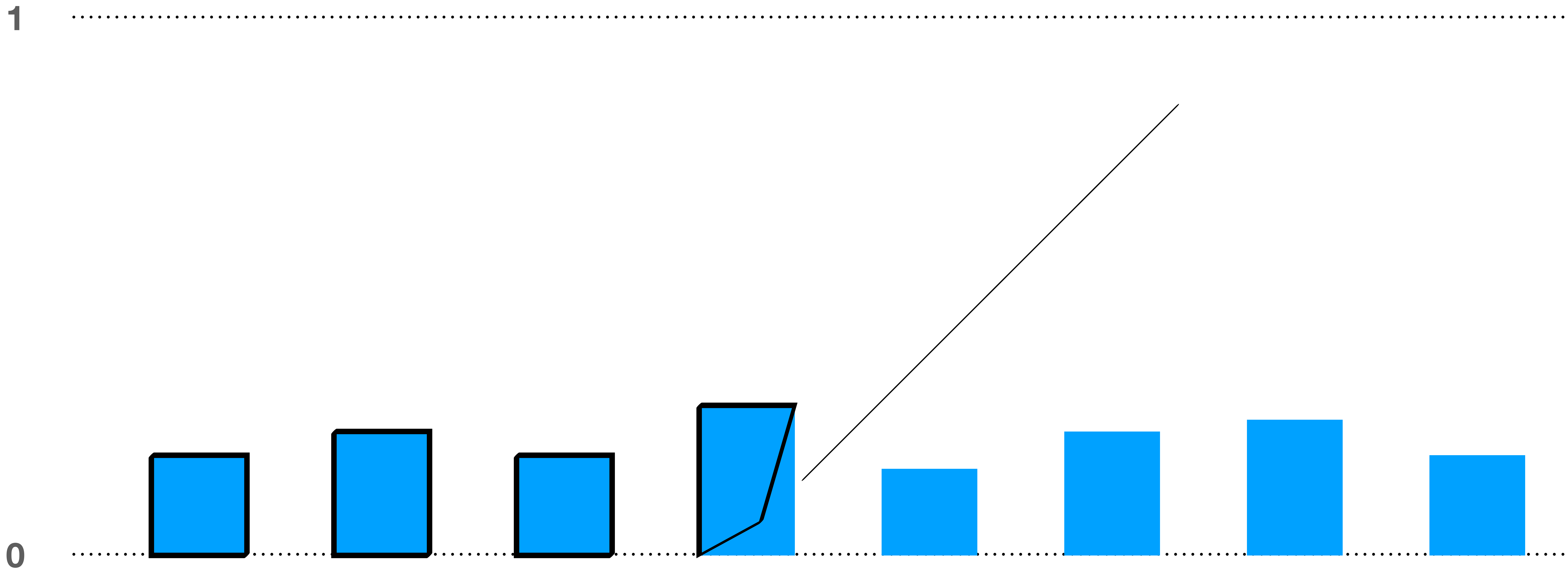
$$\frac{\log_2 n}{1 - H(p)}$$

in expectation.

