# On Average Baby PIH and Its Applications

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# Constraint Satisfaction Problem ($q$CSP)

- Variables $X = \{x_1, \cdots, x_n\}$
- Alphabet $\Sigma$
- Constraints $\Phi = \{\varphi_1, \cdots, \varphi_m\}$, each depends on $q$ variables

- Decide: whether it's satisfiable or not.

**NP-Complete.**

# The PCP Theorem [AS-ALMSS'98] [Dinur'07]

- **NP-hard** to decide whether a $q$CSP instance is
  - Satisfiable, or
  - Cannot satisfy $s$-fraction of constraints simultaneously.

$(0 < s < 1)$

# Relaxation: Multi-Assignment

- Assign each variable a **set** of values.

$x_1$: { 1, 5, 7, 9 }

$x_2$: { 2, 3, 4 }

$x_3$: { 2, 6 }

$x_4$: { 4, 5, 6, 8 }

$\varphi_1 = (x_1 x_2, C_1)$

$\varphi_2 = (x_2 x_3, C_2)$

$\varphi_3 = (x_2 x_4, C_3)$

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# Multi-Assignment PCP [Arora,Moshkovitz,Safra'06]

- **NP-hard** to decide whether a $q$CSP instance is
  - Satisfiable, or
  - Cannot satisfy $s$-fraction of constraints simultaneously **even** when **each** variable assigned $\leq t$ values.

$$(0 < s < 1,\ t > 1)$$

- Used to prove NP-hardness of approximating SetCover.

# Parameterized Inapprox. Hypo. (PIH)

- Hypothesis[Lokshtanov,Ramanujan,Saurabh,Zehavi'20]:

  No FPT algorithm decide a 2CSP parameterized by $k = |X|$ is:
  - Satisfiable, or
  - Cannot satisfy $s$-fraction of constraints simultaneously. $(0 < s < 1)$

- SOTA: Exponential Time Hypothesis -> PIH. [Guruswami,Lin,Ren,Sun,Wu'24]

- Major open problem: W[1]≠FPT -> PIH ?

# Weaken: Baby PIH [Guruswami,Ren,Sandeep'24]

- No FPT algorithm for deciding a 2CSP parameterized by $k = |X|$:
  - Being satisfiable, or
  - Cannot satisfy all constraints simultaneously
    even when **each** variable assigned $\leq t$ values. $(t > 1)$

- W[1]≠FPT -> Baby PIH. [Guruswami,Ren,Sandeep'24]
  - Following the method in [Barto,Kozik'22] showing Baby PCP without using PCP Theorem.

# Weaken: Baby PIH [Guruswami,Ren,Sandeep'24]

$x_1$: $\{0,\ 1,\ 3,\ 5,\ 7,\ 8,\ 9\}$

$x_2$: $\{2\}$

$x_3$: $\{2,\ 6\}$

$x_4$: $\{8\}$

$\leq t$

$(t = 7)$

# Question: Average Baby PIH

$|X| = 4,$

$x_1$: $\{0, 1, 3, 5, 7, 8, 9\}$

$x_2$: $\{2\}$

$x_3$: $\{2, 6\}$

$x_4$: $\{8\}$

$\leftarrow \leq t \rightarrow$

$(t = 7)$

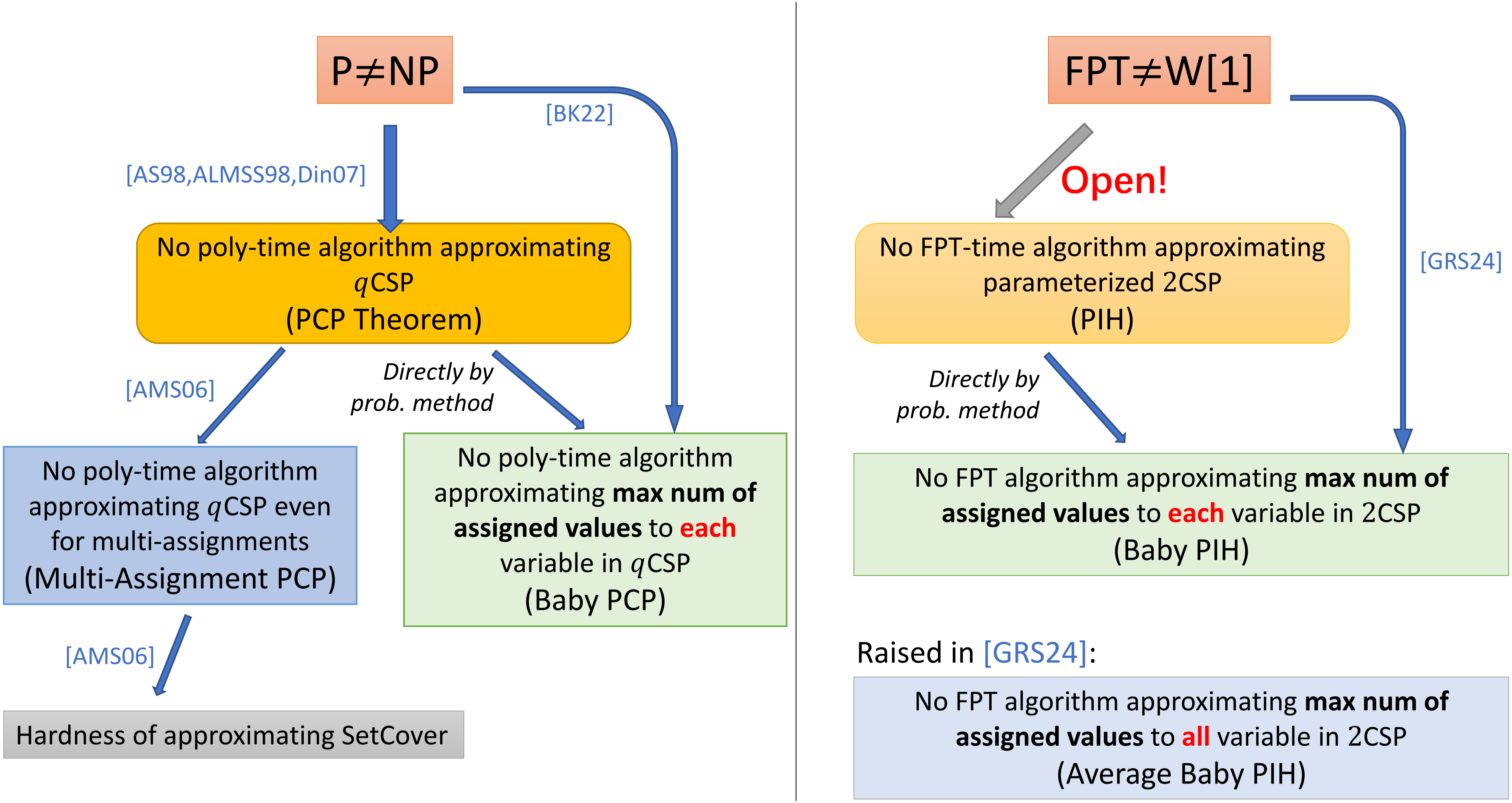
Total # of values: $7 + 1 + 2 + 1 = 11 = 2.75|X|$.

# Question: Average Baby PIH

- No FPT algorithm for deciding a 2CSP parameterized by $k = |X|$:
  - Being satisfiable, or
  - Cannot satisfy all constraints simultaneously even when assigning to $X$ less than $t|X|$ values **in total**. $(t > 1)$

$\ell_1$ instead of $\ell_\infty$

- Raised in [Guruswami,Ren,Sandeep'24].


P≠NP
[BK22]
[AS98,ALMSS98,Din07]
No poly-time algorithm approximating
$q$CSP
(PCP Theorem)
[AMS06]
Directly by
prob. method
No poly-time algorithm
approximating $q$CSP even
for multi-assignments
(Multi-Assignment PCP)
No poly-time algorithm
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(Baby PCP)
[AMS06]
Hardness of approximating SetCover
FPT≠W[1]
Open!
No FPT-time algorithm approximating
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(Baby PIH)
Raised in [GRS24]:
No FPT algorithm approximating **max num of**
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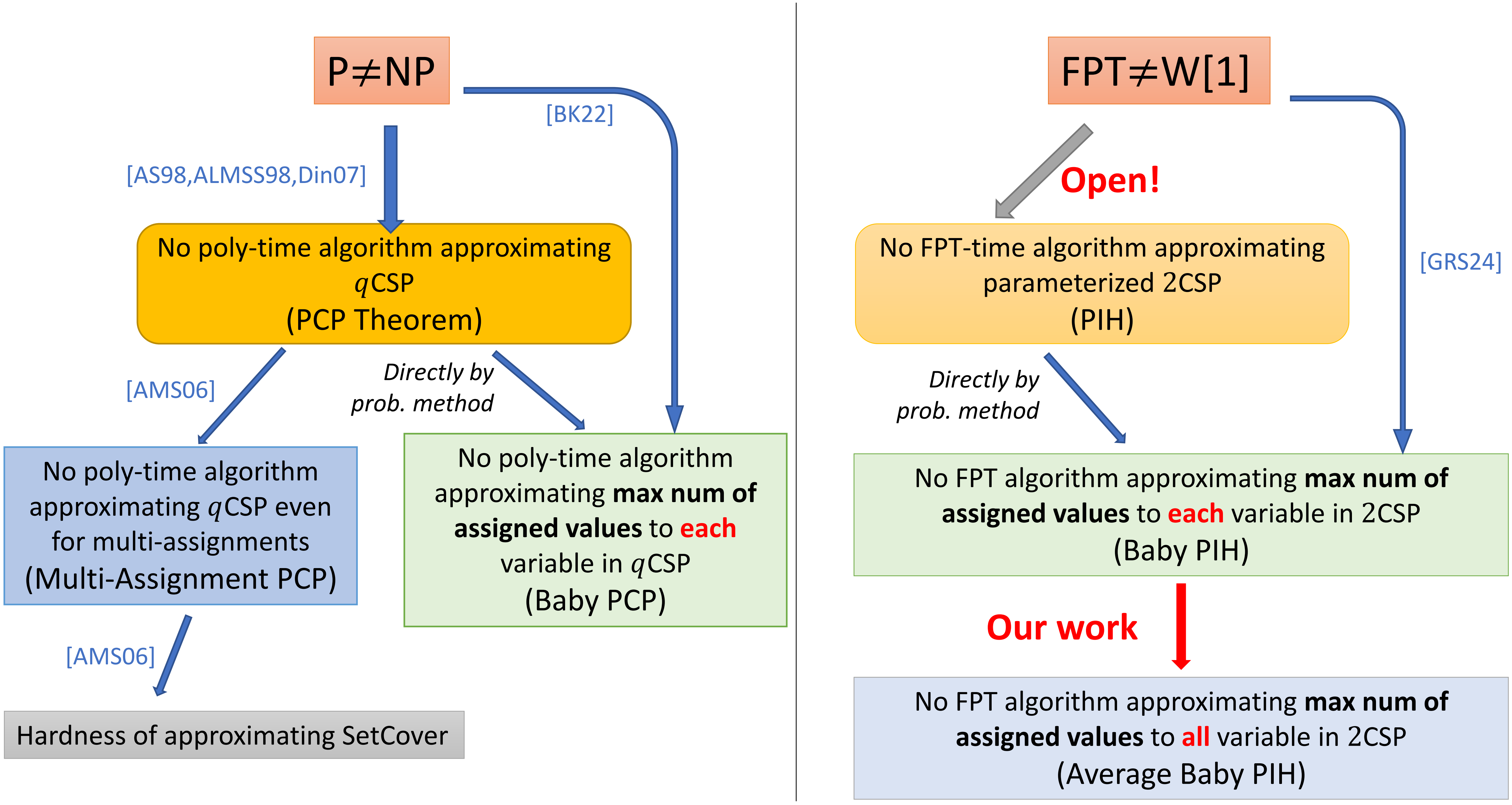
(Average Baby PIH)

# Our result

W[1]≠FPT ⟶ Average Baby PIH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No FPT algorithm approximating max num of assigned values to all variable in 2CSP
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W[1]≠FPT → Average Baby PIH

- A reduction for 2CSP instances that:

| Baby PIH | | Average Baby PIH |
|---|---|---|
| Satisfiable | → | Satisfiable |
| Can't satisfied when **each** variable assigned $\leq t$ values. | → | Can't satisfied when assigning to $X$ less than $\frac{t}{2}\lvert X\rvert$ values **in total**. |

Baby PIH —Our work→ Average Baby PIH

W[1]≠FPT → Baby PIH [Guruswami,Ren,Sandeep'24]

