Polynomial Kernel and Incompressibility for Prison-Free Edge Deletion and Completion

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Kernelization

A kernelization for a parameterized problem P is an algorithm that:

1. Reads an instance (I, k) of P, parameter k
2. Runs in time poly$(|I| + k)$
3. Outputs (I', k') where $|I'|, k' \leq f(k)$, mapping YES-instances to YES-instances and NO-instances to NO-instances

P has a polynomial kernel if there is a kernelization with $f(k) = \text{poly}(k)$

H-Free Edge Editing (Deletion, Completion)

Given (G, k), change (delete, add) k edges of G so that G has no induced copy of H

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Sample results

H	Graph class	Kernelization result
P_3	Cluster graph	PK (Gramm, Guo, Hüffner, Niedermeier 2005)
P_4	Cographs	PK (Guillemot, Havet, Paul, Perez 2014)
K_d	-	PK (folklore)
C_ℓ, $\ell \geq 4$		No PK (Cai, Cai 2015)
P_ℓ, $\ell \geq 5$		No PK (Cai, Cai 2015)

- If the graph class is **sufficiently restricted**, H-free Graph Modification can have a polynomial kernel
- For most graphs, H-free graphs have no significant structure and no polynomial kernel

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Negative news

Cai, Cai, 2015

For any 3-connected graph H:

- H-Free Edge Editing and H-Free Edge Deletion have no polynomial kernels unless H is a clique
- H-Free Edge Completion has no polynomial kernel unless H is K_d or $K_d - e$

Conjecture (Marx and Sandeep, 2022)

If H has at least five vertices, then

- H-Free Edge Editing has a polynomial kernel only if H is a clique or empty
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It suffices to verify this for a finite set of 19 graphs

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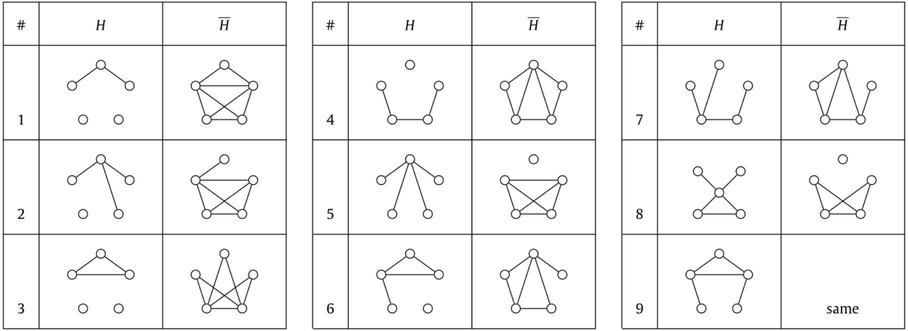
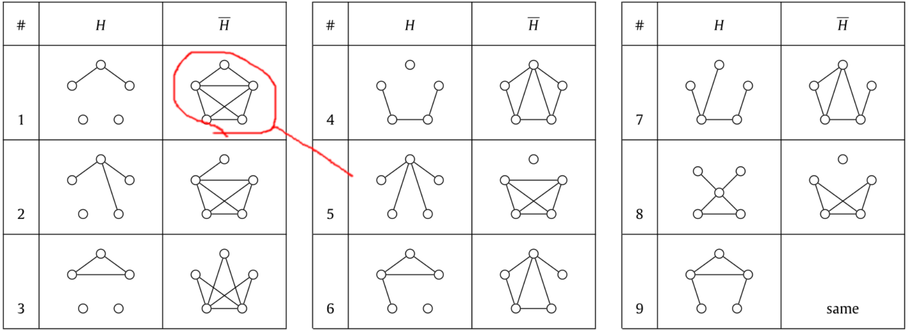