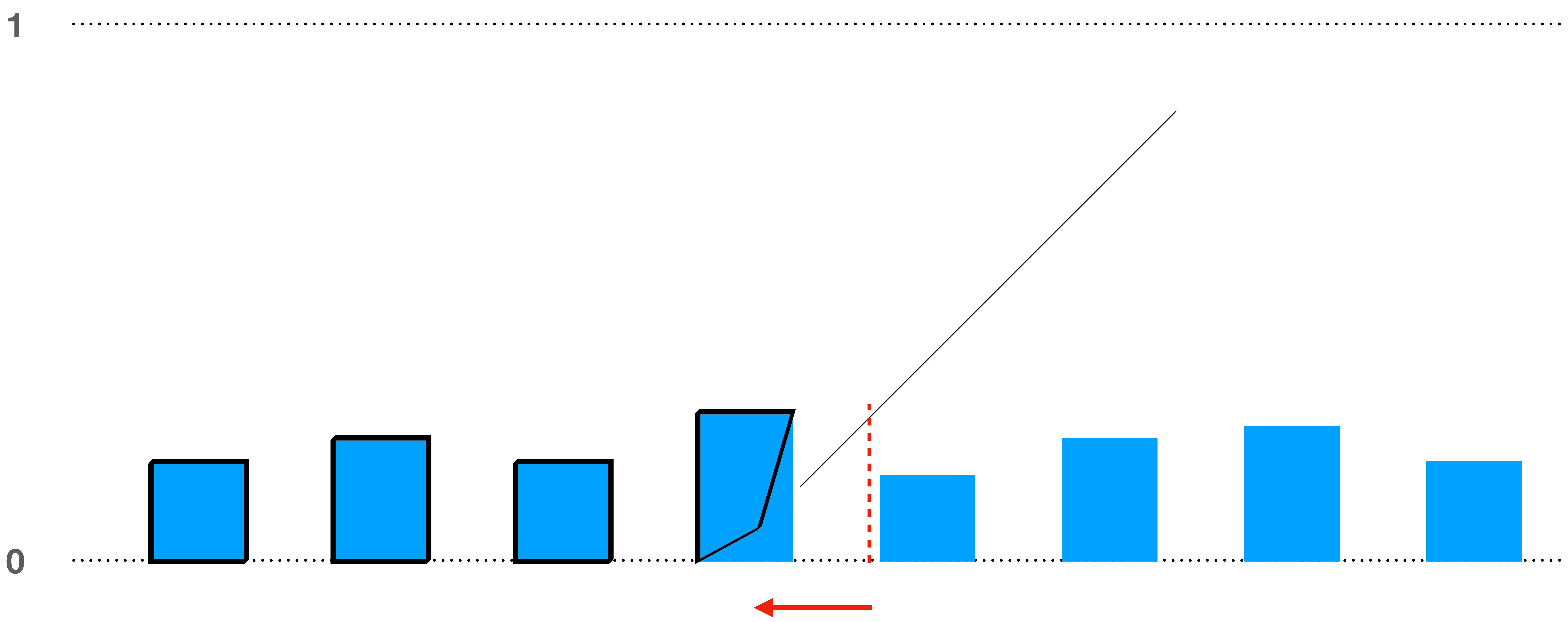
Part II

Intuition and challenges along the way

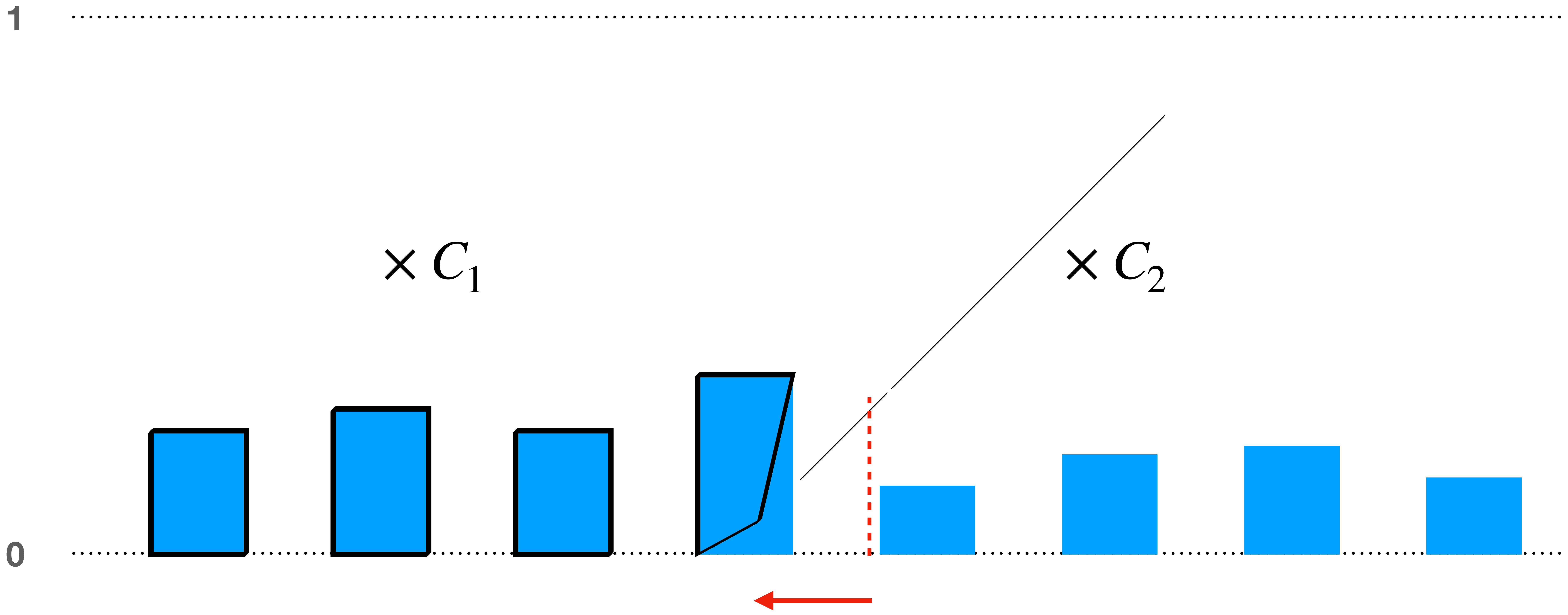
General framework - MWU



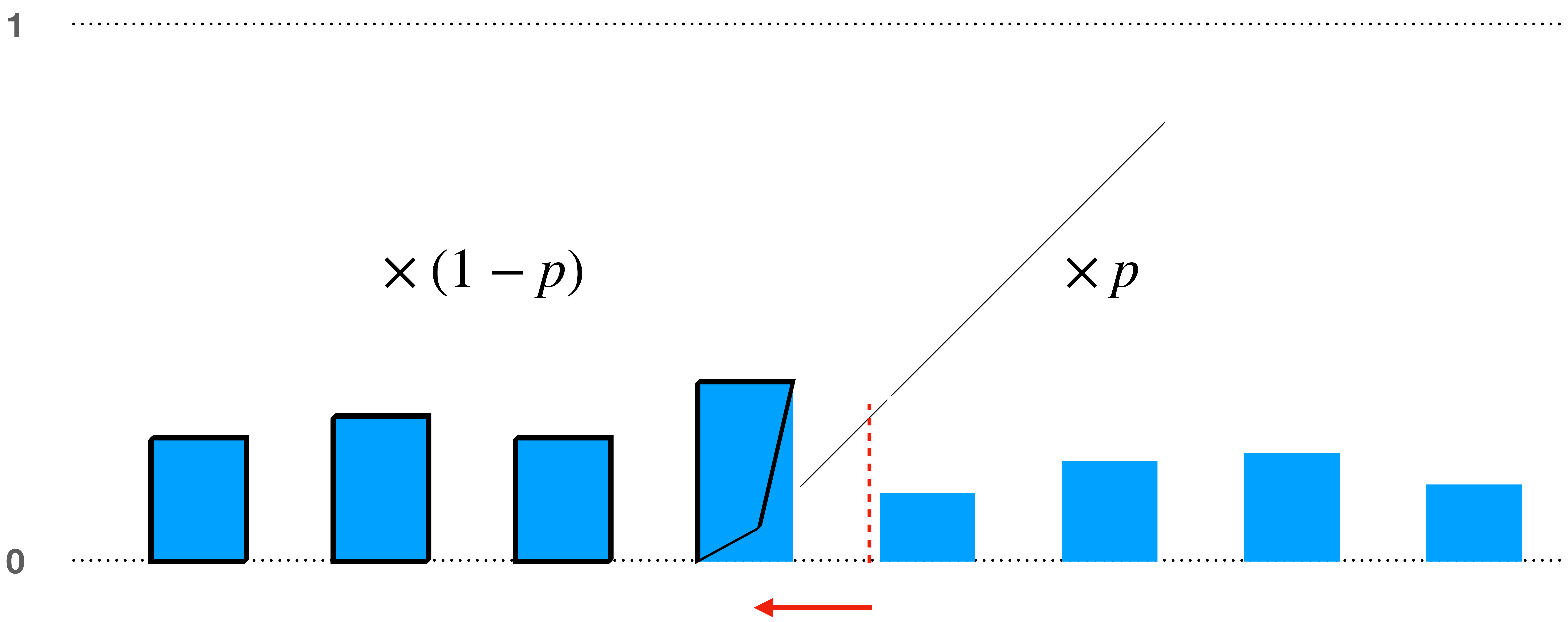
General framework - MWU



