Sampling Unlabeled Chordal Graphs in Expected Polynomial Time

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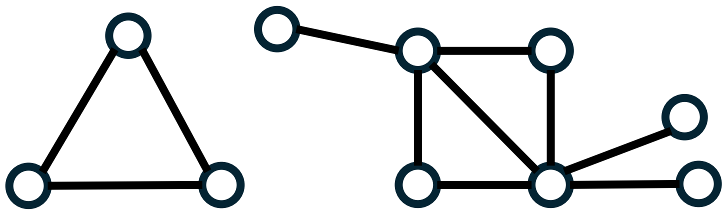
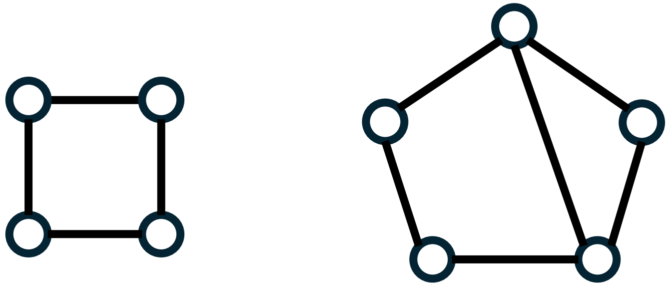
Joint work with Daniel Lokshtanov

Def: A graph is *chordal* if it has no induced cycles of length at least 4.

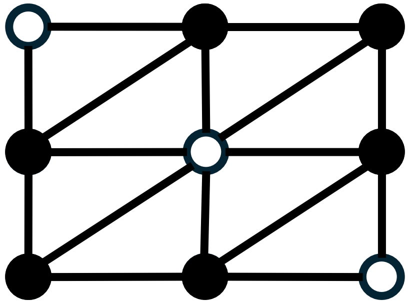


chordal



not chordal

Pop quiz: Is this graph chordal?



No

😯

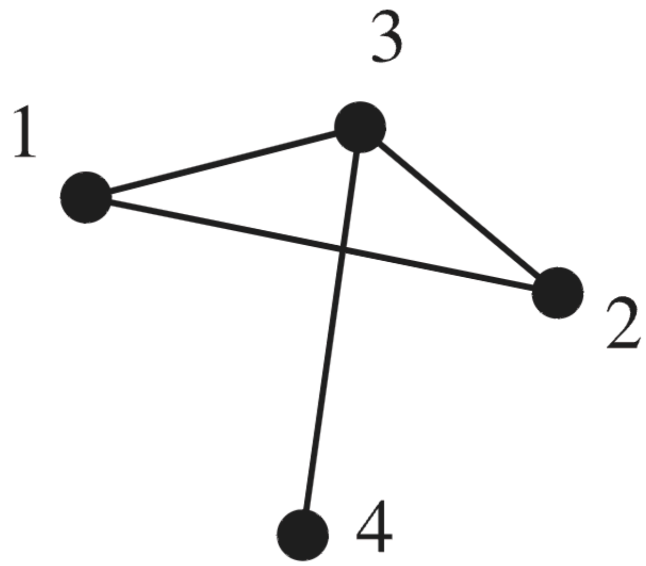
Why sample random graphs?

Graph sampling is useful for generating random test cases for software testing and can be helpful for testing conjectures.

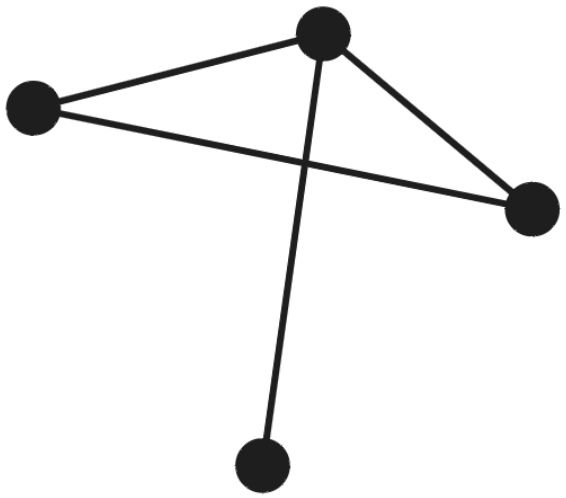
Already known…

There is an algorithm for sampling/counting n-vertex **labeled** chordal graphs that uses $O(n^7)$ arithmetic operations.