17 of Marx and Sandeep's 19 graphs

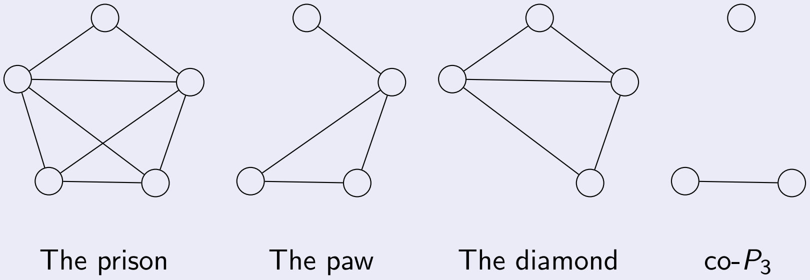


Main theorems

- **Prison-Free Edge Deletion** has a polynomial kernel
- **Prison-Free Edge Completion** has no polynomial kernel unless the PH collapses
- Marx and Sandeep's conjecture is wrong

Prison-free graphs

The prison and its subgraphs



- **Paw-Free Edge Modification** has polynomial kernels (Eiben, Lochet, Saurabh 2020; Yuan, Ke, Cao 2021)
- **Diamond-Free Edge Modification** has polynomial kernels (Sandeep, Sivadasan 2015; Cao, Rai, Sandeep, Ye 2022)

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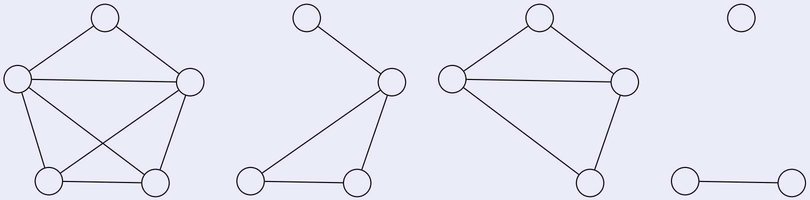

The prison | The paw | The diamond | co-P_3

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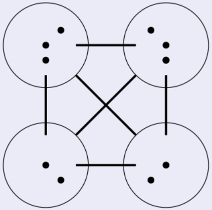
H-free graph structures

A complete multipartite graph (multiclique)



co-P_3-free	Complete multipartite (multiclique)
Paw-free	Every component is multiclique or triangle-free (Olariu 1988)
Diamond-free	Every edge is in a unique maximal clique (Wallis, Zhang 1990)
Prison-free	Edges which occur in K_4:s are partitioned into maximal induced multicliques

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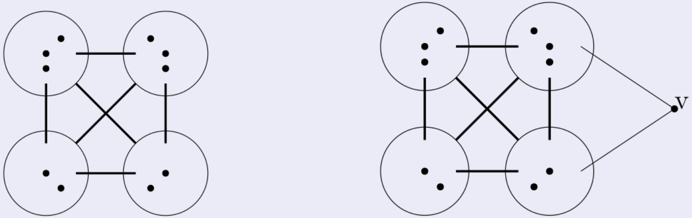

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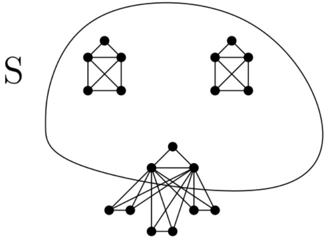
Kernelization, overview

1. Construct a modulator S
2. Decomposition of $G - S$
3. "No propagation" property
4. Marking and shrinking of components



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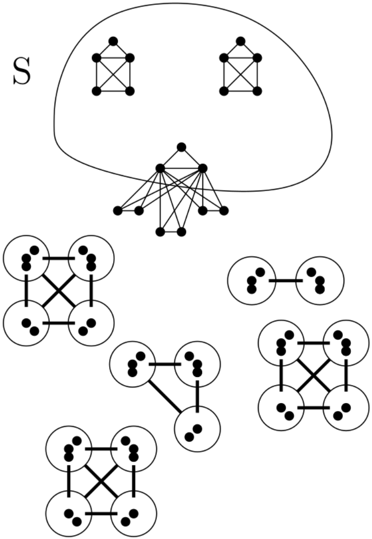