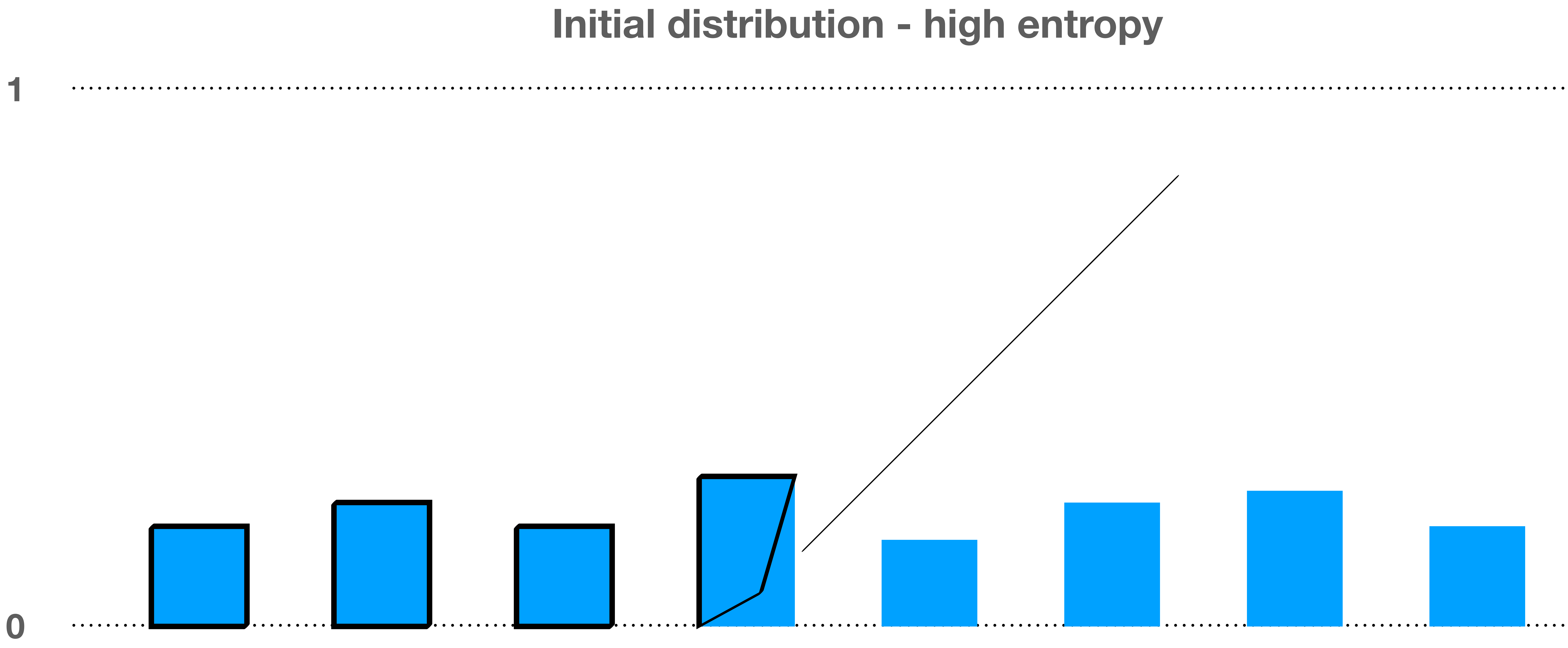
General framework - MWU



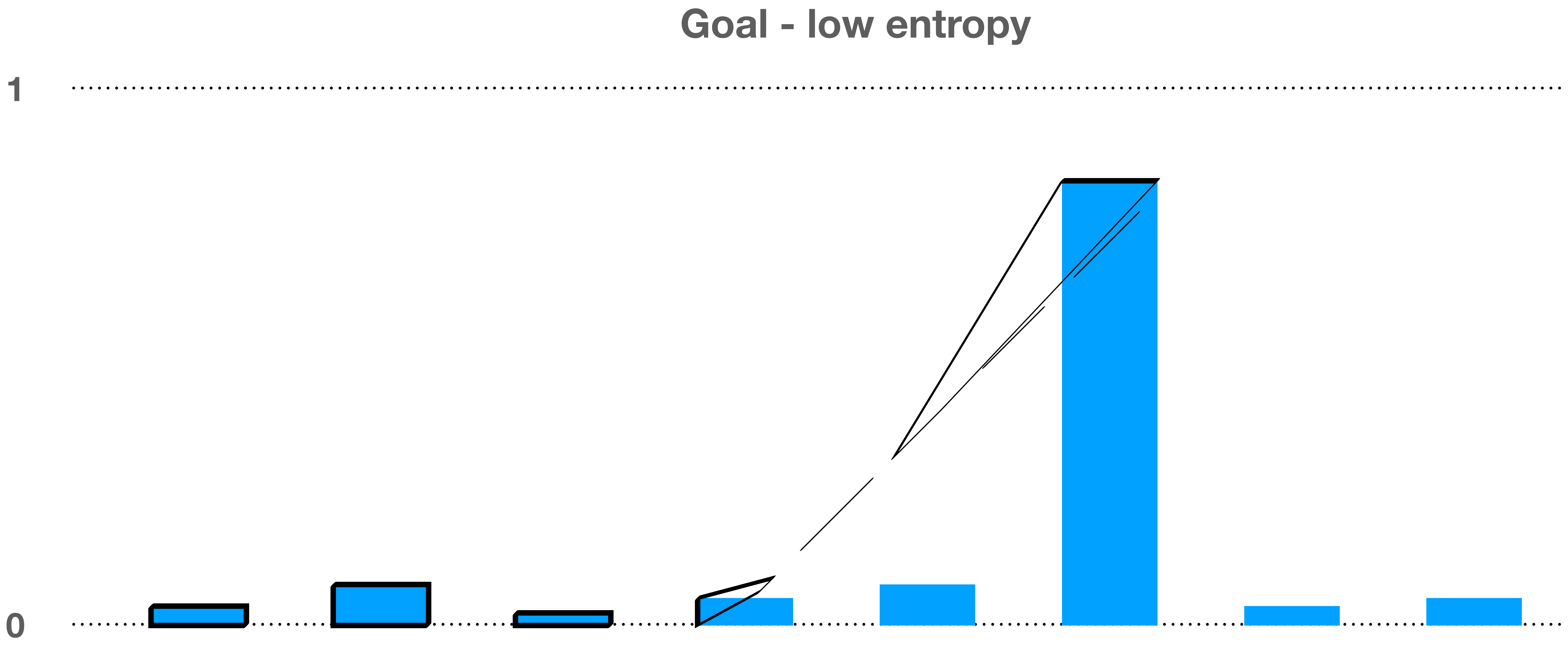
MWU - Bayesian updates



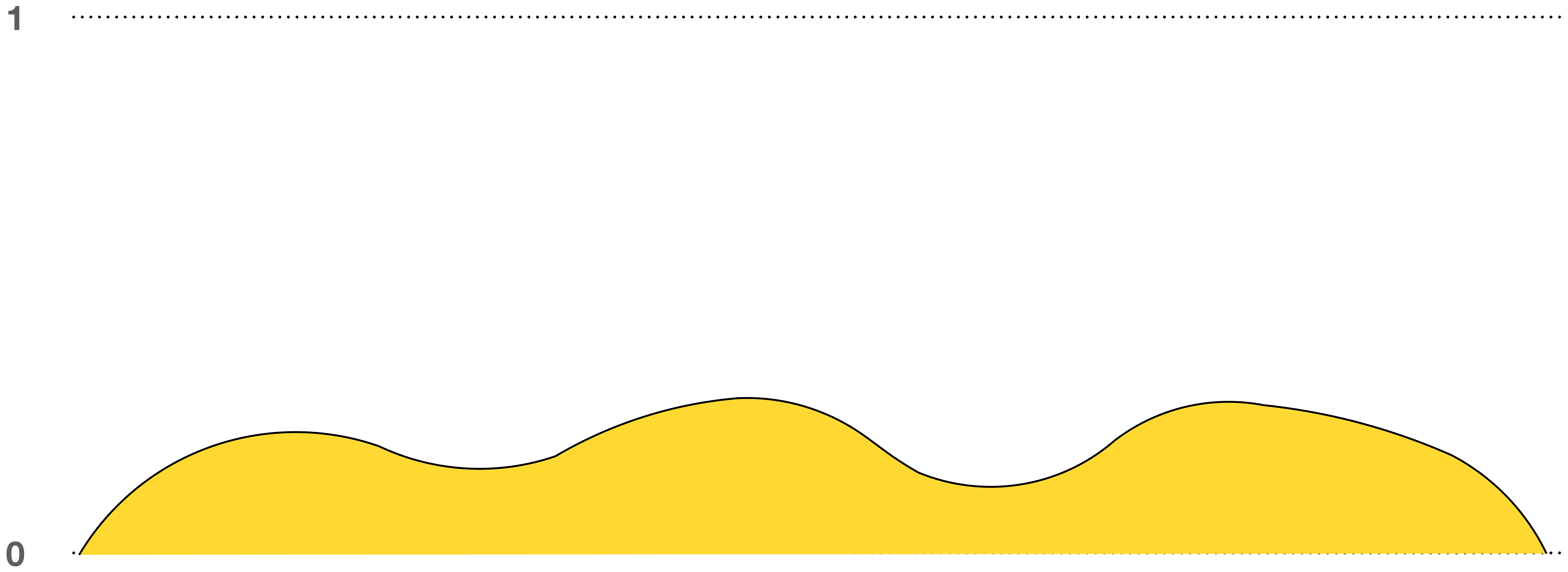
Measuring progress



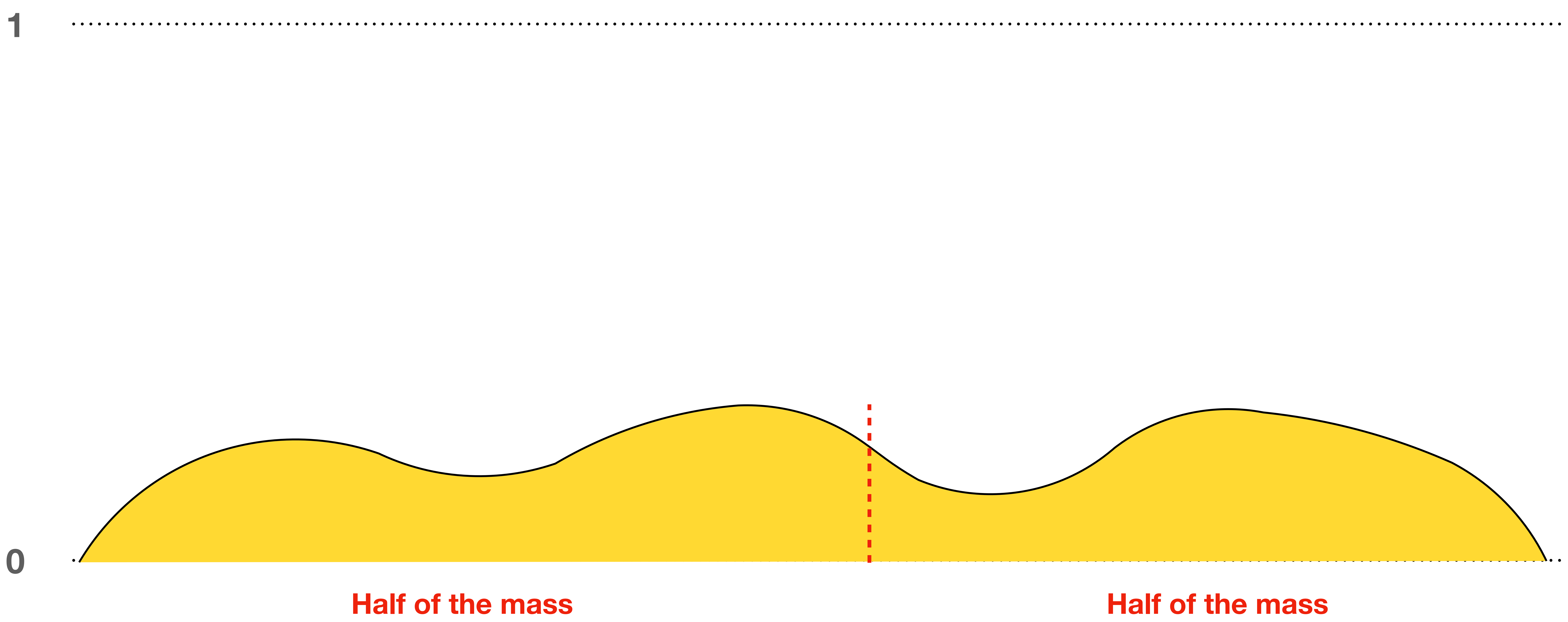