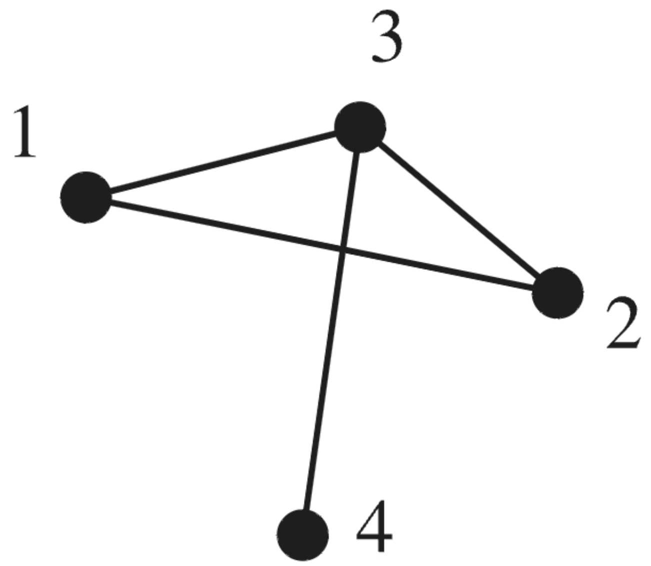
(Hébert-Johnson, Lokshtanov, and Vigoda, 2023)

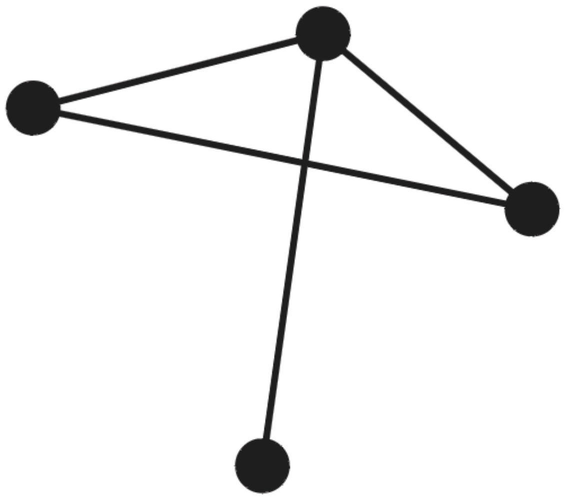


$2^{\binom{n}{2}}$ labeled graphs on n vertices



polynomial-time algorithm for **counting** unlabeled graphs?
OPEN





To sample a labeled graph: flip a coin for each of the $\binom{n}{2}$ potential edges

Expected polynomial-time **sampling** algorithm for unlabeled graphs (Wormald, 1987)

Input: n

Goal: Generate an n-vertex unlabeled chordal graph uniformly at random

Our result: A sampling algorithm that uses
$O(n^7)$ arithmetic operations in expectation

Automorphisms

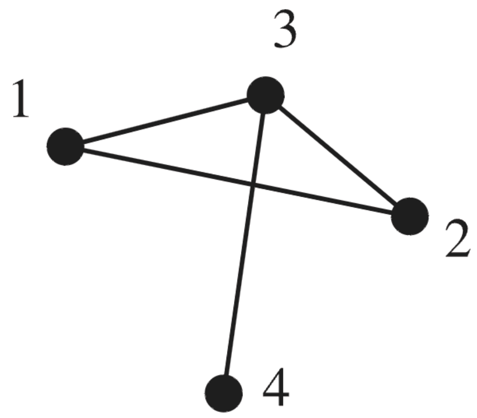
Let π be a permutation of $[n] \coloneqq \{1, 2, \dots, n\}$.

Let G be a labeled graph with vertex set $[n]$.

We say π is an *automorphism* of G if

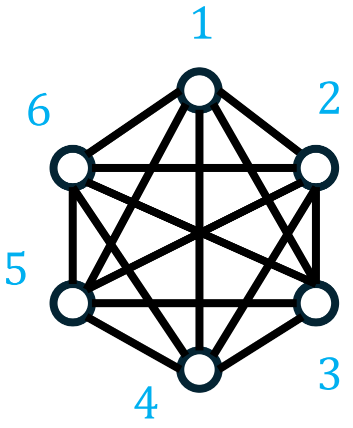
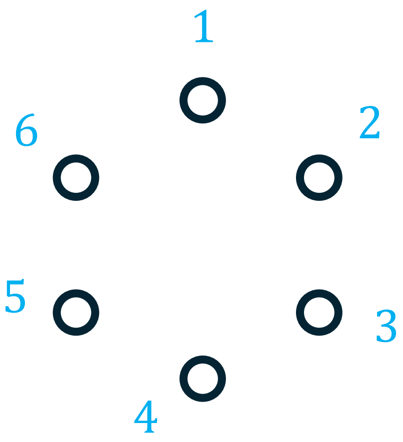
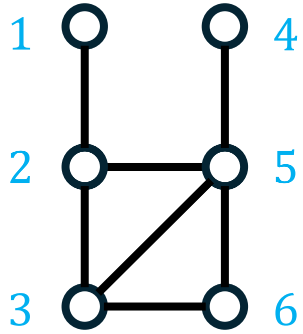
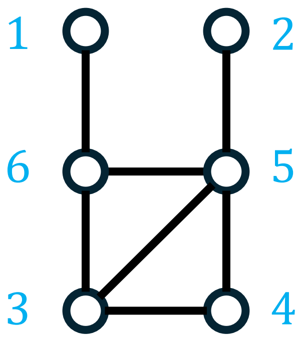
$$u \text{ and } v \text{ are adjacent} \iff \pi(u) \text{ and } \pi(v) \text{ are adjacent}$$

for all $u, v, \in V(G)$.



