Protecting the Connectivity of a Graph under Non-uniform Edge Failures

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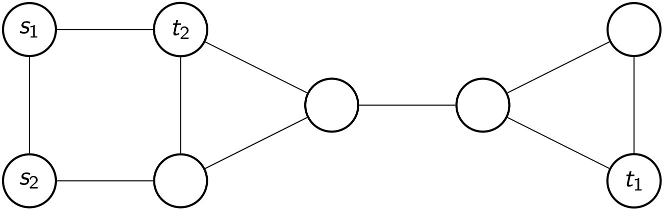
March 05, 2025

Problem Definition



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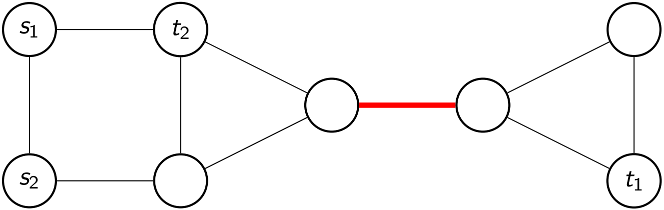

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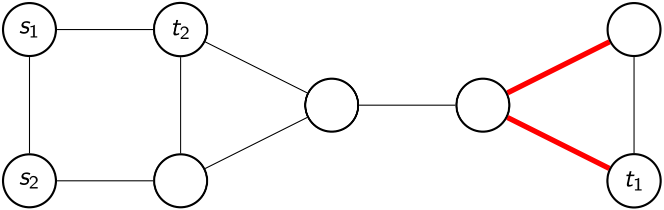

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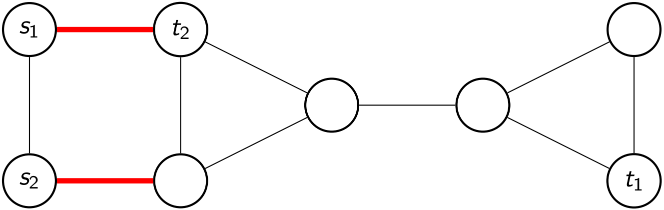

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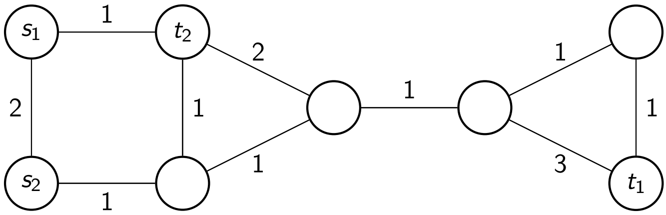

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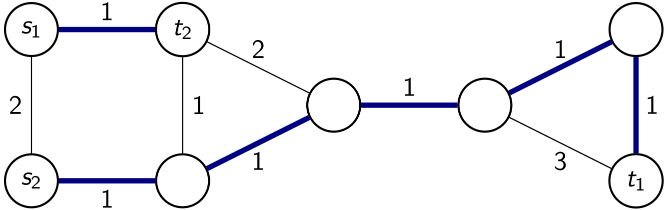

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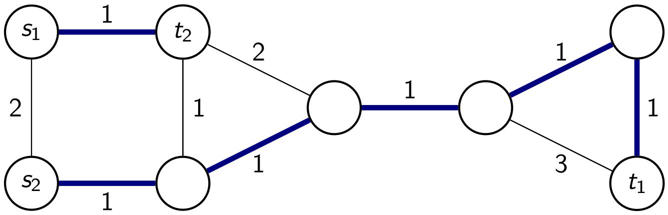

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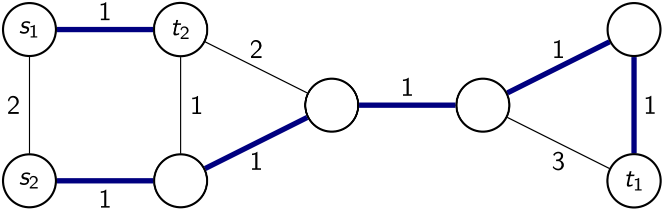