Measuring progress



Idealised scenario [Ben-Or, Hassidim, 2008]

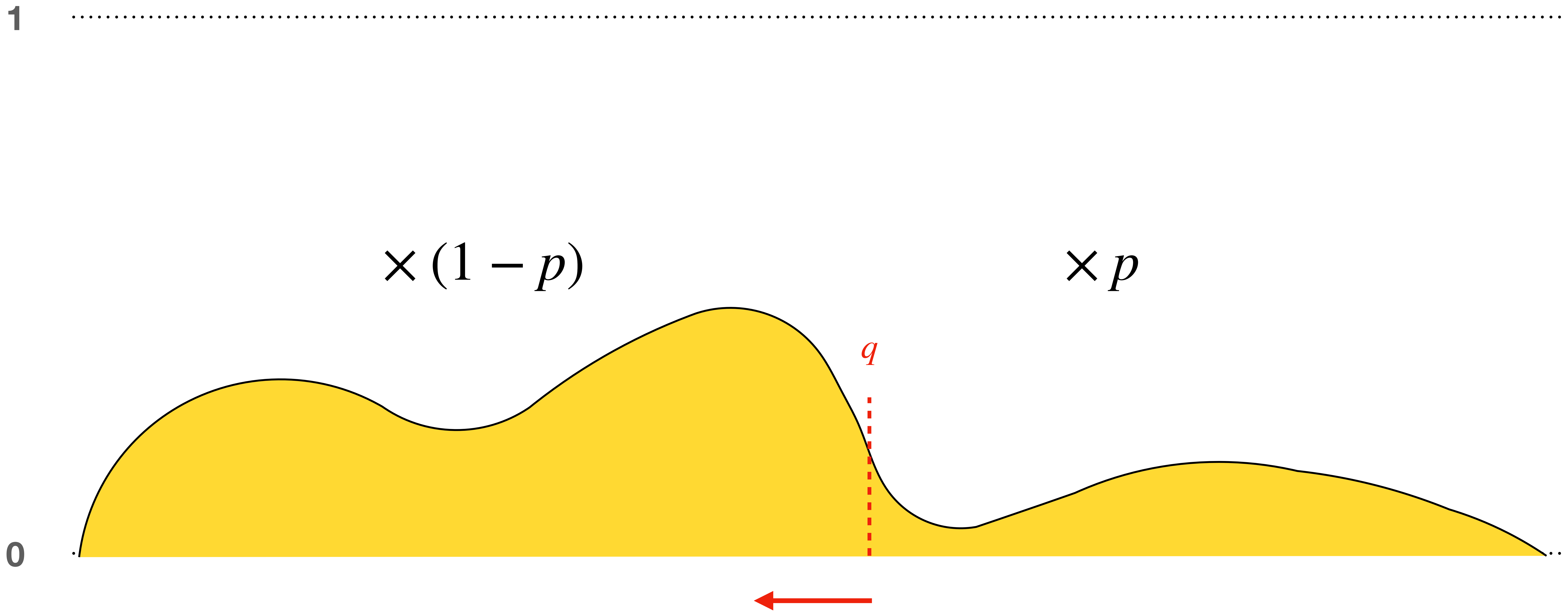


Idealised scenario [Ben-Or, Hassidim, 2008]



Idealised scenario [Ben-Or, Hassidim, 2008]