# Technical Tool

- Error-correcting codes with ***overwhelming*** (relative) distance

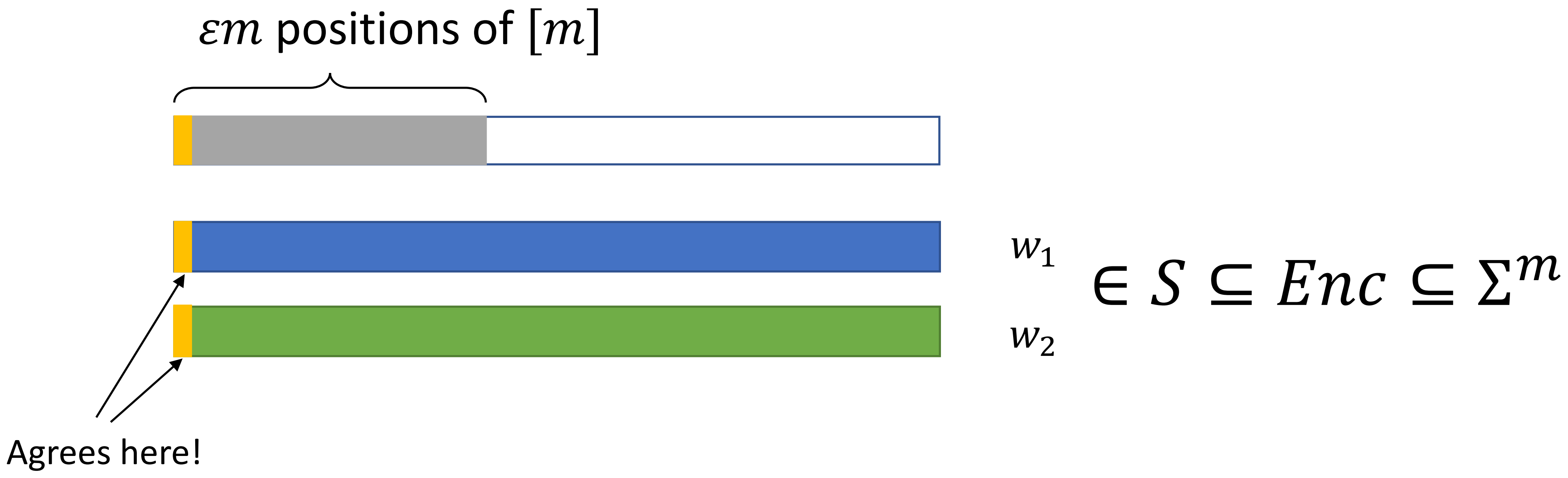

$\in Enc \subseteq \Sigma^m$

Differs on $(1 - \frac{1}{m^{0.1}})$-fraction positions

e.g. Reed-Solomon codes.

# Technical Tool

Any set $S$ of codewords that "collides" on a noticeable fraction of positions......



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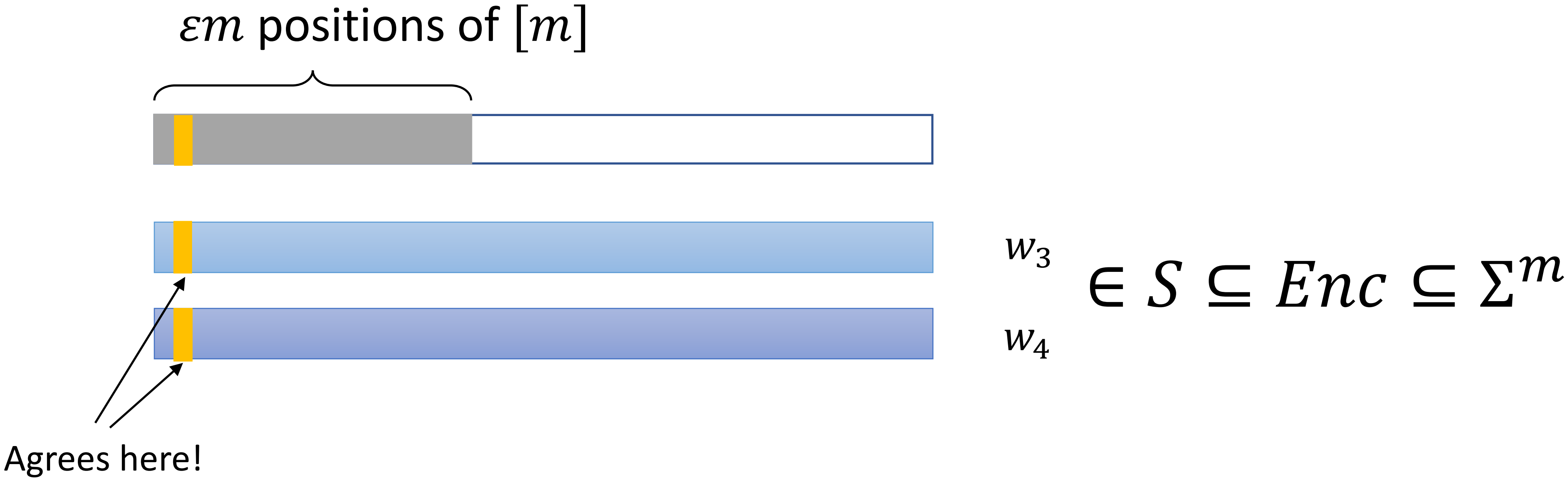

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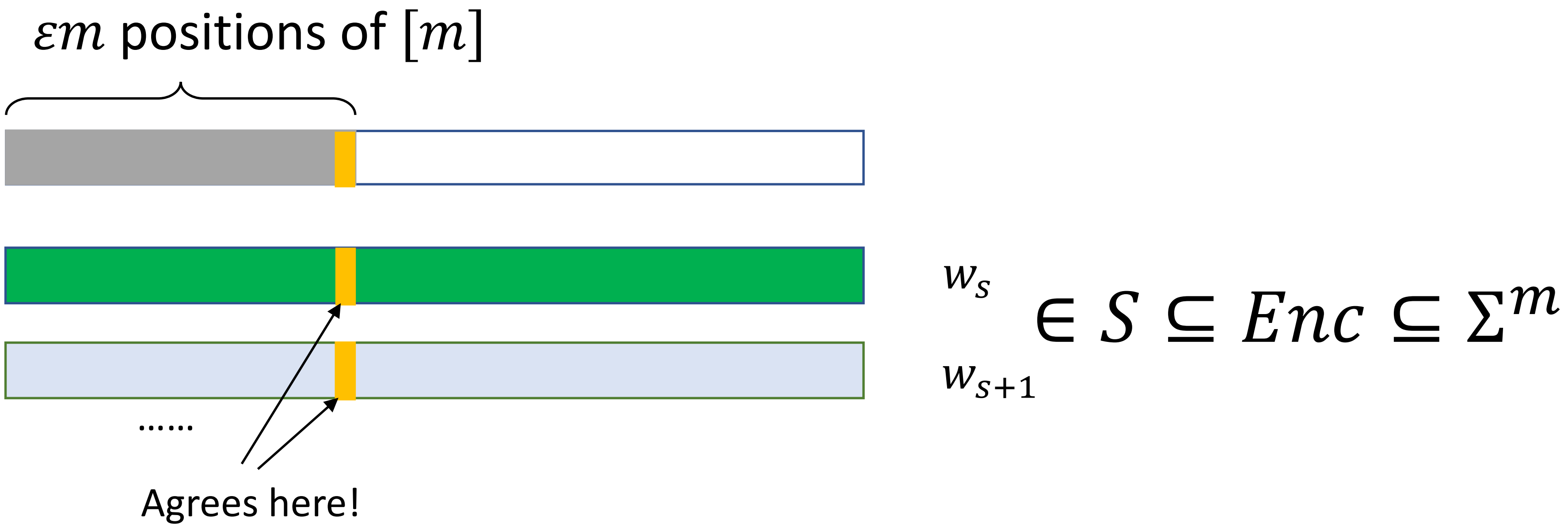
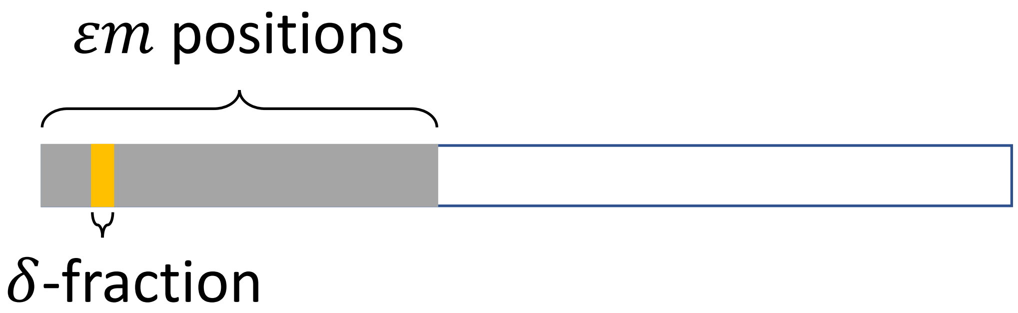


# Technical Tool




Theorem(Informal) cf. [Karthik-Navon'21, Lin-Ren-Sun-Wang'23]:
For code $Enc$ with relative distance $1-\delta$, any set of codewords "collides" on $\varepsilon m$ positions must have size $\geq \sqrt{\frac{2\varepsilon}{\delta}}$.

Recall: W[1]≠FPT → Average Baby PIH

- A reduction for 2CSP instances that:

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Baby PIH —Our work→ Average Baby PIH

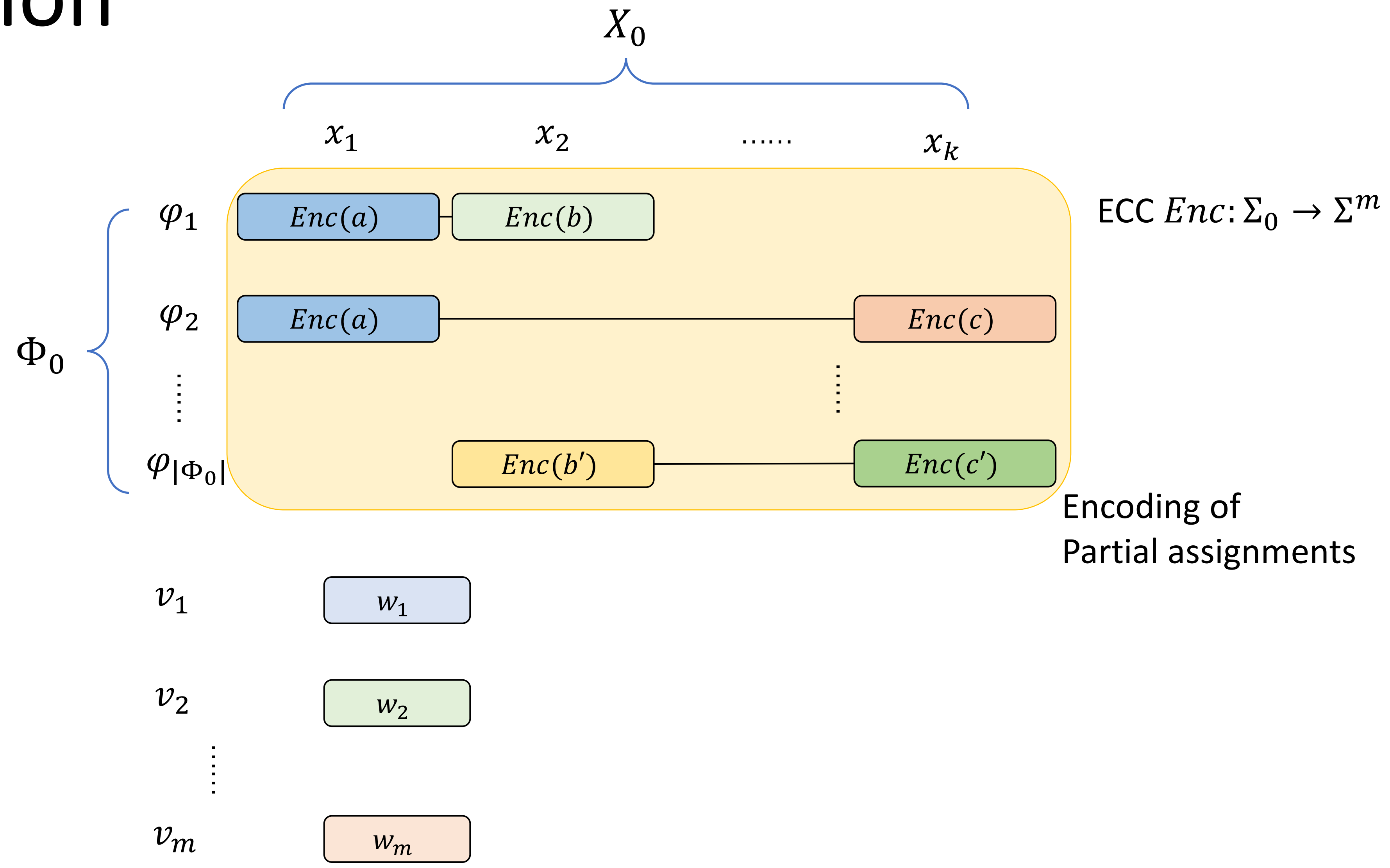
W[1]≠FPT → Baby PIH [Guruswami,Ren,Sandeep'24]

# The Reduction

Input: 2CSP instance
$\Pi_0 = (X_0, \Sigma_0, \Phi_0)$

Output: 2CSP instance $\Pi$
as shown.

