PARAMETERIZED SAGA
OF
FIRST-FIT & LAST-FIT COLORING

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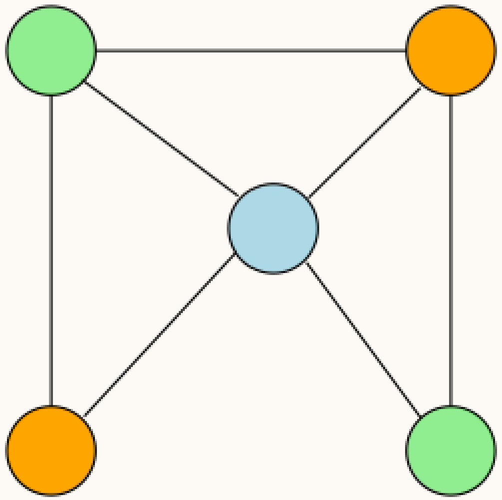
Proper Vertex Coloring

Input: A graph $G := (V, E)$

Goal: To color the vertices of the graph such that no two adjacent vertices receive the same color with as few colors as possible

Minimum number of colors used: Chromatic Number, $\chi(G)$

- NP-complete in general graphs



Graph Coloring Heuristics

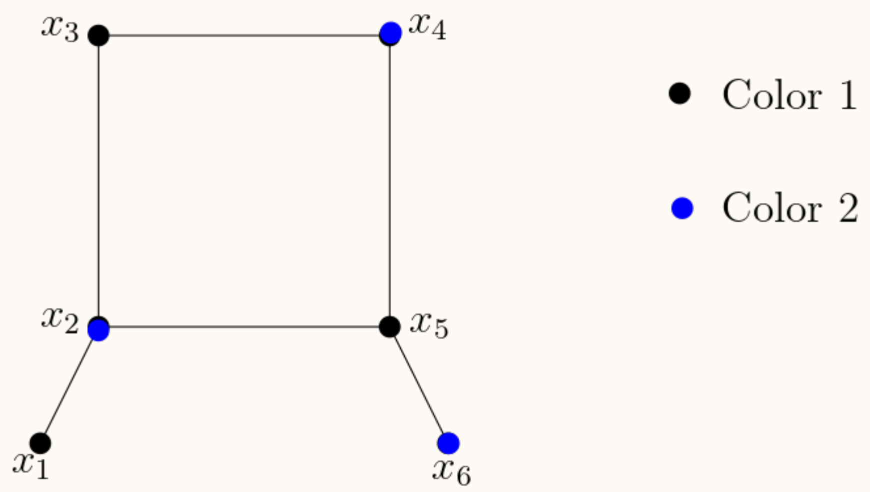
First-Fit (Greedy) Coloring Algorithm

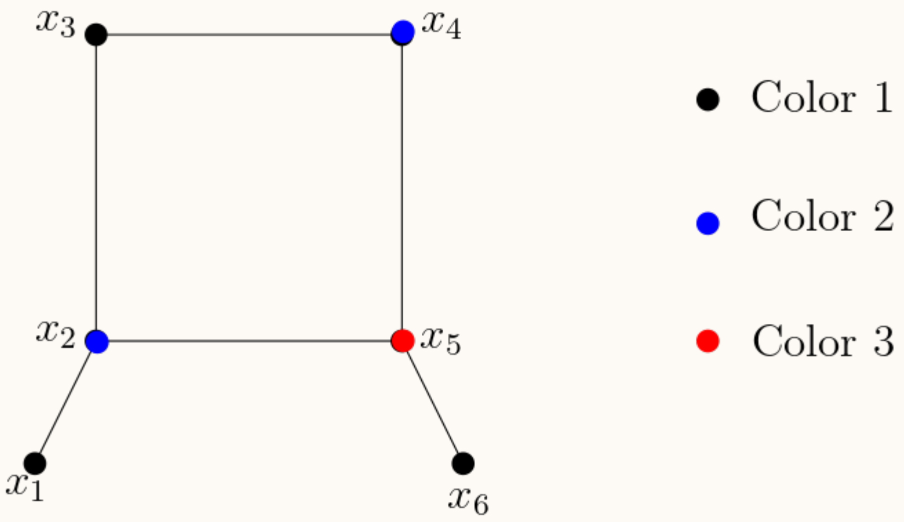
Input: $G = (V, E)$ and a vertex ordering $v_1 < v_2 < \ldots < v_n$
Output: A proper vertex coloring c

For i from 1 to n:
  Assign the smallest "available" color $c(v_i)$
return c.