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- An undirected graph $G = (V, E)$ with edge cost $c : E \to \mathbb{R}_{\geq 0}$.
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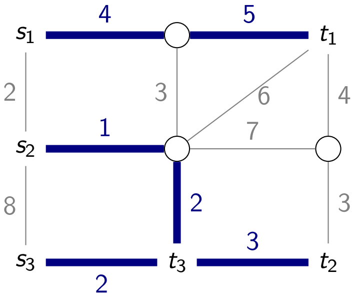

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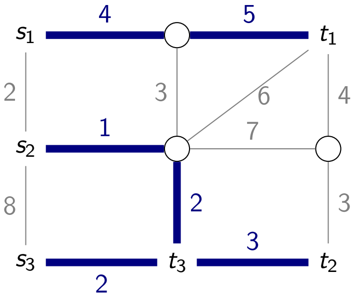

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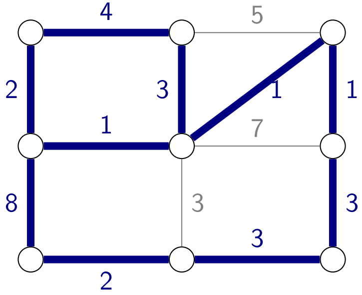

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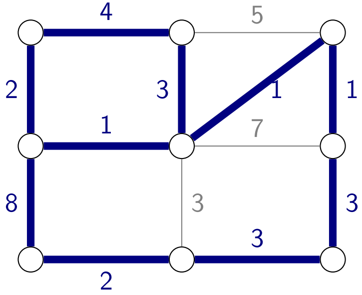

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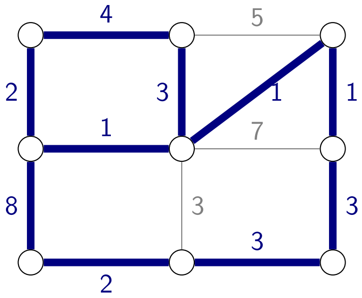
- The NP-hardness of $(1, q)$-GCP is unknown (equivalent to Minimum Spanning Tree when $q \geq |E|$).



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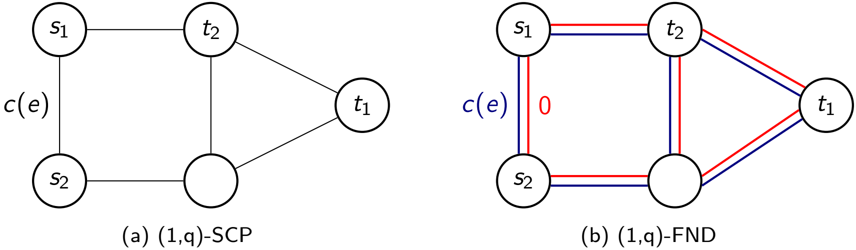

Figure: Reduction from $(1, q)$-SCP to $(1, q)$-Flexible Network Design.

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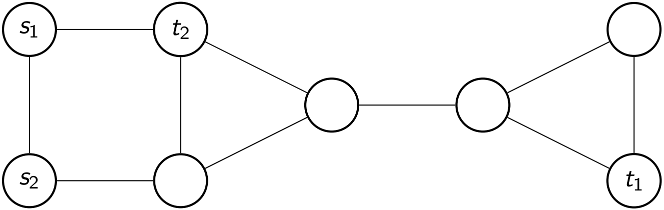

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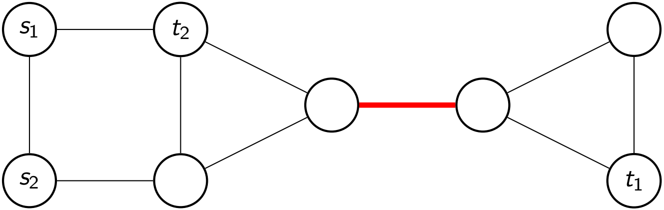