**Variables**: $\Phi_0 \cup \{v_1, \dots, v_m\}$



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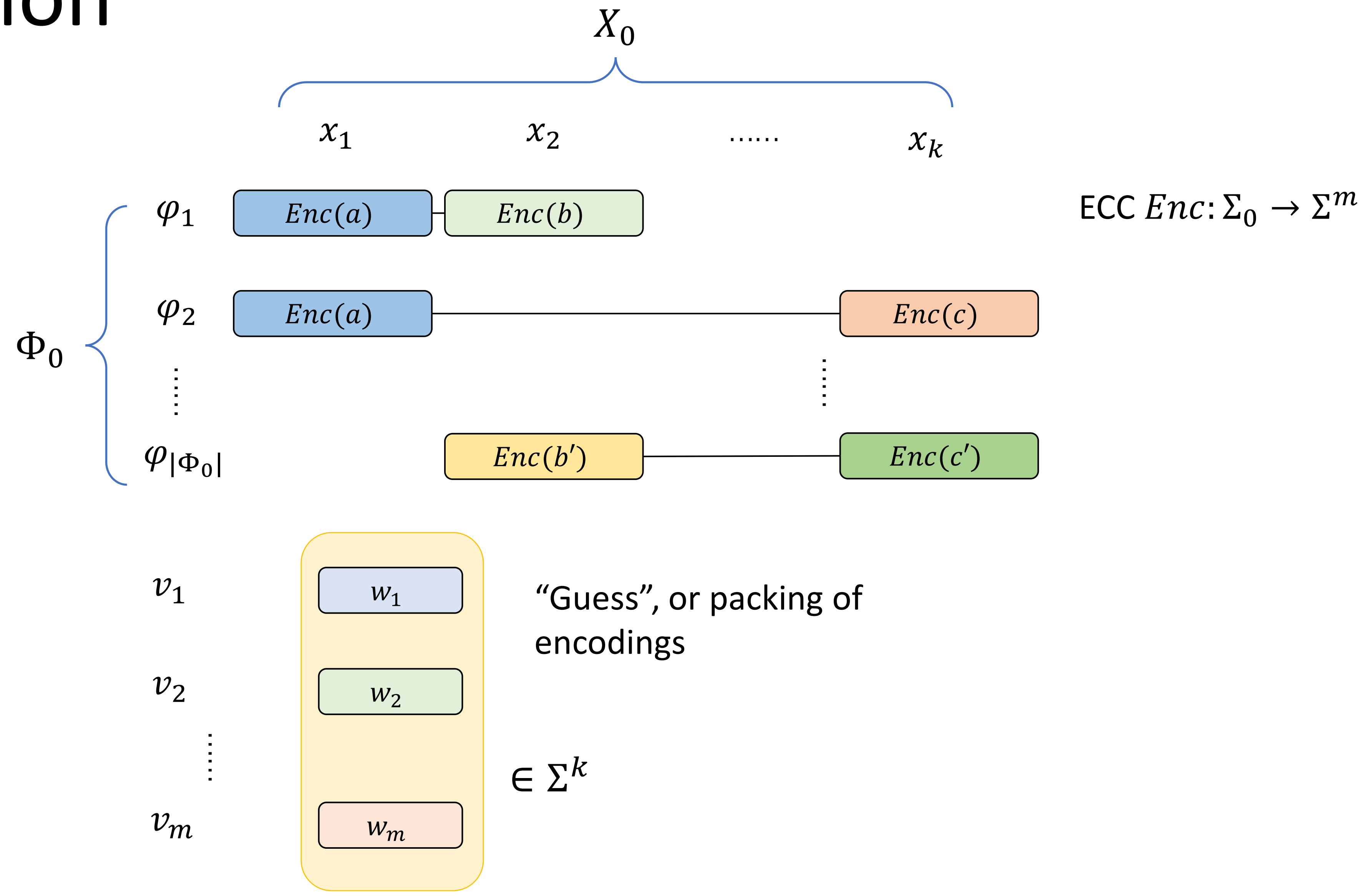

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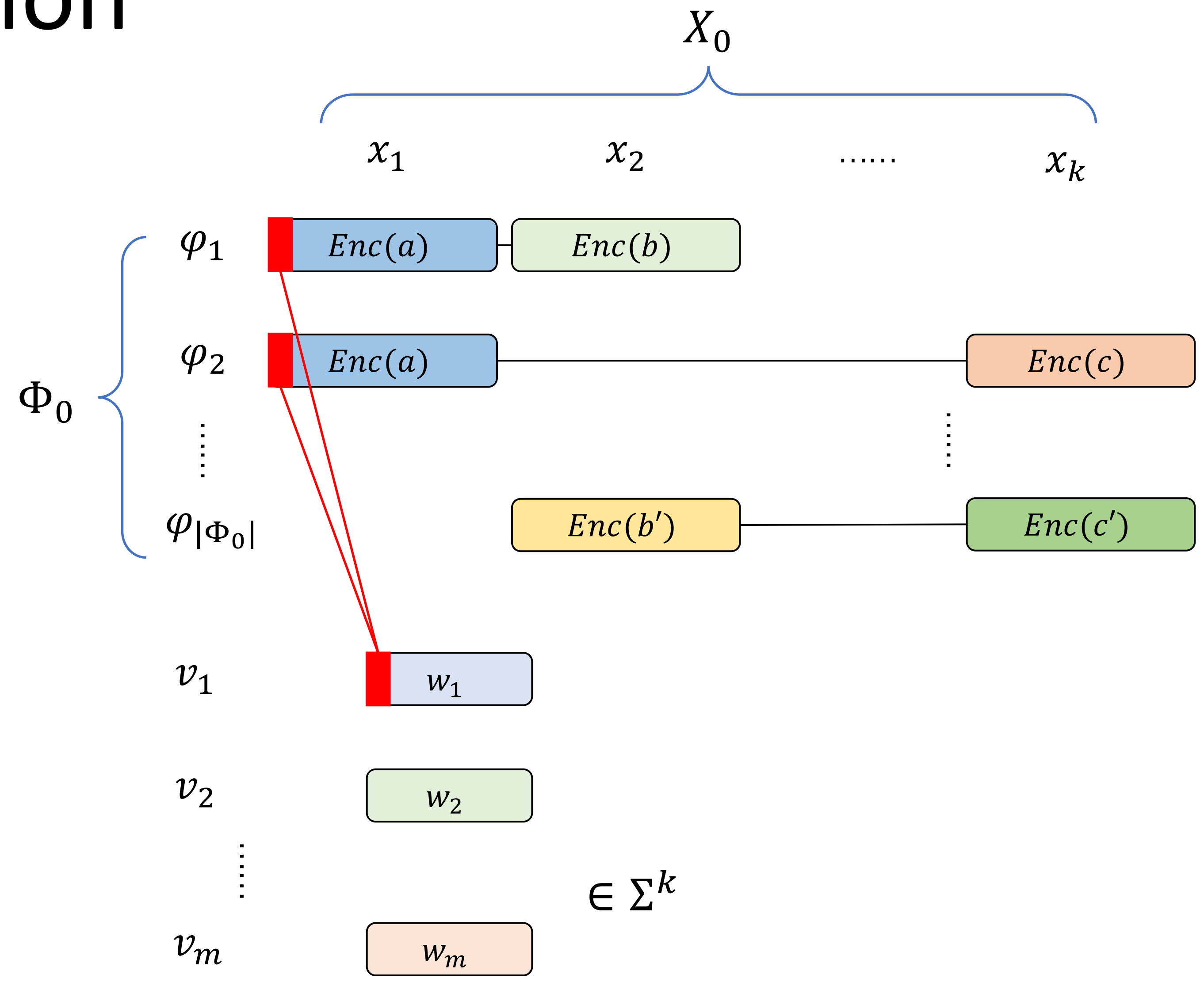
**Constraints: Equality Check**




ECC $Enc: \Sigma_0 \to \Sigma^m$

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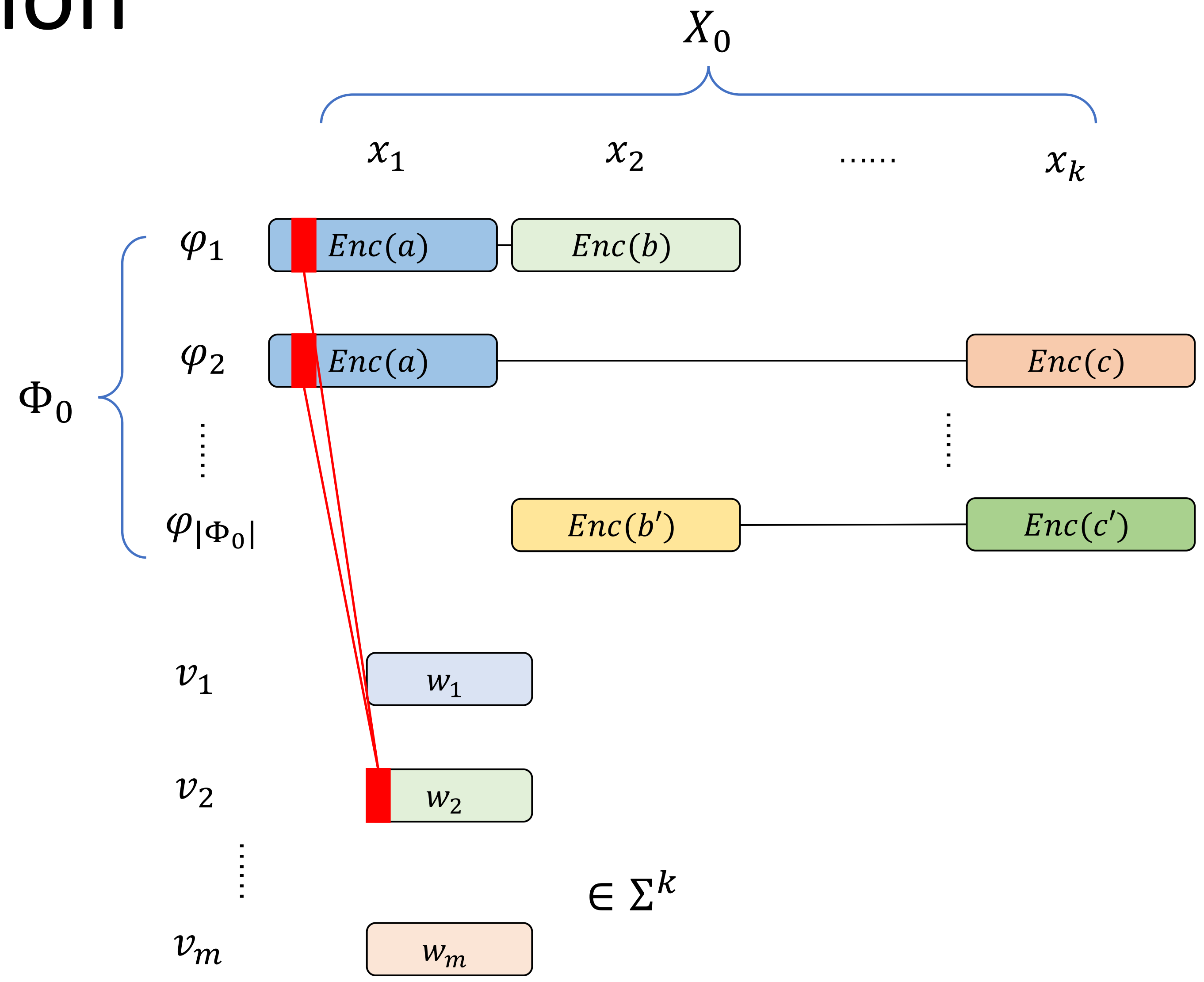