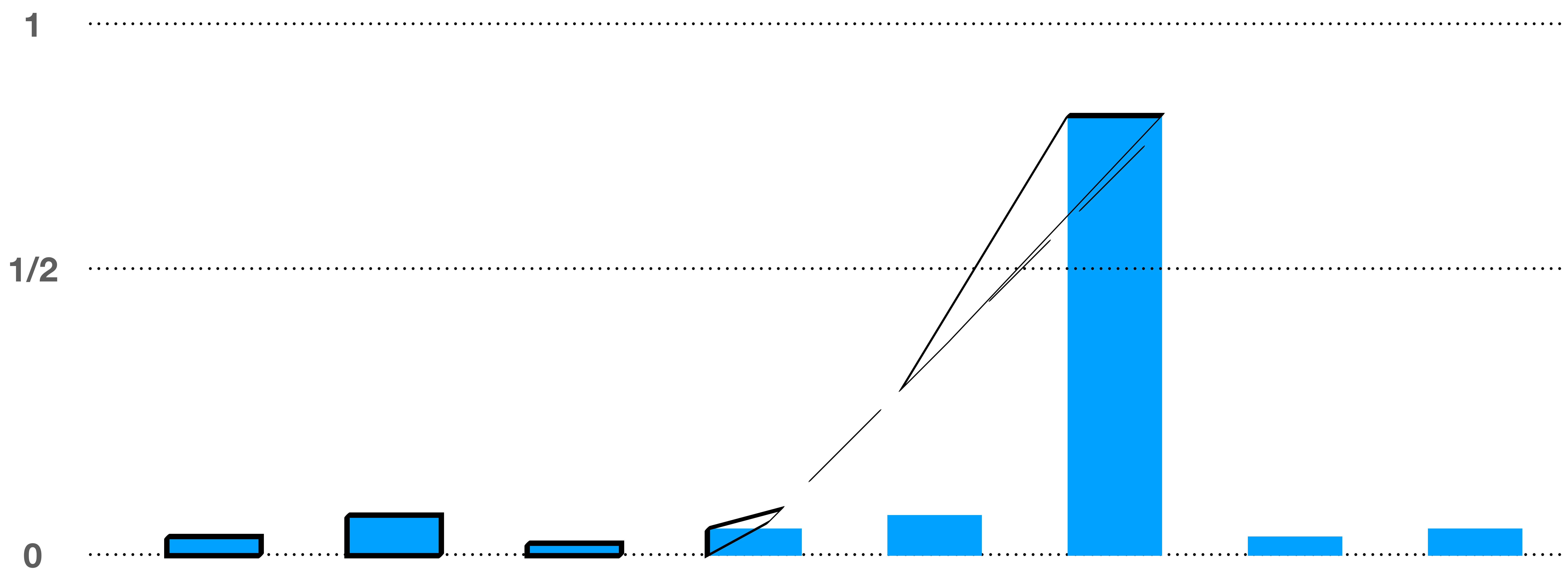
The expected entropy drop is $1 - H(p)$



But in CS we are living in a discrete world!

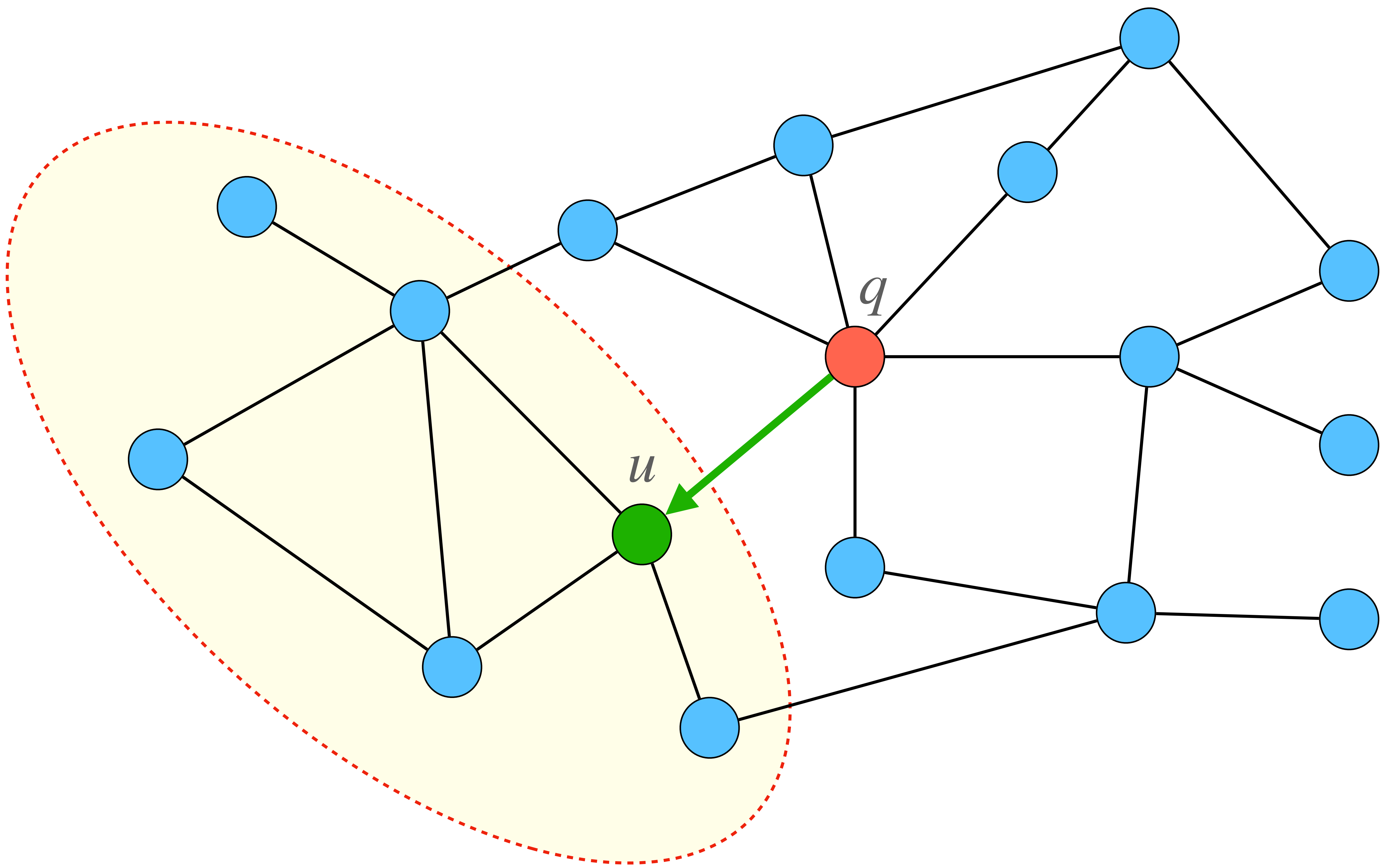
We won’t be able to get a perfect bisection all the time!

It’s a discrete, discrete, discrete world!



Bisection in graph searching

$N(q, u)$ - set of vertices consistent with query q and response u.