- For the ordering of the vertices $(x_1, x_2, x_3, x_4, x_5, x_6)$
- If we change the ordering of the vertices to $(x_1, x_2, x_3, x_4, x_6, x_5)$, then

How many colors the algorithm may use?





Grundy Coloring

- Minimum number of colors used: Chromatic Number, $\chi(G)$
- Maximum number of colors used: Grundy Number, $\Gamma(G)$

Definition. Given a graph $G = (V, E)$, a *Grundy coloring* is a proper coloring with the property: For every (i, j) with $j < i$, every vertex $v \in V_i$ has a neighbor in color class V_j.

- Such a vertex v is called a *Grundy (dominator) vertex*.

Objective: Find a Grundy coloring with maximum number of colors

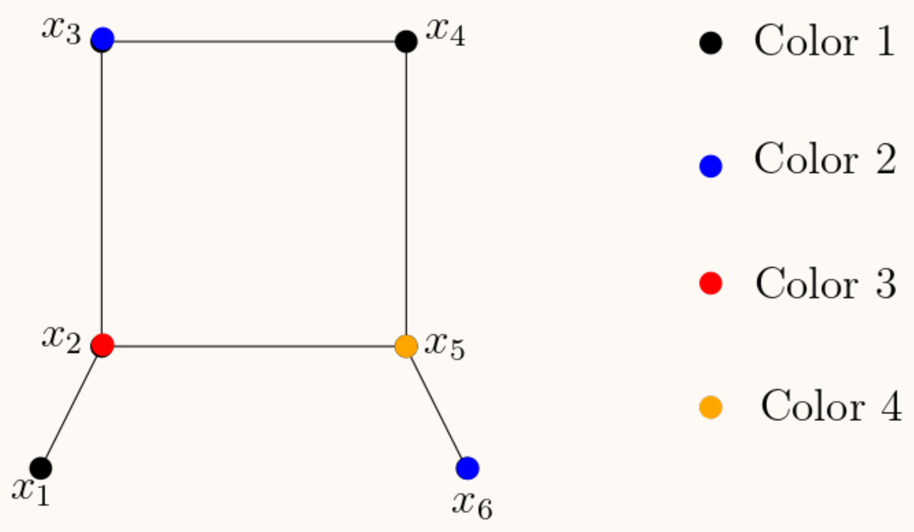
- First studied by P. M. Grundy in 1939 [Eureka, 1939]
- Formally introduced in 1979 by Christen and Selkow [J. Comb. Theory Ser. B, 1979]

Graph Coloring Heuristics

Any-available (Last-Fit) Algorithm

Input: G and a vertex ordering $v_1 < v_2 < \dots < v_n$
Output: A proper vertex coloring c.

For i from 1 to n:
    Assign to $c(v_i)$ ~~smallest~~ "available" color
return c.   any (largest)



➢ If we take the ordering to $(x_1, x_4, x_3, x_2, x_6, x_5)$

Question: Maximum number of colors used by the algorithm?
Partial Grundy number, $\delta\Gamma(G)$

Partial Grundy Coloring

Definition. Given a graph $G = (V, E)$, a *partial Grundy coloring* is a proper coloring with the property:
For every (i, j) with $j < i$, there exists a Grundy vertex $v \in V_i$, i.e., v has a neighbor in color class V_j.

Objective: Find a partial Grundy coloring using maximum number of colors

- First studied by Erdős et al. [Discrete Math., 2003]
- Natural bound: $\chi(G) \leq \Gamma(G) \leq \delta\Gamma(G) \leq \Delta(G) + 1$.
- Both the problems are NP-complete for general graphs [Goyal & Vishwanathan, 1997] [Z. Shi et al., 2005]

Parameterized Complexity Of The Problems

- GRUNDY COLORING is W[1]-hard in general graphs parameterized by solution size k
  [Aboulker, Bonnet, Kim, Sikora, STACS 2020]

Ques: Is PARTIAL GRUNDY COLORING W[1]-hard in general graphs parameterized by solution size k ?

- PARTIAL GRUNDY COLORING is FPT for $K_{t,t}$-free graphs parameterized by solution size k

Ques: Design an FPT algorithm for GRUNDY COLORING for $K_{i,j}$-free graphs parameterized by solution size k.