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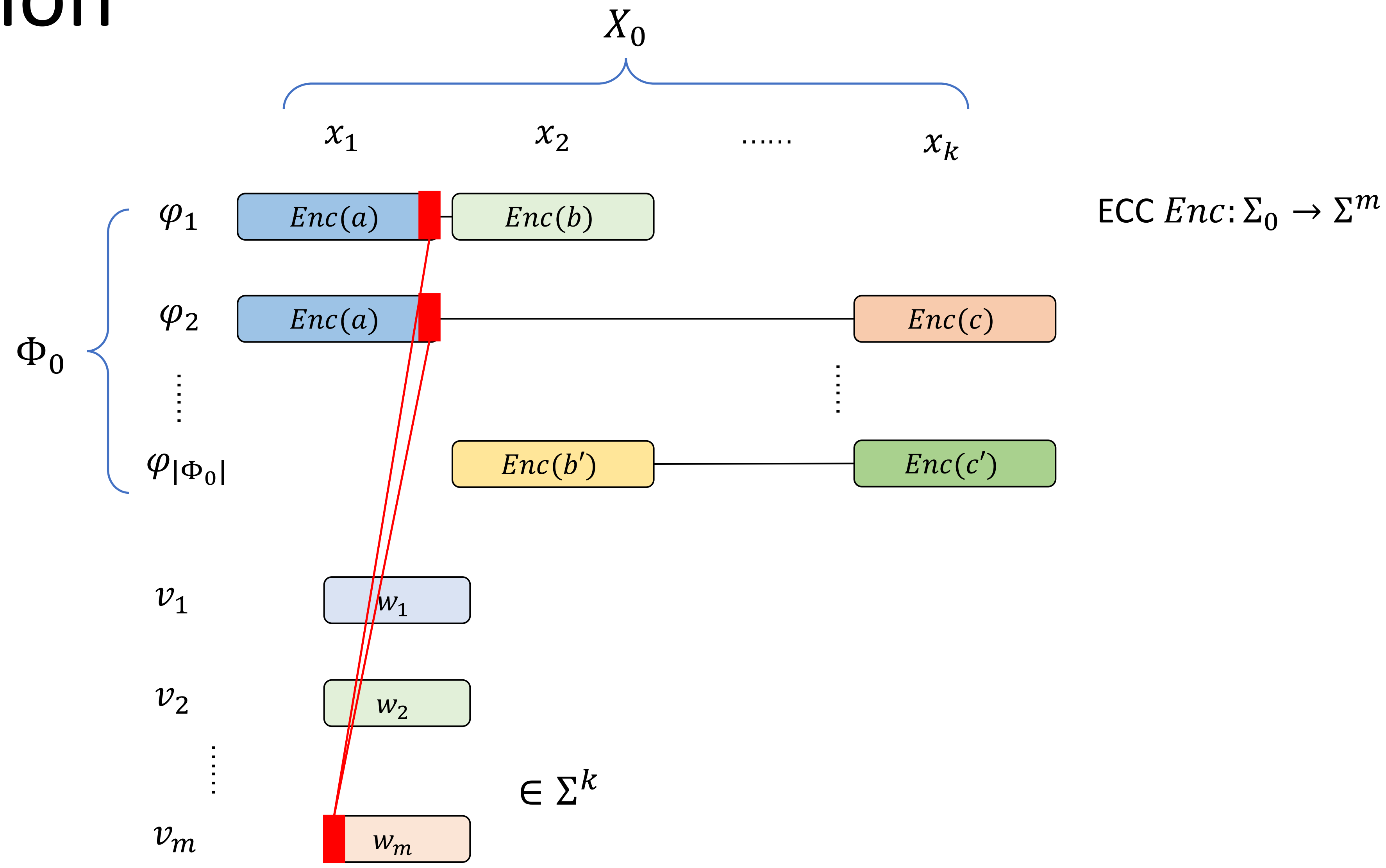

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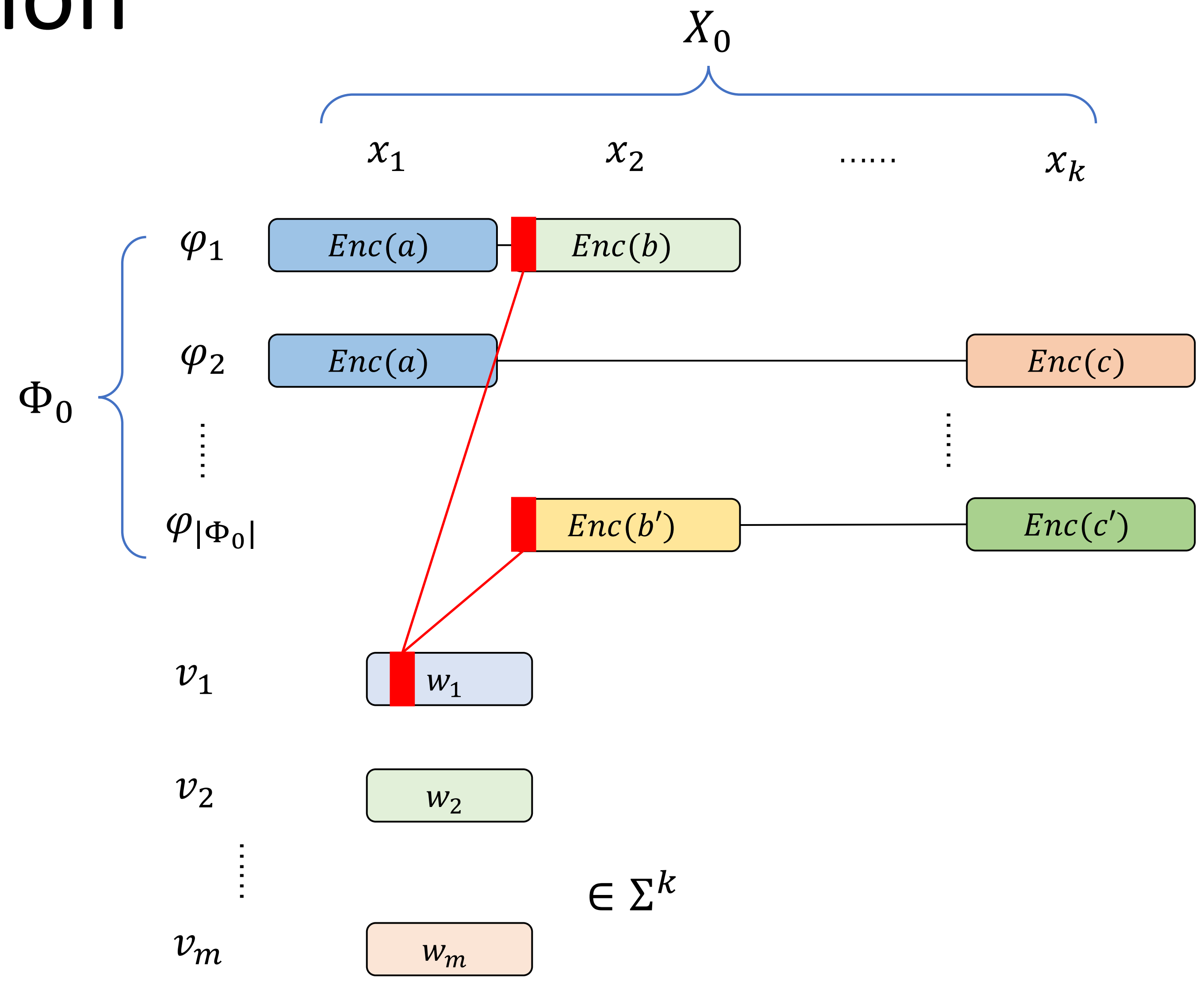


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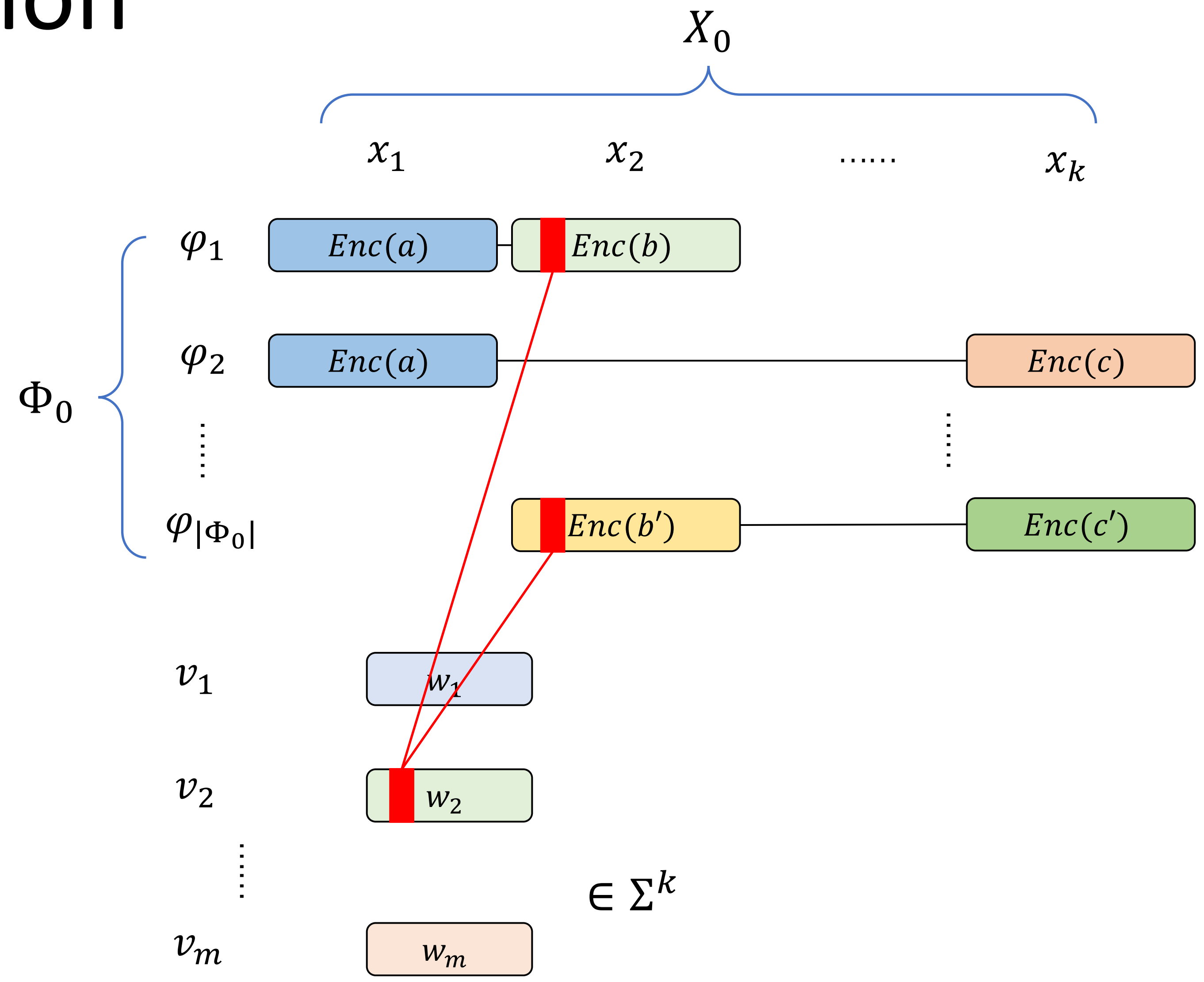