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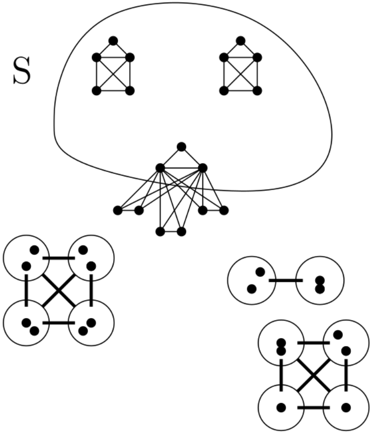
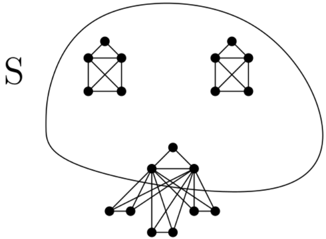


Modulator

Sunflower lemma (consequence)

Let $\mathcal{F}$ be a set family of sets of size d. There is a subfamily $\mathcal{F}' \subseteq \mathcal{F}$ such that $|\mathcal{F}| = O(k^d)$ and every set X, $|X| \leq k$ that intersects every $F \in \mathcal{F}'$ also intersects every $F \in \mathcal{F}$

- Let $\mathcal{F}$ contain all edge sets of induced prison in G



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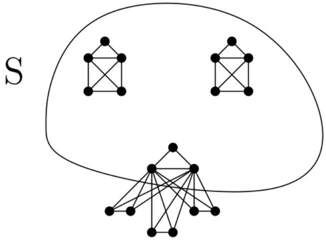
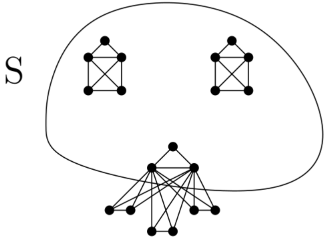
- Let $\mathcal{F}$ contain all edge sets of induced prison in G
- Let $S = V(\mathcal{F}')$; then $|S| = O(k^8)$
- $G - S$ is prison-free and S handles all prisons in G



Modulator

Reduction rule

If any prison has only one edge in S, delete that edge and decrease k



Modulator

Reduction rule

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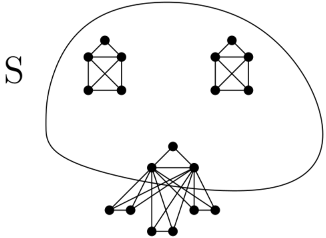
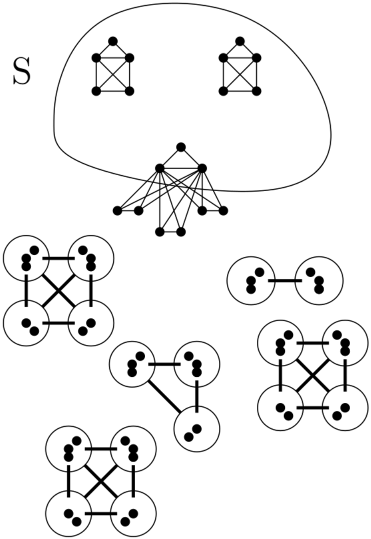
Reduction rule

If any edge e is not in S and not in a K_5 or $K_5 - e$, discard e



Edge decomposition of $G - S$



Edge decomposition

The edges of $G - S$ partition into maximal induced multicliques

Hints:

- All edges occur in dense subgraphs with S (at least K_4's)
- Multicliques with ≥ 4 parts can't share edges without creating a prison