Our Results

Theorem. There is a randomized algorithm for *PARTIAL GRUNDY COLORING* with running time $2^{O(k^4)}\, n^{O(1)}$. In particular, if (G, k) is a no-instance then the algorithm outputs **No**; and if (G, k) is a yes-instance then with probability 2/3 the algorithm outputs **Yes**.

Theorem: For any fixed $i, j \in N$, there is an FPT algorithm that given a $K_{i,j}$-free graph G and a positive integer k, decides if there is Grundy coloring of G using at least k colors.

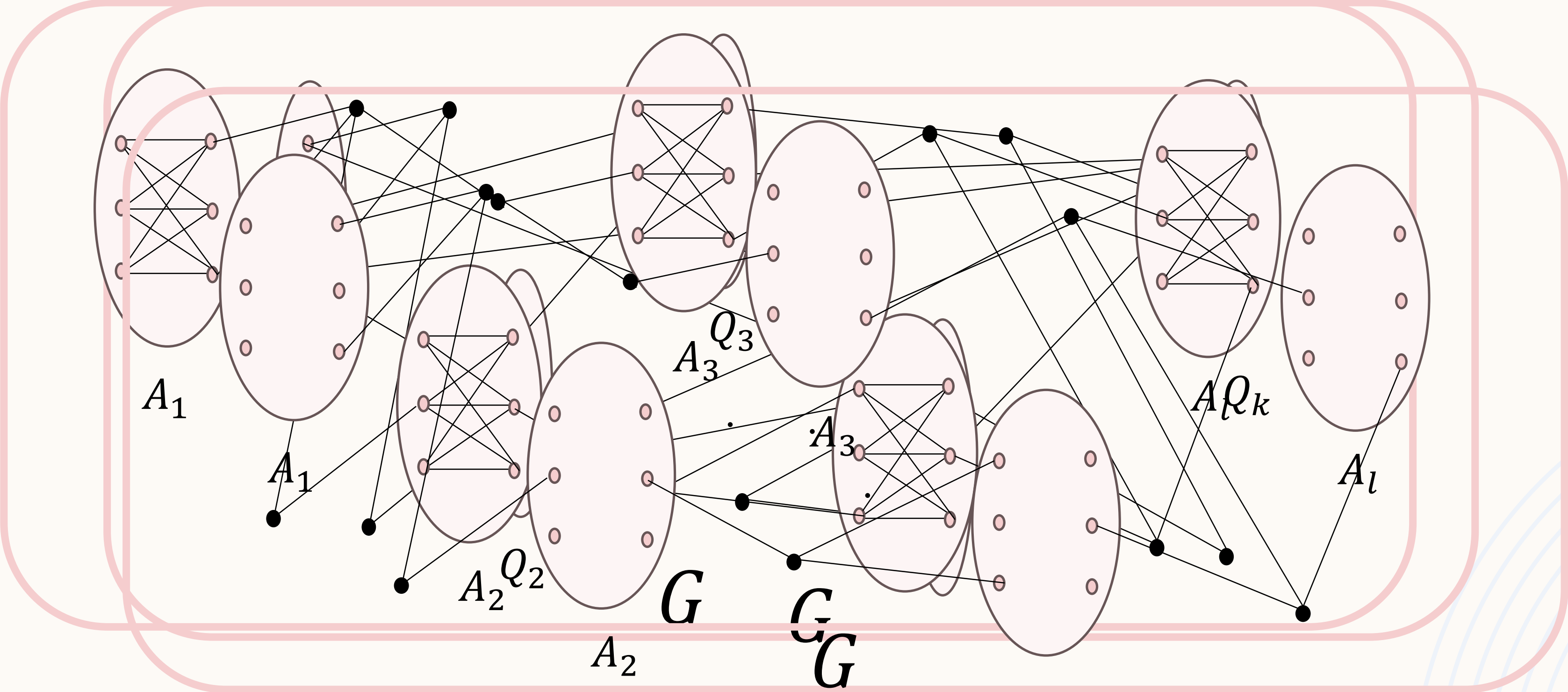
FPT ALGORITHM FOR PARTIAL GRUNDY COLORING

FPT Algorithm for Partial Grundy Coloring

Input: A graph G, positive integer k **Parameter:** k
Question: Does G have a partial Grundy coloring with at least k colors?