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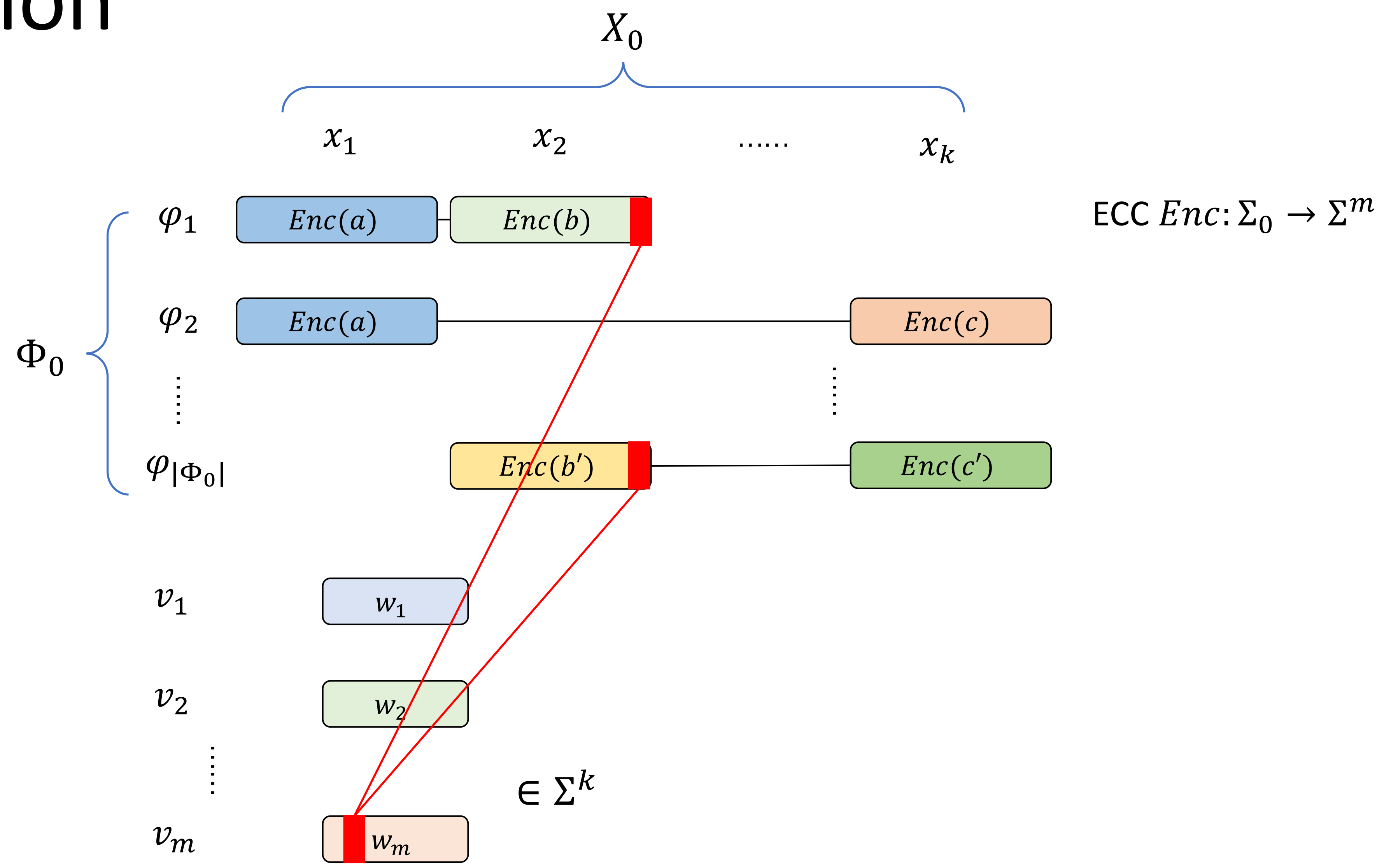

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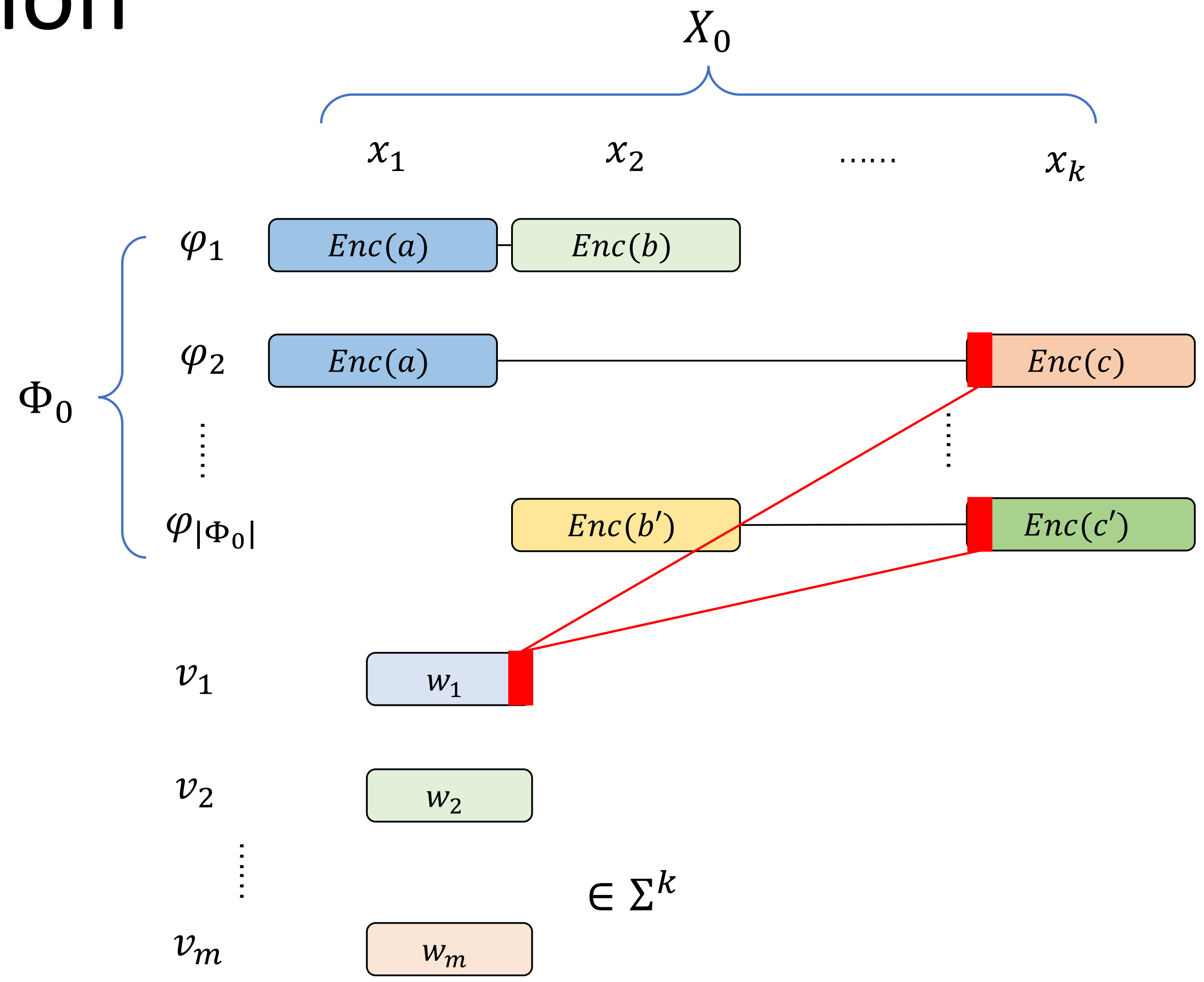


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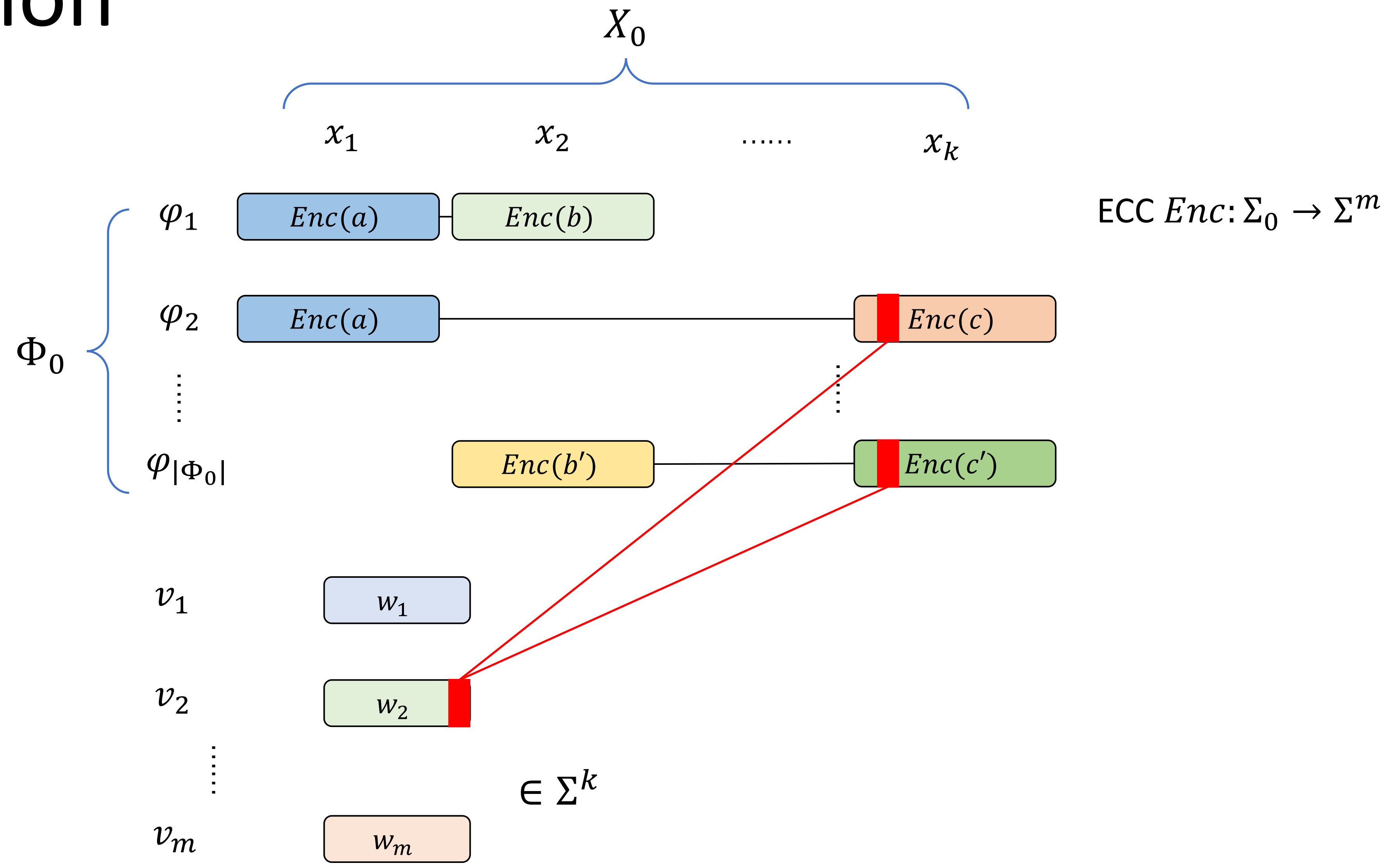

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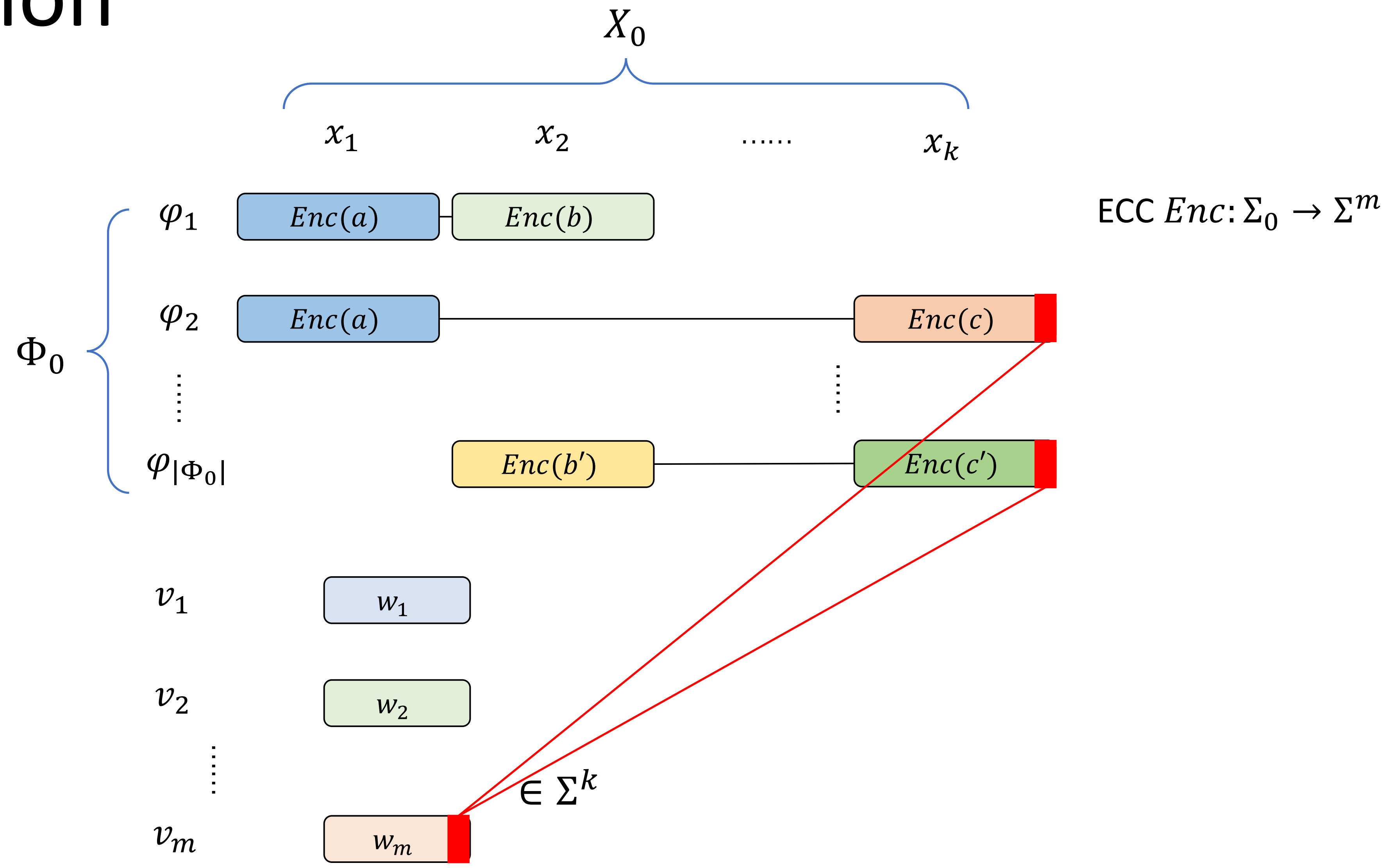


