On read-k projections of the determinant

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The permanent versus determinant problem

X – an $n \times n$ matrix with variables $x_{i,j}$ as entries.

$$\begin{aligned} \mathrm{det}_n(X) &= \sum_{\sigma} \mathrm{sgn}(\sigma) \prod_{i=1}^{n} x_{i,\sigma(i)} \,, \\ \mathrm{perm}_n(X) &= \sum_{\sigma} \prod_{i=1}^{n} x_{i,\sigma(i)} \,. \end{aligned}$$

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$$\mathrm{perm}_n(X) = \sum_{\sigma} \prod_{i=1}^{n} x_{i,\sigma(i)} .$$

Question. [Pólya] Is there a simple expression of permanent in terms of the determinant?

Question. [Valiant] What is the smallest $m = m(n)$ so that we can express

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where M is an $m \times m$ matrix with variables or scalars as entries?

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- $m(n)$ is conjectured to be exponential in n.
- $m(n) \geq \Omega(n^2)$ [Mignon-Ressayre].

Our question. Can we have a representation

$$\mathrm{perm}_n(X) = \det_m(M)\,,$$

such that every variable appears a small number of times in M?

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- We also present an explicit n-variate multilinear polynomial for which the same bound on determinantal representation holds.

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- Given a polynomial f, is it easier to express f as $\mathrm{perm}(M)$ than $\det(M)$?

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1. *There exists a multilinear polynomial $f \in \mathbb{F}[x_1, \ldots, x_n]$ such that every M with*

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2. *If $\mathrm{char}(\mathbb{F}) \neq 2$ then every multilinear $f \in \mathbb{F}[x_1, \ldots, x_n]$ can be expressed as*
$$f = \mathrm{perm}(M)$$
where M is an $m \times m$ matrix with $m \leq O(2^n)$ and each variable x_i appears in M exactly once.

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where M is an $m \times m$ matrix with $m \leq O(2^n)$ and each variable x_i appears in M exactly once. Moreover, every row and column of M contains at most one variable.

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- Suppose $perm_n(X) = \det_m(M)$.
- Set $k \sim \log n$. Every multilinear $f \in \mathbb{F}[x_{11}, \ldots, x_{kk}]$ can be written as

$$f = \mathrm{perm}_n(X_f)$$

  where X_f is obtained by setting variables *outside* of $\{x_{11}, \ldots, x_{kk}\}$ to constants in X.

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- But some such f requires determinantal representation of variable size $\Omega(2^{k/2}) \implies$ the variables $x_{11}, \ldots, x_{kk}$ must appear in M at least $\Omega(2^{k/2})$ times.

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- The same applies to any subset of X not sharing a row or column $\implies M$ contains $\Omega(\frac{n^2}{k}2^{k/2})$ variables.

Thank you