Theorem [**Degeneracy Reduction**]. There is a polynomial-time algorithm that given a graph G and a positive integer k, does one of the following:
(i) Correctly concludes that there is a partial Grundy coloring of G with at least k colors, or
(ii) Outputs at most $2k^3$ induced bicliques $A_1, \cdots, A_l$ in G such that the following holds. For any $v \in V(G)$, the degree of v in $G - F$ is at most k^3, where F is the union of the edges in the above bicliques.

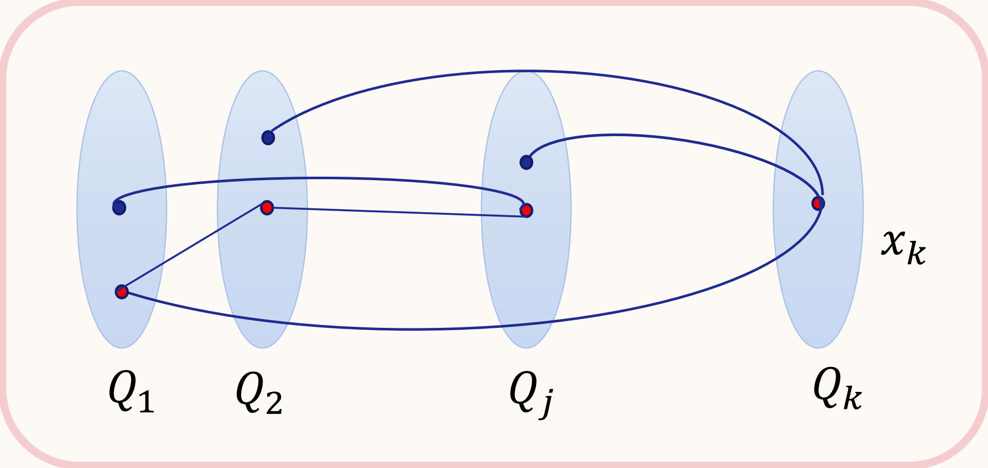
Degeneracy Reduction



Partial Grundy Witness

Def. A sequence of k-pairwise disjoint independent sets $(Q_1, Q_2, \dots, Q_k)$ of a given graph G is a **k-PG witness** if:

For any $i \in [k]$, $\exists\ v \in Q_i$ such that $\forall\ j \in [i-1]$, $Q_j \cap N_G(v) \neq \emptyset$. ($\exists$ such that $|Q_i| \leq k, \forall\ i$)



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Observation. Given a graph G, an induced subgraph H of G, and a partial Grundy coloring of H using k colors, we can find a partial Grundy coloring of G using at least k colors in linear time.

FPT Algorithm Idea

Input: A graph G, an integer k

Step 1. Use Color Coding

Color $V(G)$ u.