Bisection in graph searching

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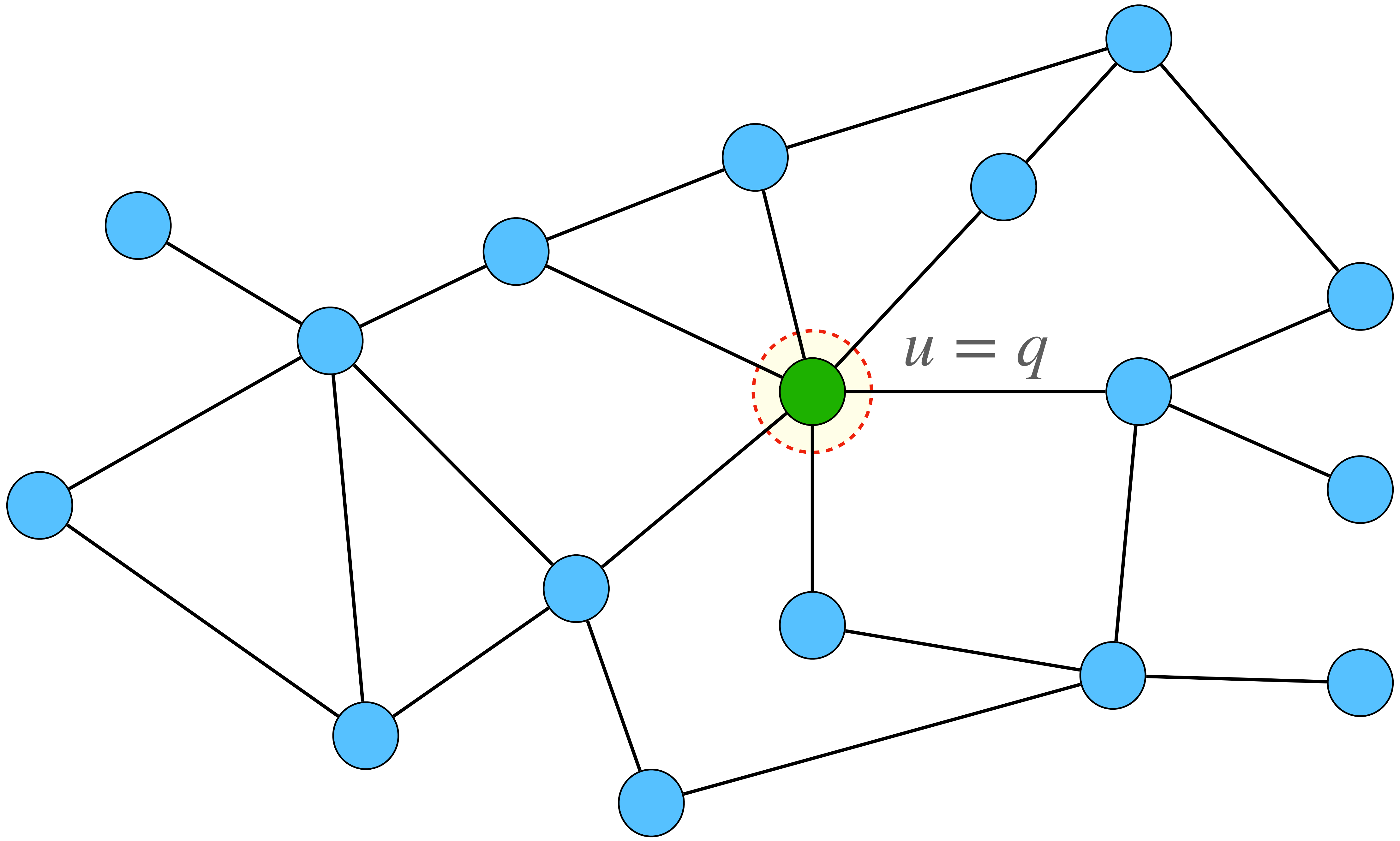
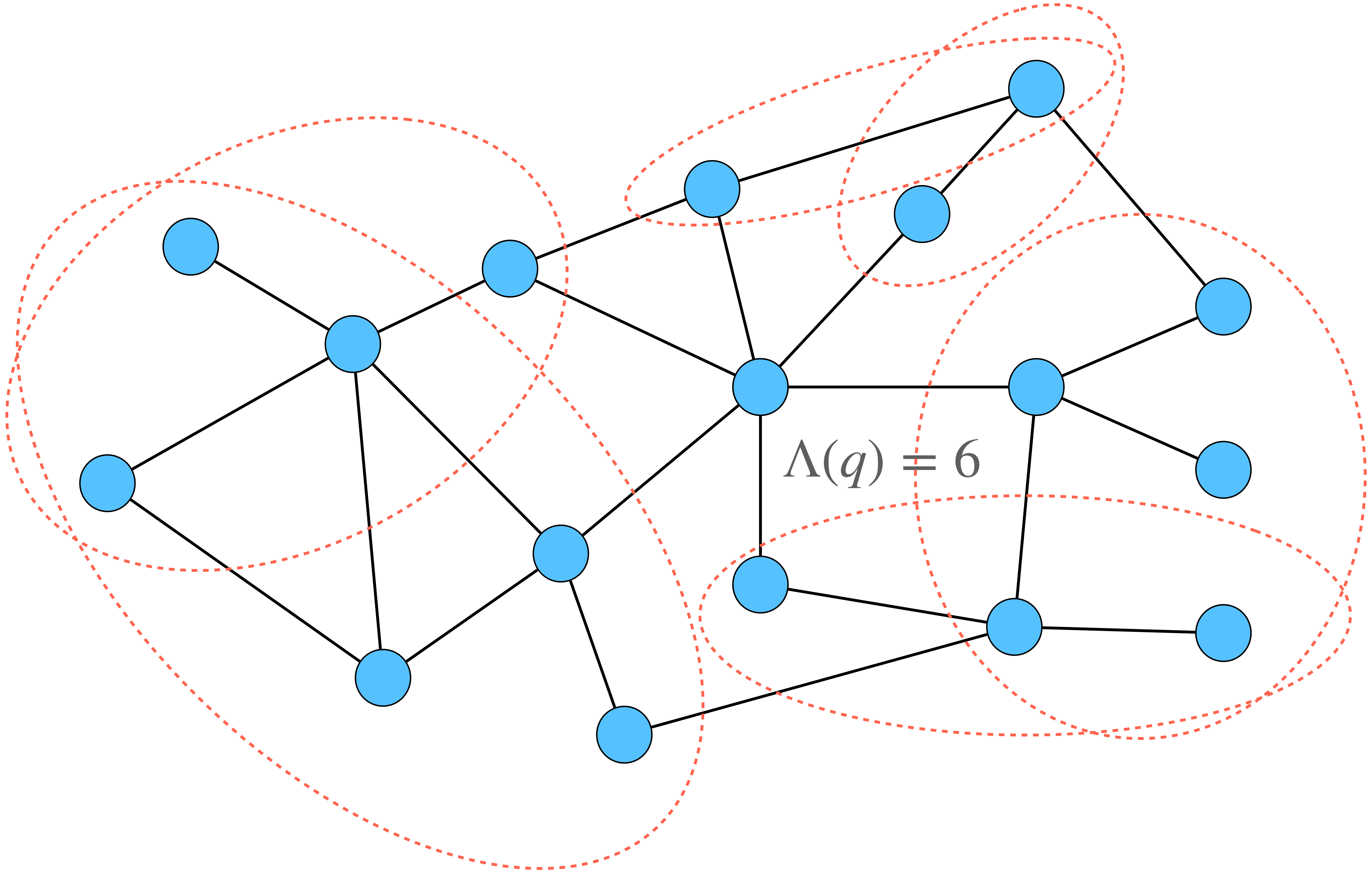