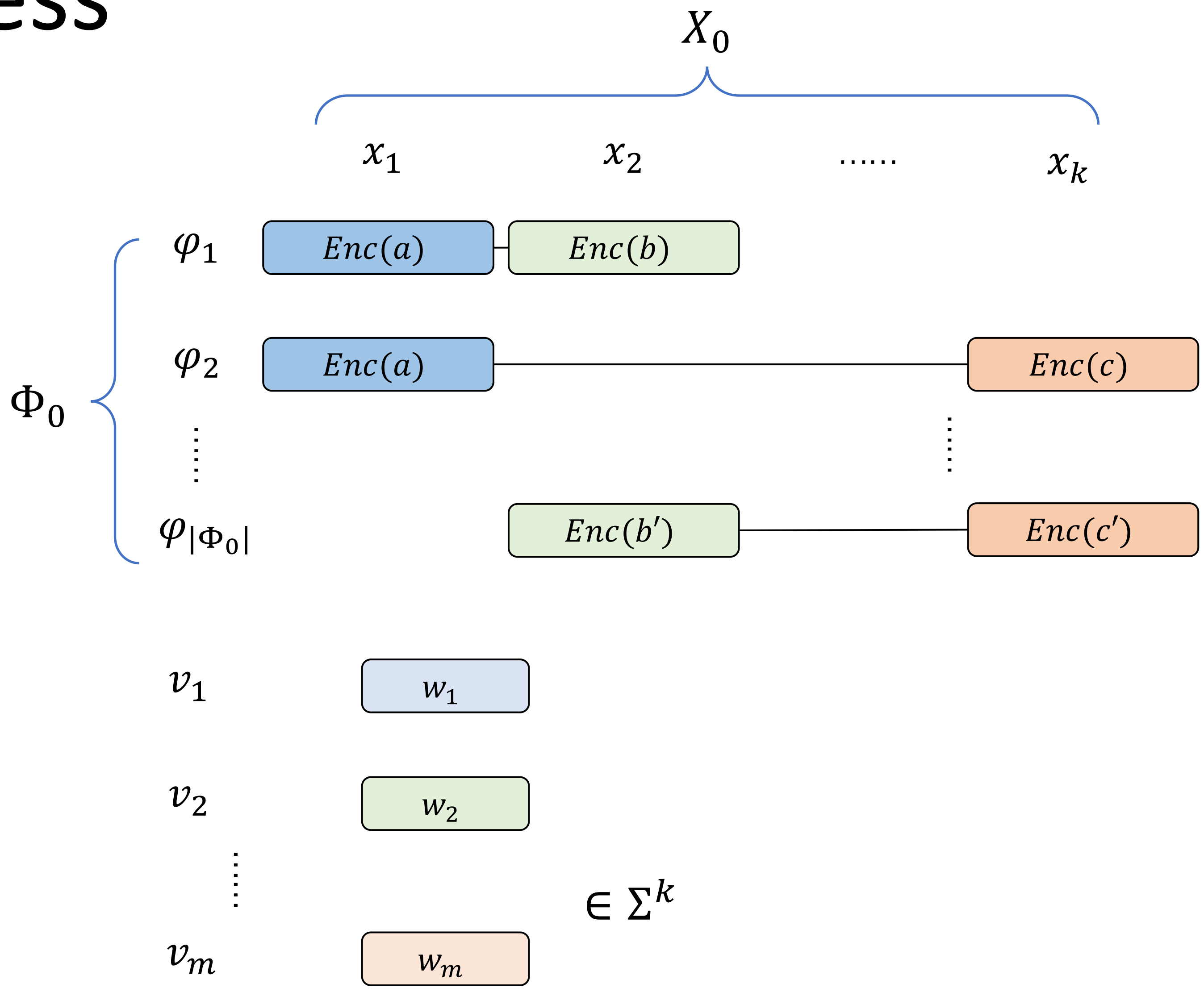
# Completeness

$\Pi_0 = (X_0, \Sigma_0, \Phi_0)$

Satisfiable

Direct

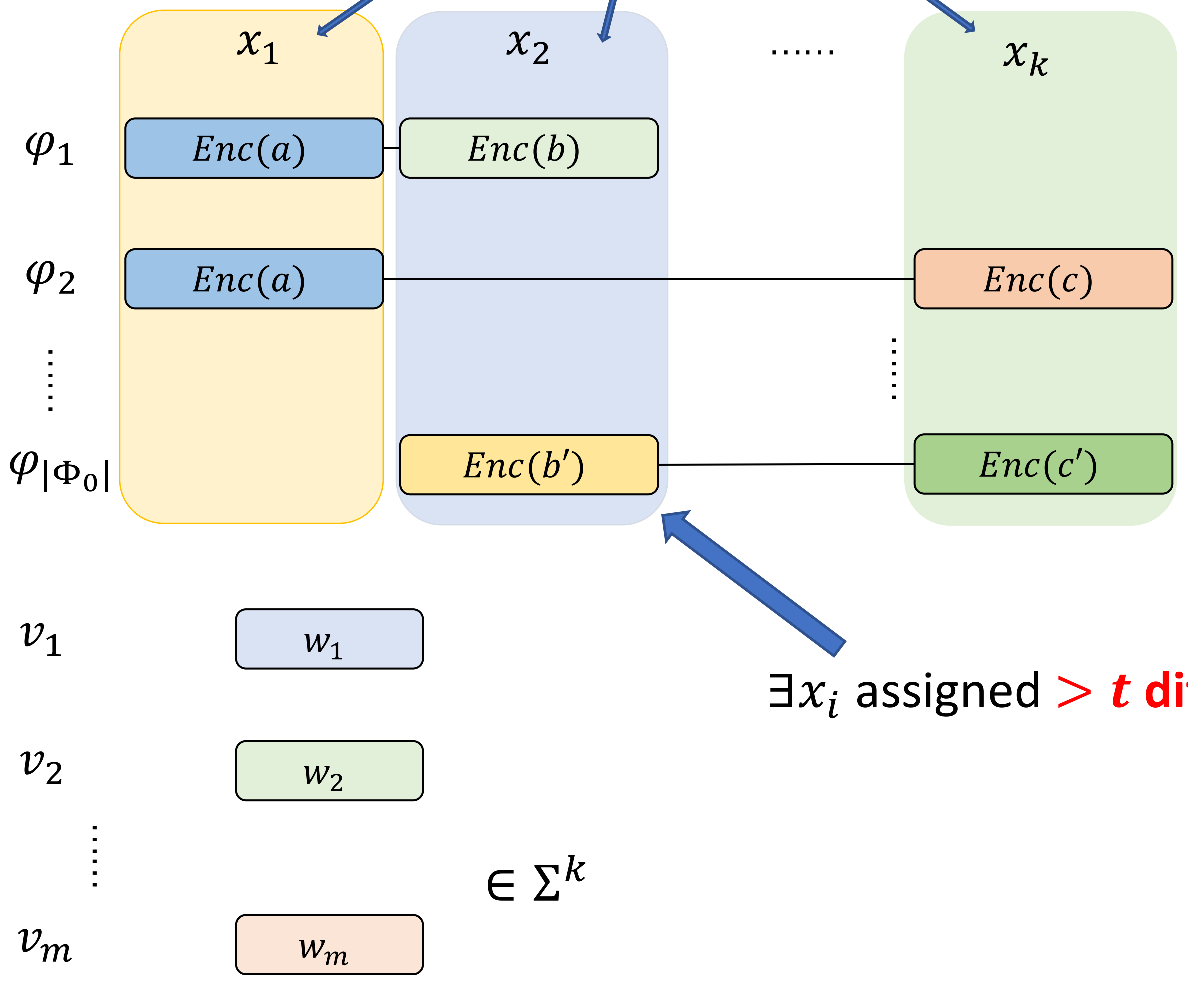
$\Pi$ Satisfiable



# Soundness

$\Pi_0 = (X_0, \Sigma_0, \Phi_0)$
Can't satisfied when **each** variable assigned $\leq t$ values

A **satisfying** multi-assignment for $\Pi_0$




**Recall**: Encoding of Partial assignments

$\exists x_i$ assigned $>$ $t$ **different** values

# Soundness

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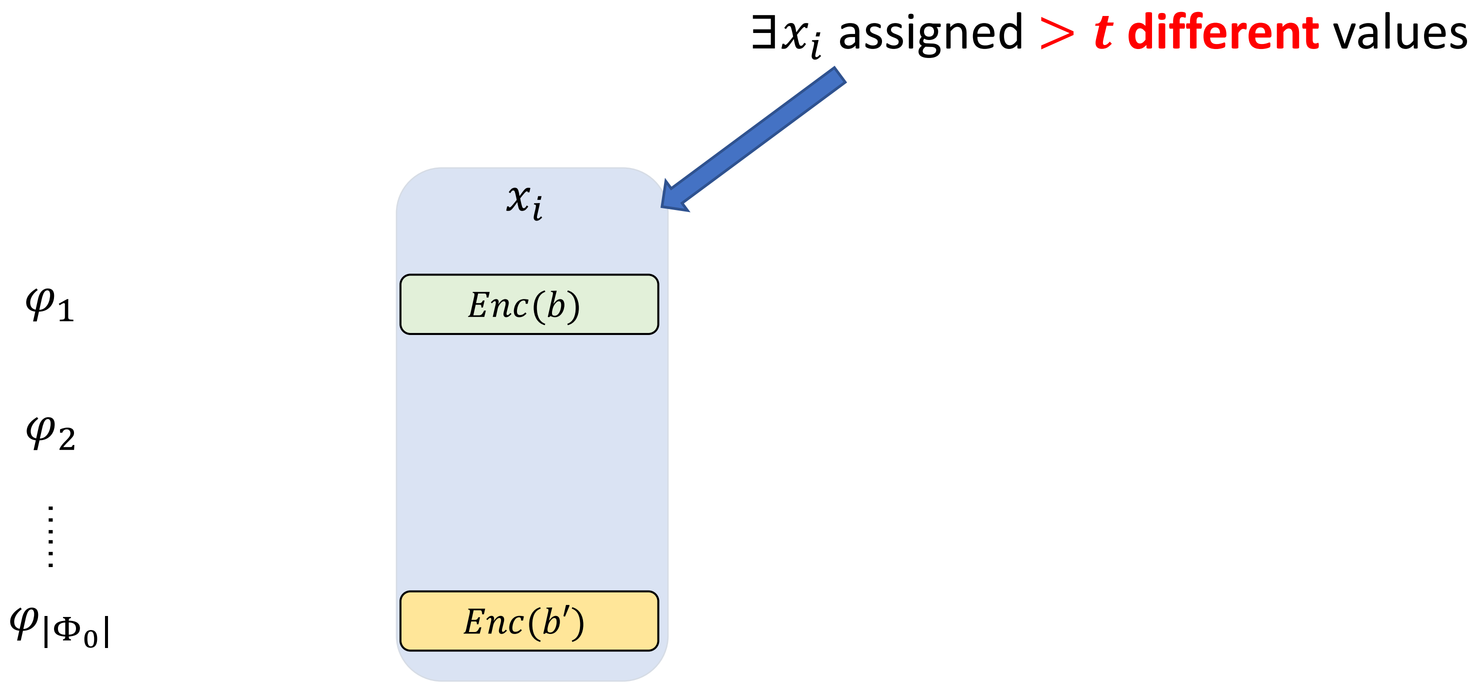
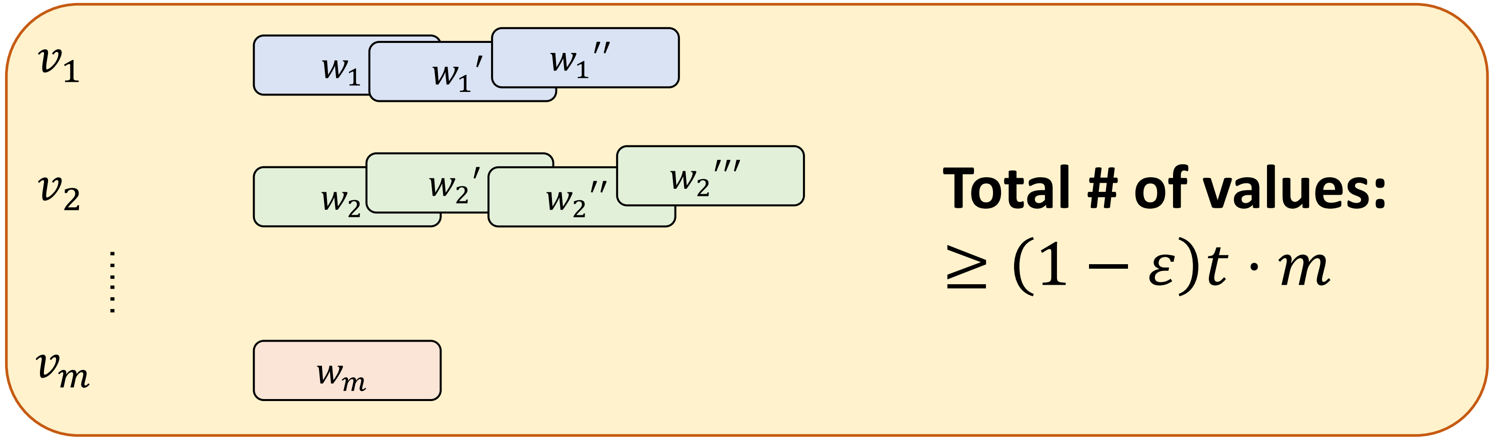