Edge decomposition of $G - S$

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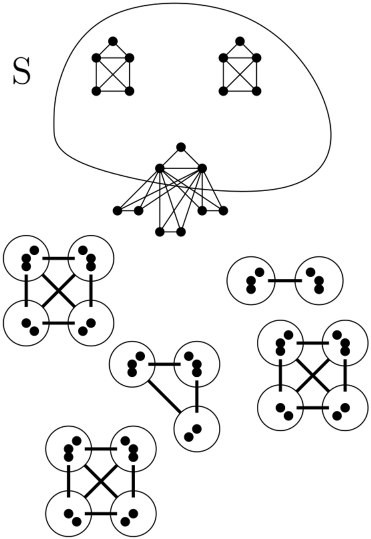


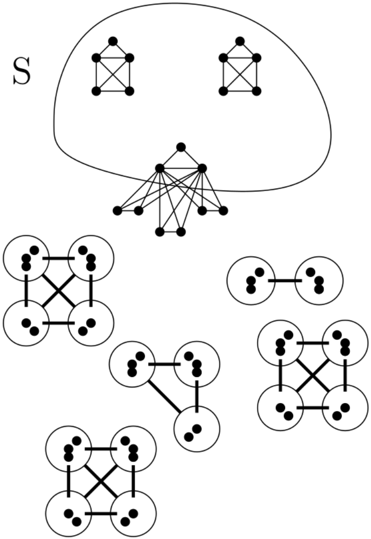
Hints:

- All edges occur in dense subgraphs with S (at least K_4's)
- Multicliques with ≥ 4 parts can't share edges without creating a prison
- For every edge uv in S, $N(uv) \setminus S$ is co-P_3-free – a multiclique!

Propagation

Every K_5 or $K_5 - e$ in G occurs inside $F \cup S$ for some multiclique F of $G - S$

- Propagation: Deleting e in prison creates another prison, etc.



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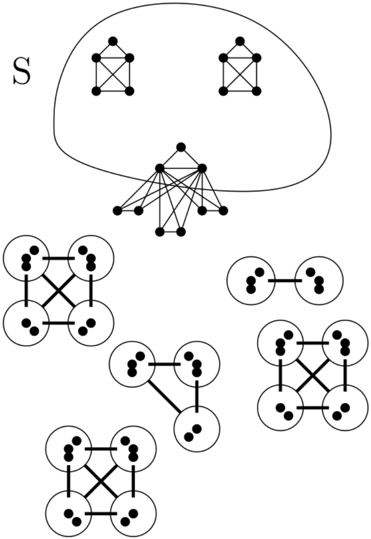

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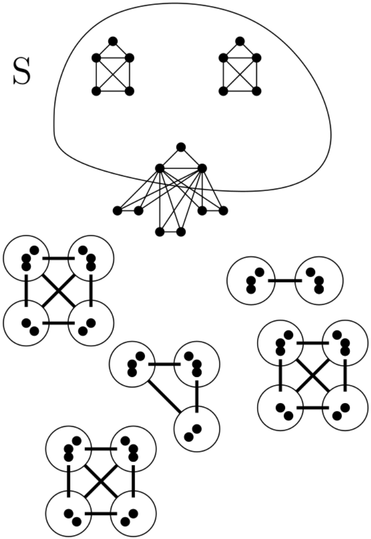
- **No propagation** – multicliques act independently



Summary

- **Prison-Free Edge Deletion** has a polynomial kernel
- **Prison-Free Edge Completion** does not (see paper)
- Open 1: Prison-Free Edge Editing?

Summary

Conjecture (Cai and Cai, 2015)

If H is 2-connected with at least six vertices, then

- H-Free Edge Editing has a polynomial kernel only if H is a clique
- H-Free Edge Deletion has a polynomial kernel only if H is a clique
- H-Free Edge Completion has a polynomial kernel only if H is missing at most one edge

- **Prison-Free Edge Deletion** has a polynomial kernel
- **Prison-Free Edge Completion** does not (see paper)
- Open 1: Prison-Free Edge Editing?
- Open 2: Is the conjecture of Cai and Cai true?