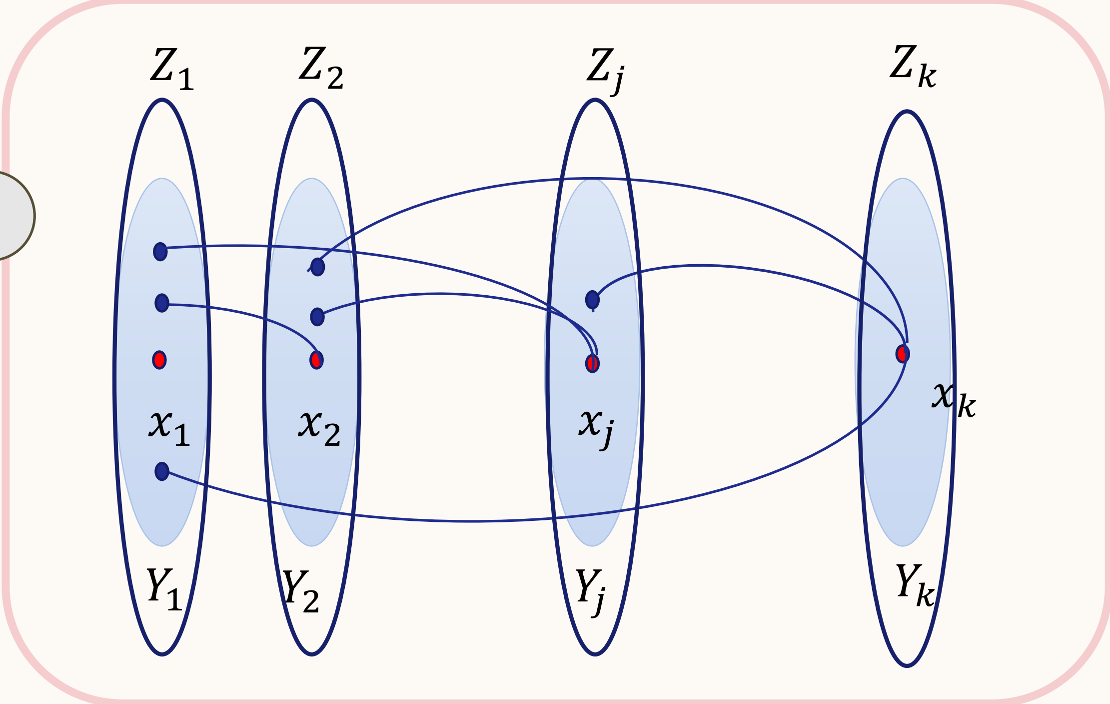
→ V(G

⇒ a k-PG witness

with probability k^{-k^2}

Ind. Covering lemma

Given a d-degenerate graph,
it returns an ind set Y such that any ind set X of size at most k, $X \subseteq Y$ with probability $f(d,k)$
(D. Lokshtanov et al. SODA 2018, ACM Trans. 2020)



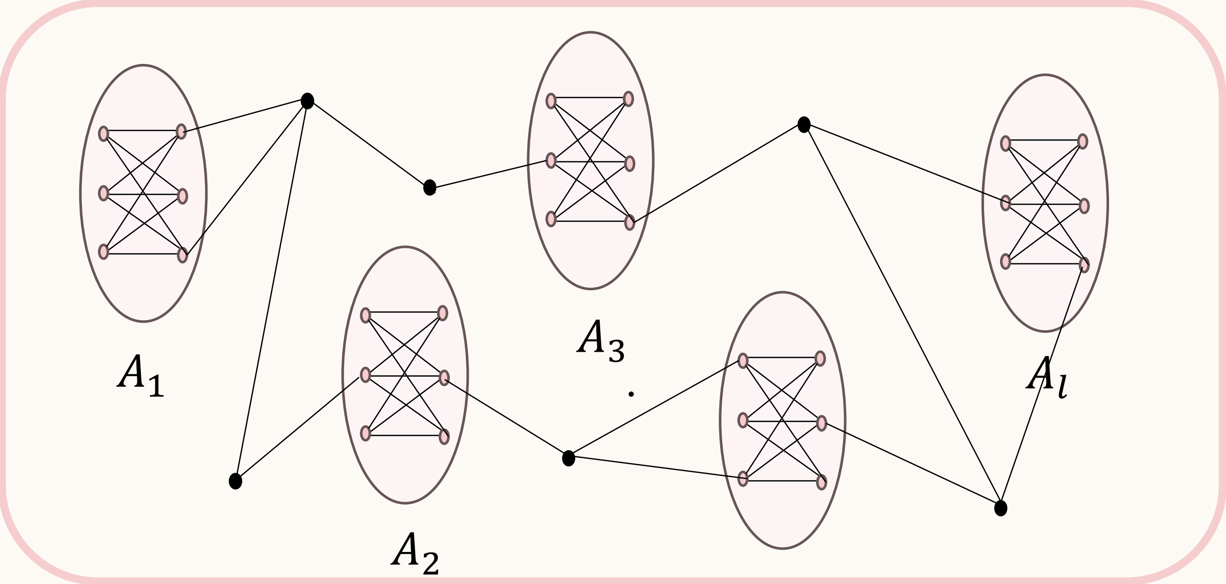
FPT Algorithm Idea

Step 2. Run algorithm of Degeneracy Reduction on instance (G, k)

Either returns a k-PG witness;

or

Returns $A_1, A_2, \dots, A_l$ bicliques, $l \leq 2k^3$.



FPT Algorithm Idea

Step 3. Use "Independence Covering Lemma" for each

$(G[Z_i - D_b], k)$

Return $Y_{i,b}$

Let $F_i = \cup_{b \in \{0,1\}^l} Y_{i,b}$

Union of one part of each biclique

Given a d-degenerate graph,
it returns an ind set Y such that any ind set X of size at most k, $X \subseteq Y$ with probability $f(d,k)$
(D. Lokshtanov et al. SODA 2018, ACM Trans. 2020)

Step 4. If $(F_1 \times F_2 \times \cdots \times F_k)$ has a tuple that is a k-PG witness,

return YES;

Else return NO.

Thank You for your attention!