Case 1:
**More than $(1-\varepsilon)$ fraction** of $v$'s, each assigned **$t+1$ values**



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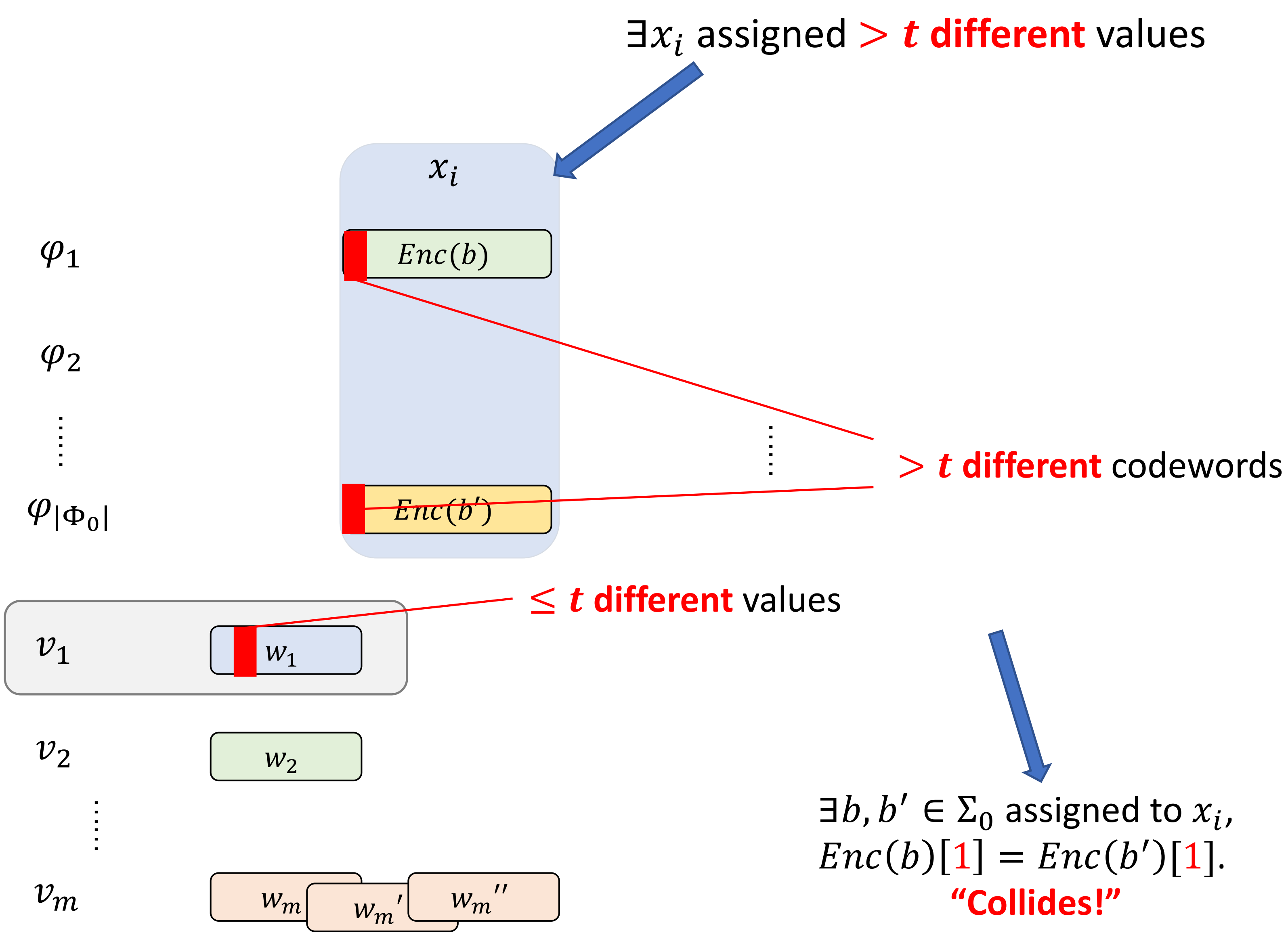


Case 2:
**More than $\varepsilon$ fraction** of $v$'s, assigned $\leq$ ***t* values**

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$\Pi_0 = (X_0, \Sigma_0, \Phi_0)$
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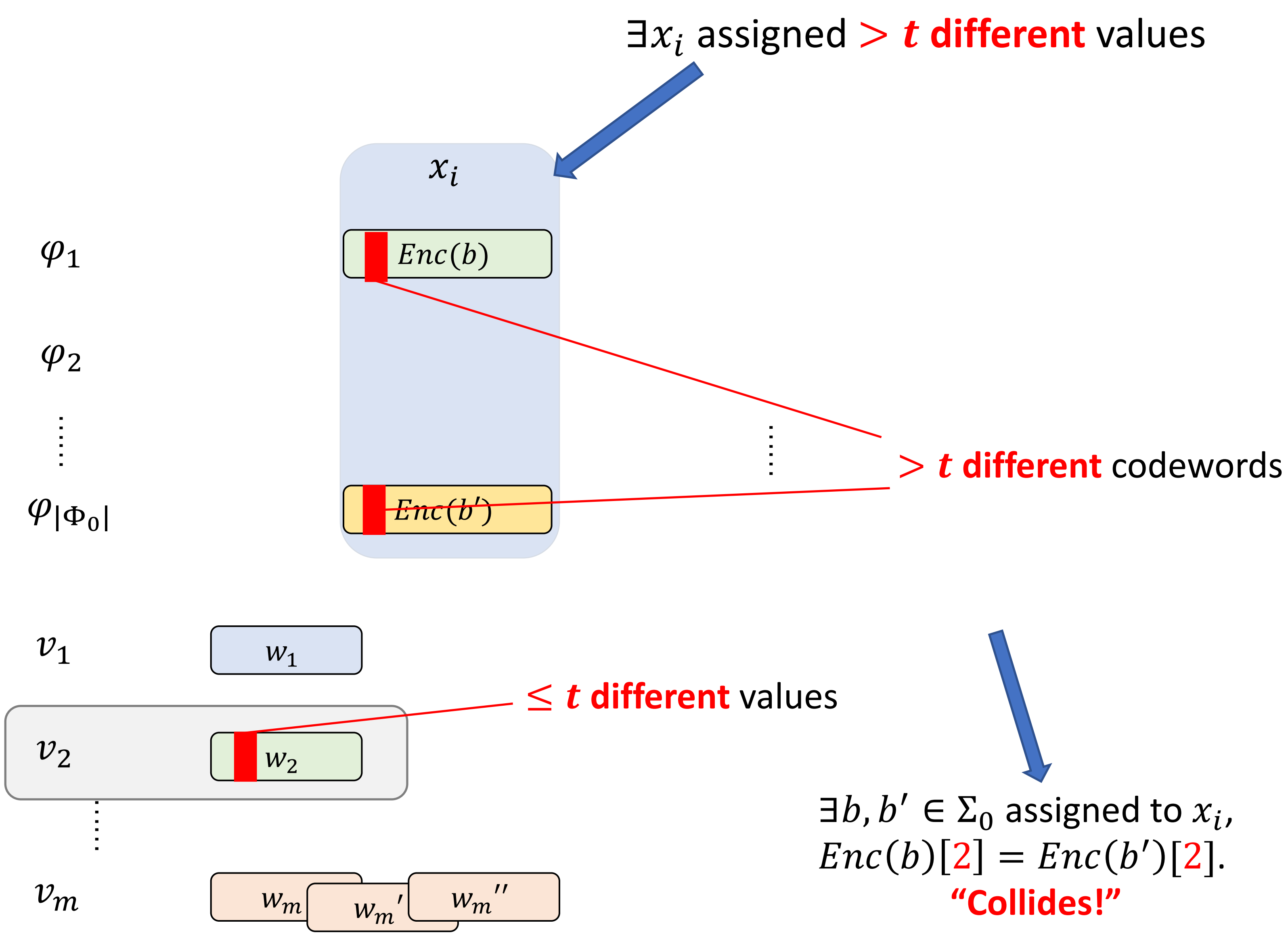
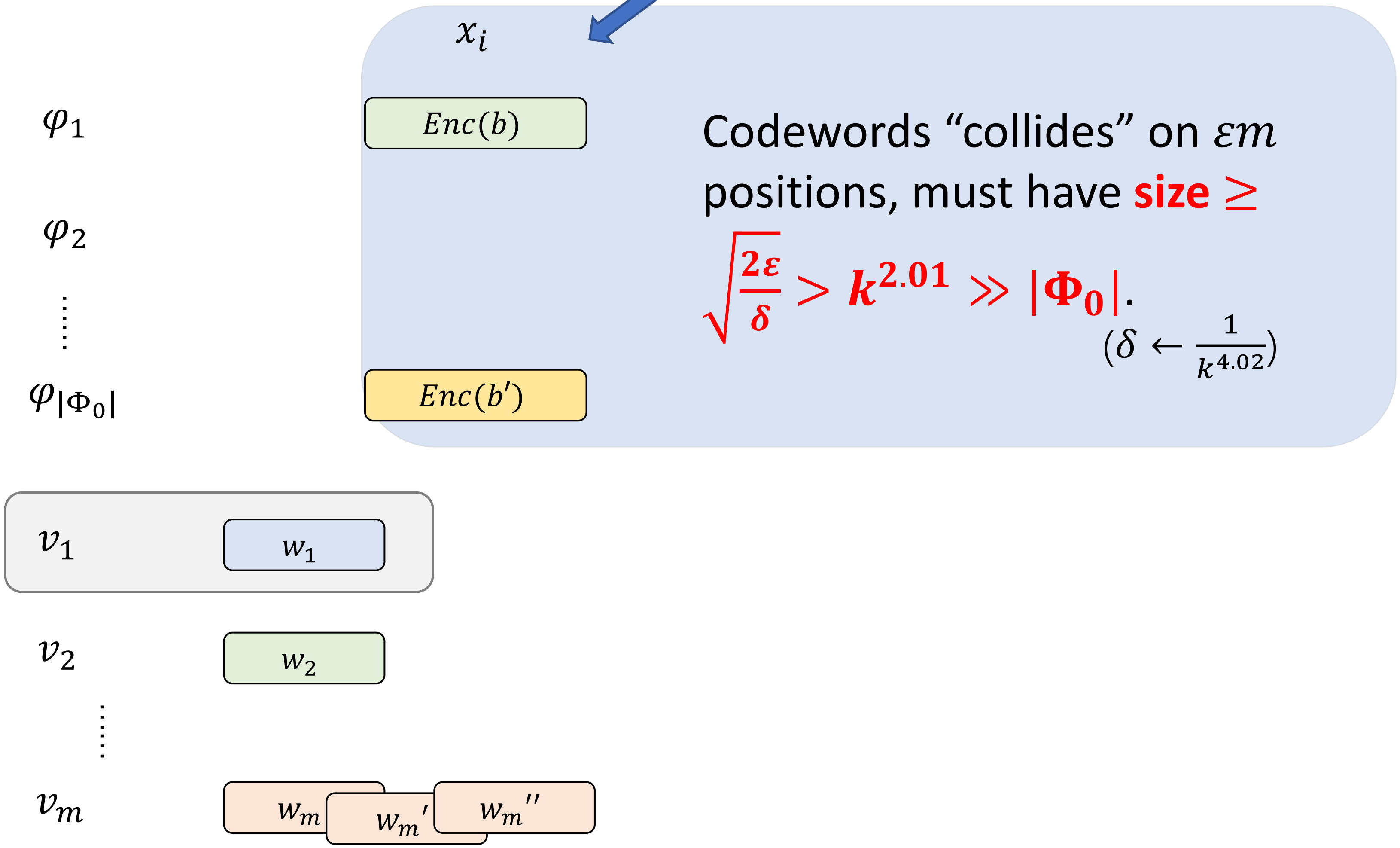
Case 2:
**More than $\varepsilon$ fraction** of $v$'s, assigned **$\leq t$ values**