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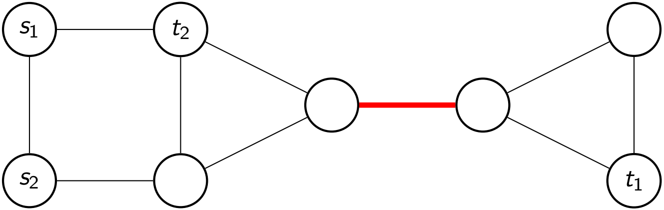

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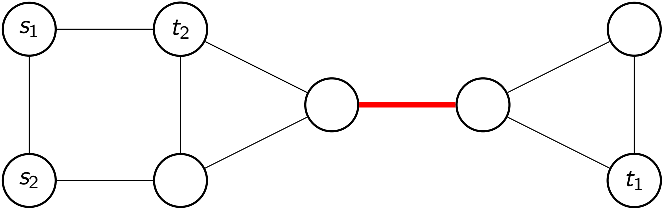

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The optimal solution is to protect all bridges that separates some (s_i, t_i)!

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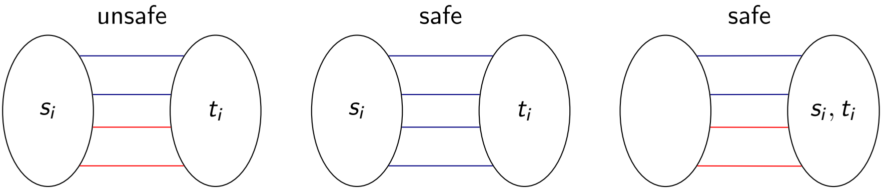
- Safe cuts: not critical or already have p protected edges.



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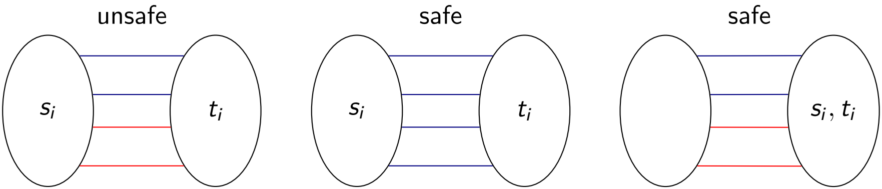

Idea: Iteratively find an unsafe cut and protect the edges in the cut.

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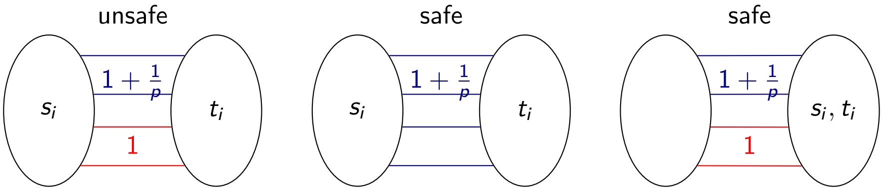
The capacity function:

- $u(e) = 1$ if e is unprotected.
- $u(e) = 1 + \frac{1}{p}$ if e is protected.



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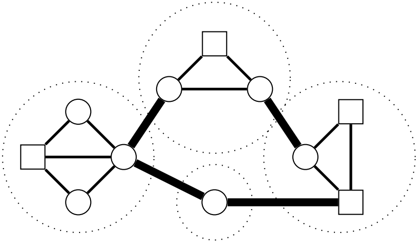
A decomposition lemma

There is a polynomial-time algorithm which decompose a 2EC graph G into disjoint 2EC subgraphs $G_1, \ldots, G_k$ s.t. $G / \bigcup_{i=1}^{k} G_i$ forms a cycle.