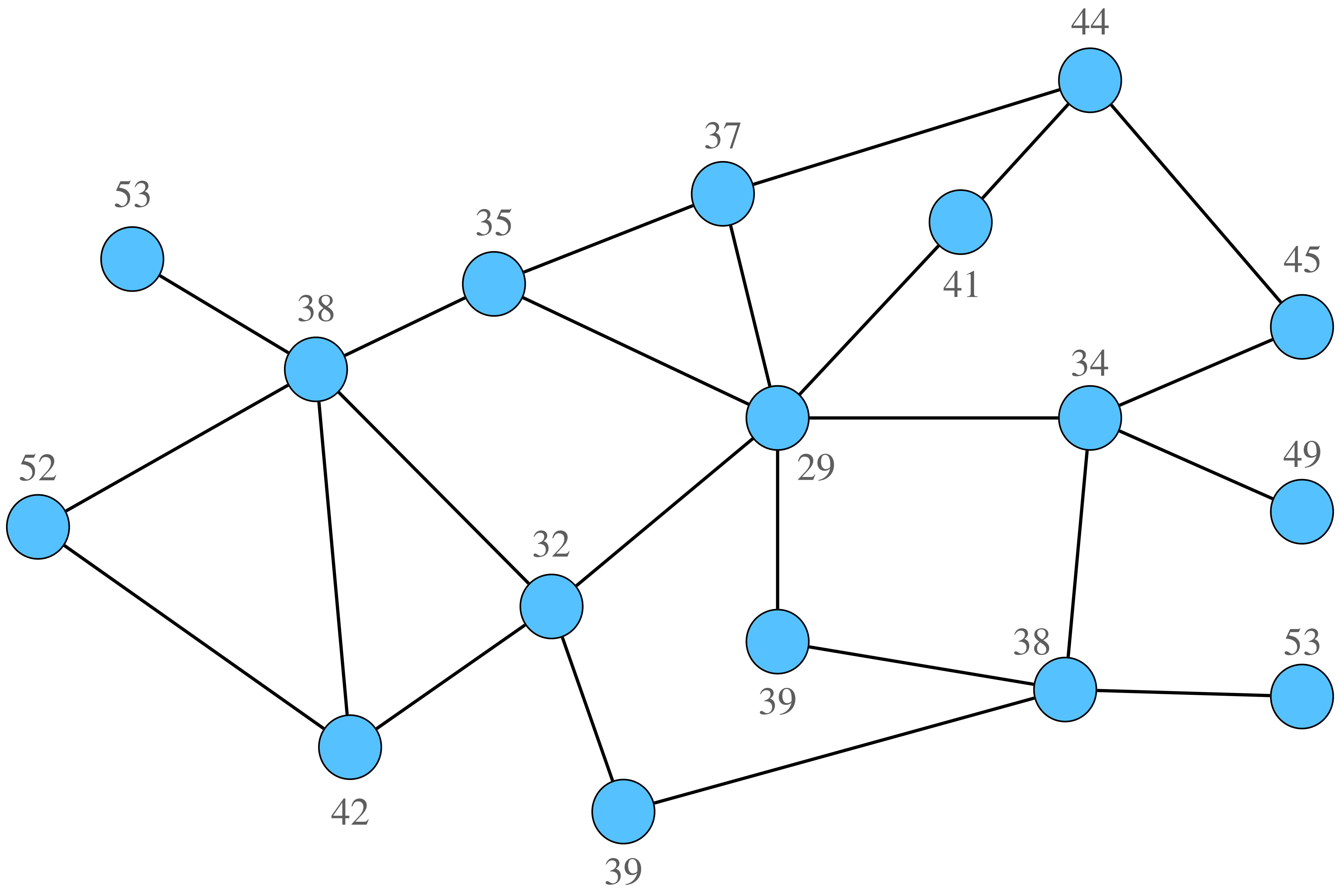
Bisection in graph searching

$$\Lambda(v) = \max_{u \in N(v)} \omega(N(v, u))$$



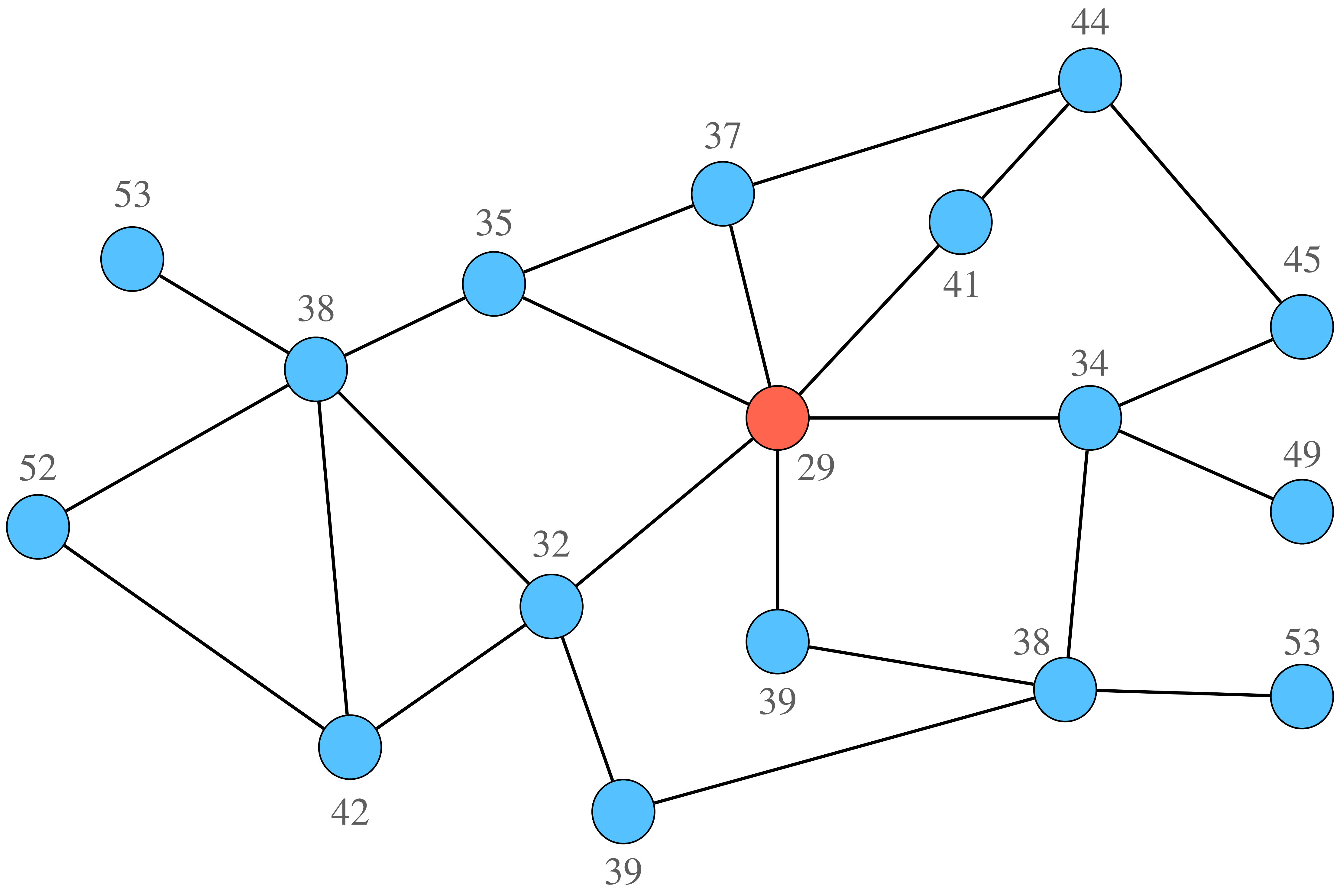
Bisection in graph searching

$$\Phi(v) = \sum_{u \in V} \omega(u) \cdot d(u, v)$$



Bisection in graph searching

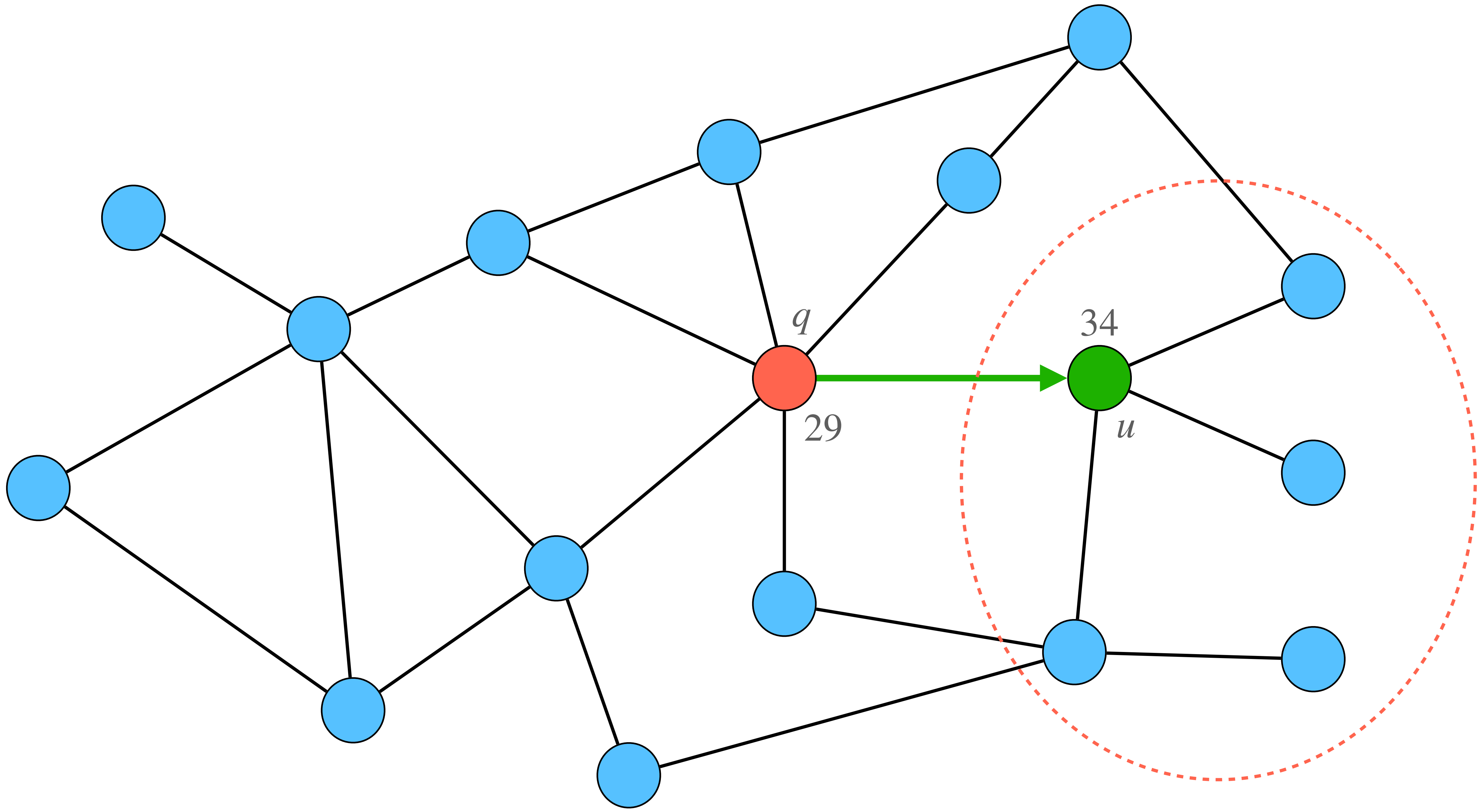
Median of a graph: $q = \arg\min_{v \in V} \Phi(v)$



Bisection in graph searching

Lemma [Emamjomeh-Zadeh et al. '16]:

if q is a median, then $\Lambda(q) \leq \dfrac{\omega(V)}{2}$



Part III

Our results

Previous (close to optimal) results - noisy binary search

Paper	Setting	Query complexity
Ben-Or, Hassidim [FOCS 2008]	Expected	$\frac{1}{1-H(p)}(\log_2 n + O(\log\log n) + O(\log \delta^{-1}))$
Gan et al. [SWAT 2022]	Worst-case Constant p	$\frac{1}{1-H(p)}(\log_2 n + O(\sqrt{\log n \log \delta^{-1}} \log \frac{\log n}{\log \delta^{-1}}) + O(\log \delta^{-1}))$
Gretta, Price [ICALP 2024]	Worst-case More general problem	$\frac{1}{1-H(p)}(\log_2 n + O(\log^{2/3} n \log^{1/3} \delta^{-1}) + O(\log \delta^{-1}))$

Previous (close to optimal) results - noisy binary search

Paper	Setting	Query complexity
Burnashev, Zigangirov [Problemy Peredachi Informatsii, 1974]	Expected Target unif. random	$\frac{1}{1-H(p)}(\log_2 n + \log_2 \delta^{-1} + \log_2 \frac{1-p}{p})$
Ben-Or, Hassidim [FOCS 2008]	Expected	$\frac{1}{1-H(p)}(\log_2 n + O(\log\log n) + O(\log \delta^{-1}))$
Gan et al. [SWAT 2022]	Worst-case Constant p	$\frac{1}{1-H(p)}(\log_2 n + O(\sqrt{\log n \log \delta^{-1}} \log \frac{\log n}{\log \delta^{-1}}) + O(\log \delta^{-1}))$