# Soundness

$\exists x_i$ assigned $>$ ***t* different** values

$\Pi_0 = (X_0, \Sigma_0, \Phi_0)$
Can't satisfied when **each** variable assigned $\leq t$ values

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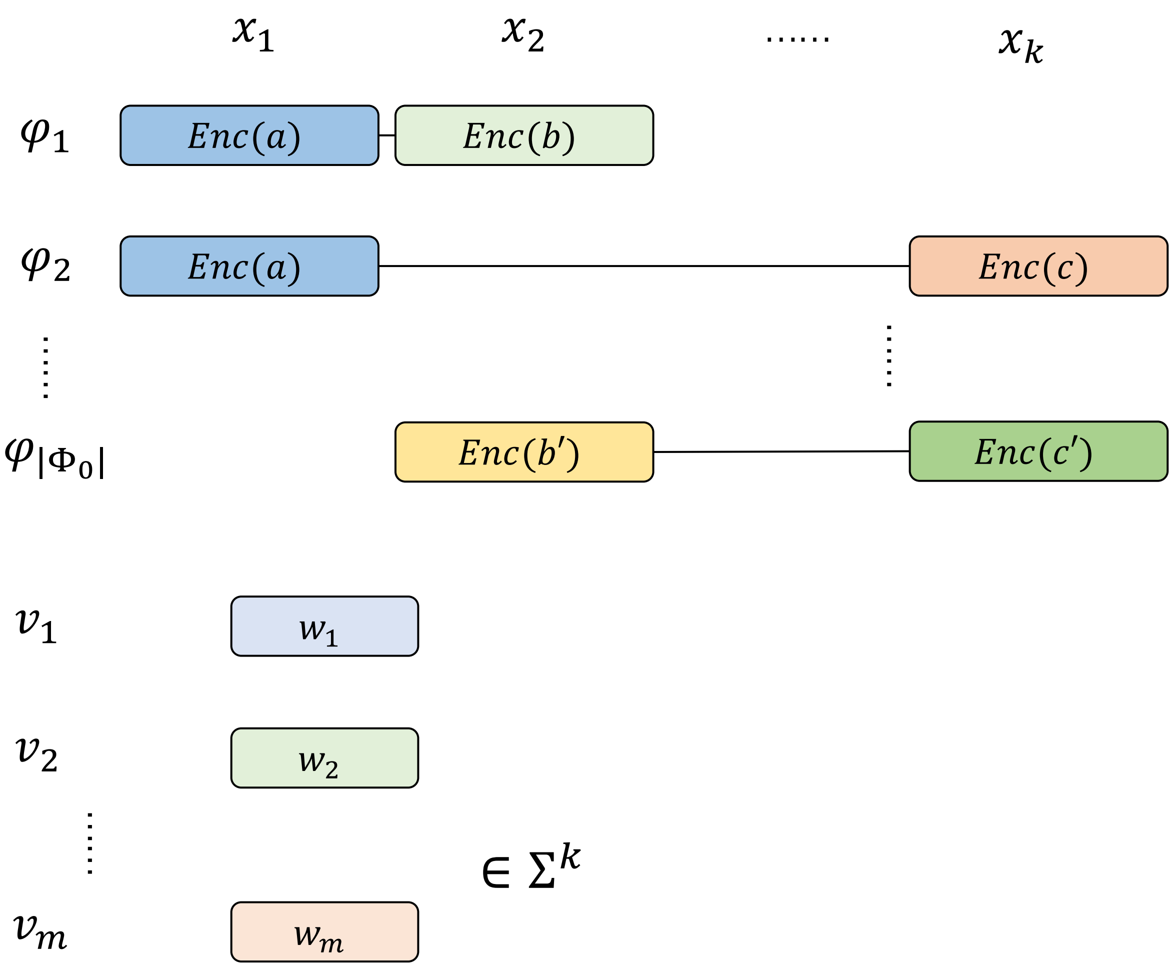
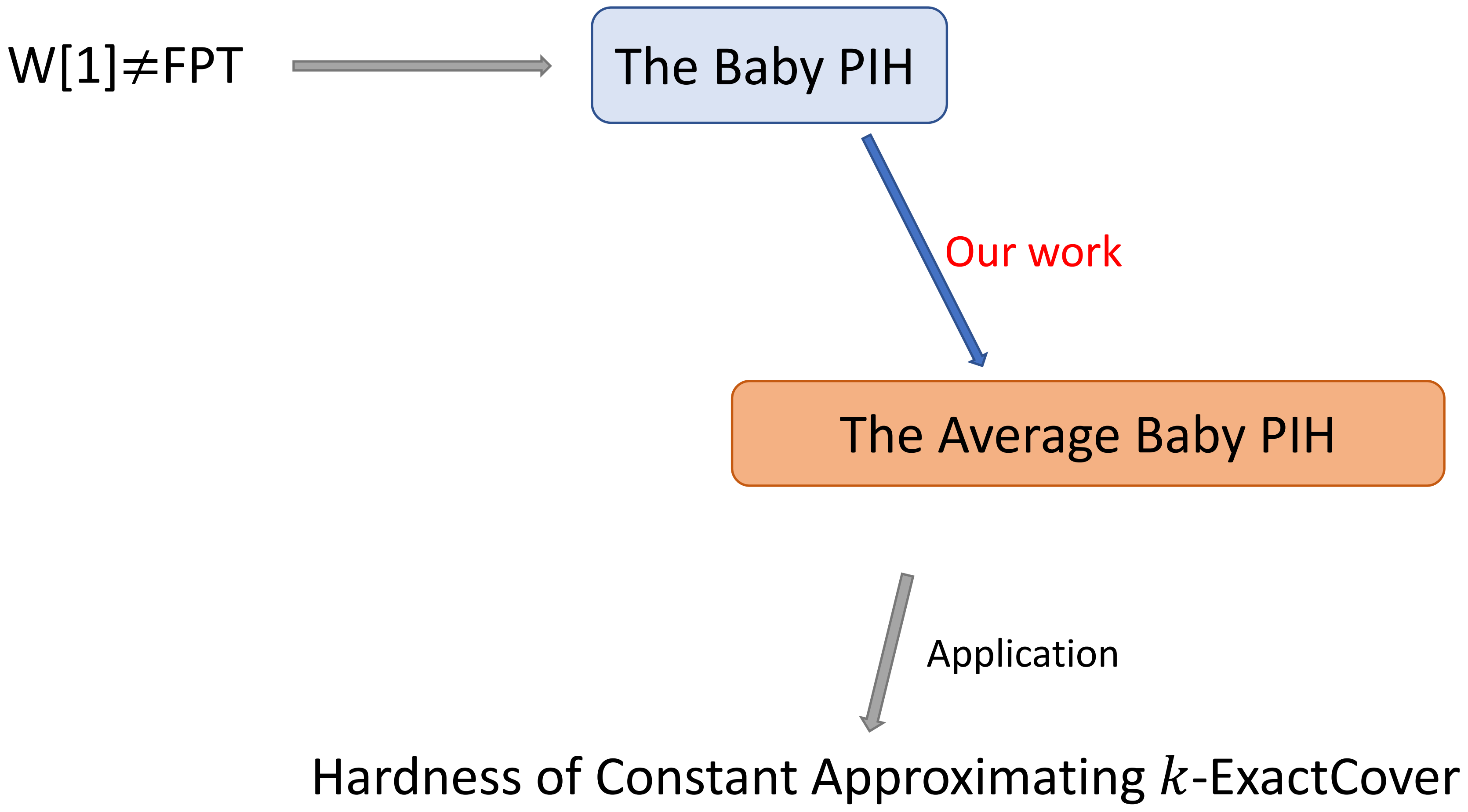
# Soundness

$\Pi_0 = (X_0, \Sigma_0, \Phi_0)$
Can't satisfied when **each** variable assigned $\leq t$ values

$\Pi$ Can't satisfied when assigning to $X$ less than $\min(\frac{t}{2}|X|, k^2)$ values **in total**.



# Conclusion


W[1]≠FPT
The Baby PIH
Our work
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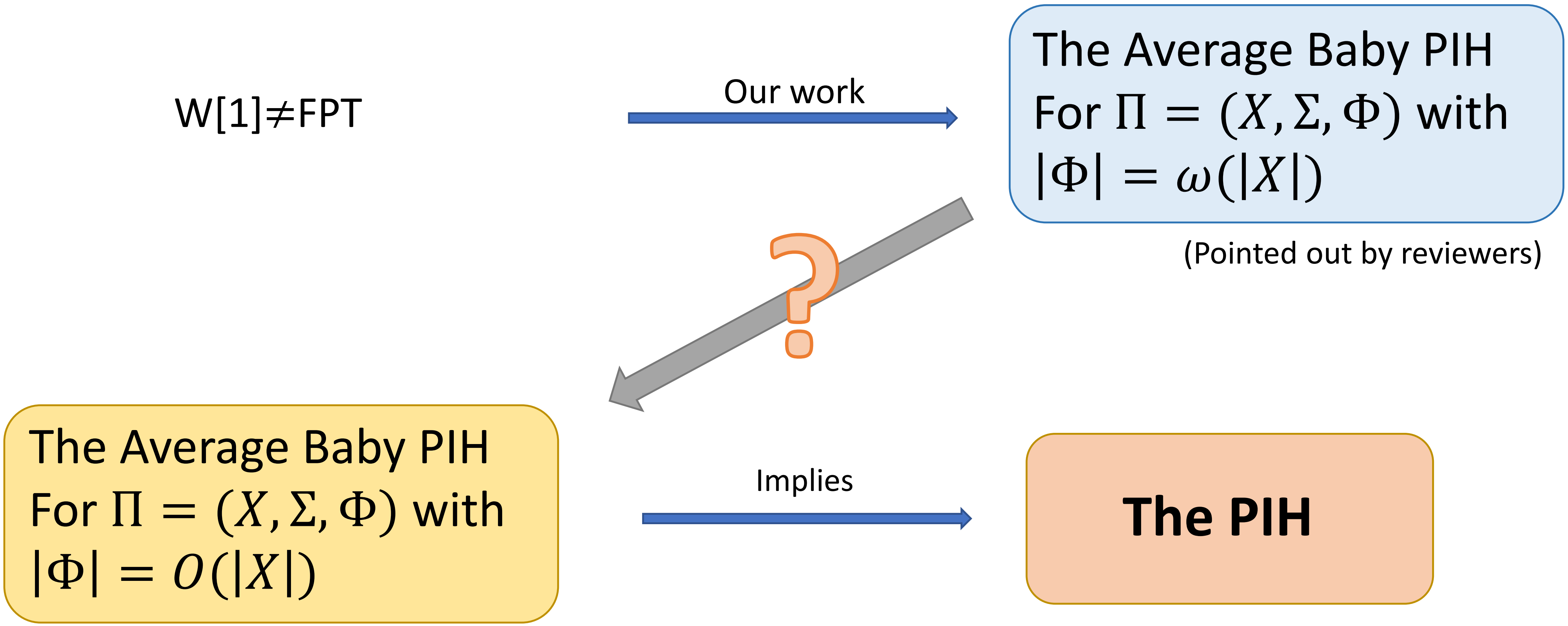
Application
Hardness of Constant Approximating $k$-ExactCover

# Open Question



# Thank You!