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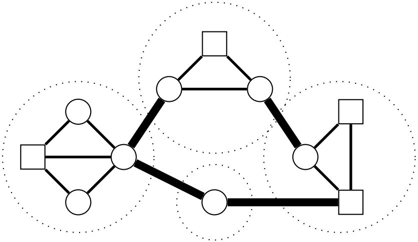

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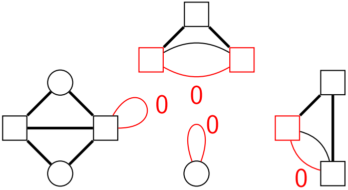

(b) Independent sub-instances

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Augmentation-based approximation algorithms

Critical cuts: $\mathcal{S} := \{S \subset V \mid |\delta(S)| \leq p + q - 1, S \text{ separates some terminal pair}\}$.

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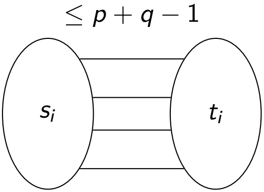

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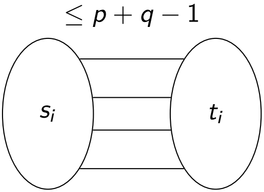

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- Our primal-dual algorithm has p phases [Williamson et al. 1995].
- The augmentation problem: given that each critical cut contains $\geq i - 1$ protected edges, we protect more edges to ensure $\geq i$ protected edges.

$$\begin{aligned} \min \quad & \sum_{e \in E \setminus X_{i-1}} c_e x_e \\ \text{s.t.} \quad & \sum_{e \in \delta(S) \setminus X_{i-1}} x_e \geq 1 \quad \forall S \in \mathcal{S}_i \\ & x_e \geq 0 \quad \forall e \in E \setminus X_{i-1} \end{aligned}$$

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Augmentation-based approximation

(informal) Dual mapping [Williamson et al. 1995]

Given a dual feasible solution $\{y_S^{(i)}\}$ of the ith phase, we can construct a dual feasible solution $\{y_S, z_e\}$ to the main LP s.t.

$$\sum_{S \in \mathcal{S}_i} y_S^{(i)} \leq \frac{1}{p-i+1}\Big(\sum_{S \in \mathcal{S}} p \cdot y_S - \sum_{e \in E} z_e\Big) \leq \frac{1}{p-i+1}\textsc{Opt}\ .$$

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(informal) Total cost of p phases

Given a K-approximation algorithm for the augmentation problem, the total cost of is

$$\sum_{i=1}^{p} cost(phase_i) \le K\sum_{S\in\mathcal{S}_i} y_S^{(i)} \le K\sum_{i=1}^{p}\frac{1}{p-i+1}\textsc{Opt} = \mathcal{O}(K\log p\cdot\textsc{Opt}).$$

Overview of our results

Hardness:

- $(1, q)$-GCP is NP-hard even if $c(e) \equiv 1$.
- Verifying a solution to (p, q)-stCP is NP-complete.
- This implies that there is no α-approximation, which also holds for (p, q)-Flexible Network Design.

Exact algorithms for small values of p or q.

- $(p, 1)$-SCP.
- $(1, 2)$-SCP, $(2, 2)$-GCP.

Approximation algorithms for general values of p, q.

- $\mathcal{O}(q \cdot \log p)$-approximation for (p, q)-SCP assuming p is constant.
- $\mathcal{O}(\log p \cdot \min\{\log n, p + q\})$-approximation for (p, q)-GCP.

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- $q = mincut$, i.e. find minimum-cost edge set that intersects with all minimum cuts.
- FPT results?

Augmentation-based approximation

We solve the augmentation problems approximately.

- $(p + q - 1)$-approximation for (p, q)-SCP, assuming p is constant.
- $\mathcal{O}(\min\{\log n, p + q\})$-approximation for (p, q)-GCP.

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- Starting from $y=0$, iteratively find a violating cut S and increase y_S.
- Use the "capacity trick" to distinguish violating and non-violating cuts.
- For (p,q)-GCP, set u s.t. $u(\text{violating cuts}) \leq 2u(\text{mincut})$.
- The number of 2-approximate mincuts is polynomial and they can be enumerated in polynomial time [Karger 1993].