Gretta, Price [ICALP 2024]	Worst-case More general problem	$\frac{1}{1-H(p)}(\log_2 n + O(\log^{2/3} n \log^{1/3} \delta^{-1}) + O(\log \delta^{-1}))$

Previous results - noisy graph search

Paper	Setting	Query complexity
Emamjomeh-Zadeh et al. [STOC 2016]	Expected	$\frac{1}{1-H(p)}(\log_2 n + o(\log n) + O(\log^2 \delta^{-1}))$
Dereniowski et al. [SOSA 2019]	Worst-case	$\frac{1}{1-H(p)}(\log_2 n + O(\sqrt{\log n \log \delta^{-1}} \log \frac{\log n}{\log \delta^{-1}}) + O(\log \delta^{-1}))$

Our results

Model	Binary search	Graph search
Worst case complexity	$\frac{1}{1-H(p)}(\log_2 n + O(\sqrt{\log n \log \delta^{-1}}) + O(\log \delta^{-1}))$	
Expected complexity	$\frac{1}{1-H(p)}(\log_2 n + O(\log \delta^{-1}))$	$\frac{1}{1-H(p)}(\log_2 n + O(\log \log n) + O(\log \delta^{-1}))$

Contributions

- **Simple** - query the median until its over.
- **Fast** - close to optimal, slightly improving previous complexities.
- **Correct** - we correct errors from previous literature.

Techniques

- New measure of progress (in analysis): total weight minus the weight of the heaviest element.
- Random choice of a query [Burnashev, Zigangirov, 1974] with tighter analysis.

Open problems

- Closing the small gap between lower and upper bounds.
- Graph (and tree) searching with permanent probabilistic noise, see [Boczkowski et al. ESA 2018].
- Improving lower order terms in noisy binary search with monotonic probabilities, see [Greta & Price, ICALP 2024].