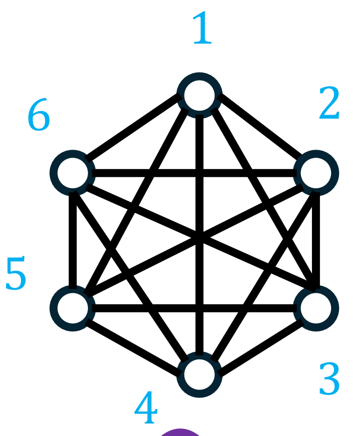
$\pi = (12)$
is an automorphism of this graph

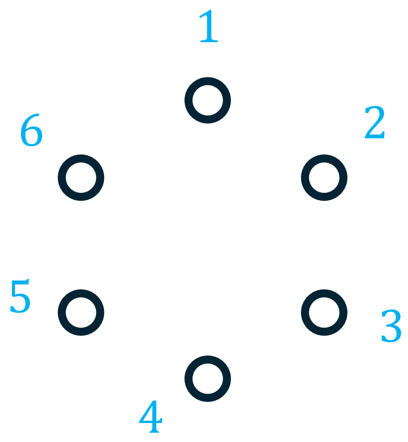
labeled chordal graphs

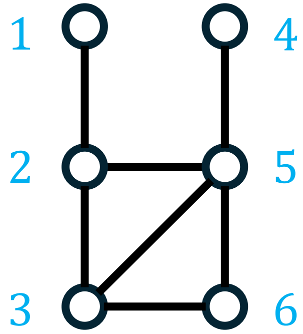
permutations			$\cdots$		$\cdots$	
identity	1	1	$\cdots$	1	$\cdots$	1
(12)	1	1		0		0
(13)	1	1		0		0
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\ddots$	$\vdots$
(12)(6543)	1	1	$\cdots$	0	$\cdots$	0

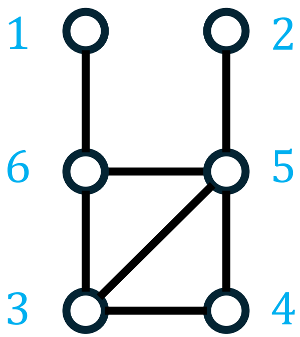
An entry of **1** for (π, G) means π is an automorphism of G.

labeled chordal graphs









permutations						
identity	1	1	$\cdots$	1	$\cdots$	1
(12)	1	1	6!	0		0
(13)	1	1		0		0
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\ddots$	$\vdots$
(12)(6543)	1	1	$\cdots$	0	$\cdots$	0

6!

By Burnside's lemma, the number of **1**s for each **unlabeled** graph is $n!$

Fixed points and moved points

Let π be a permutation of $[n] := \{1, 2, \dots, n\}$.

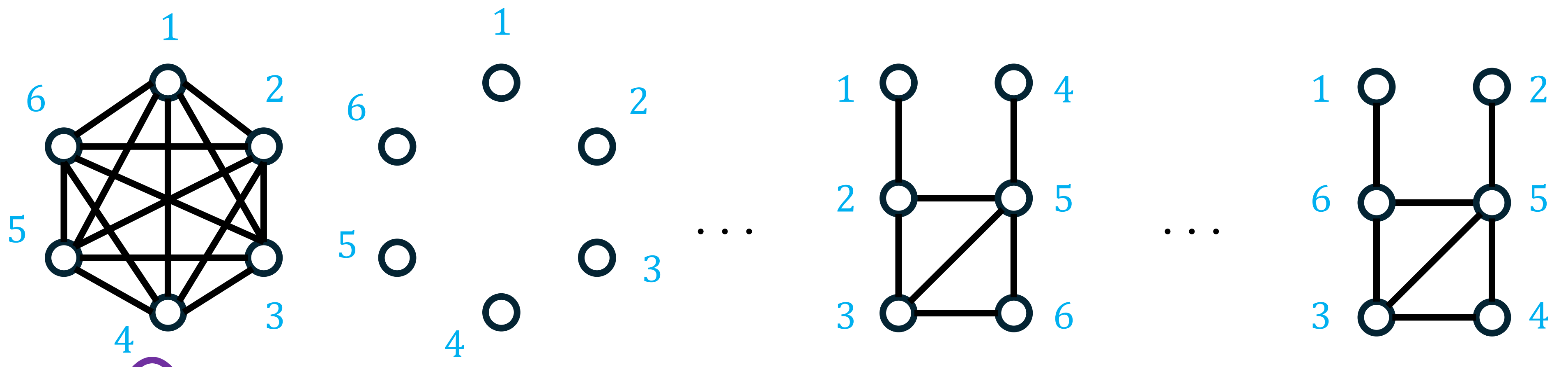
$\text{fix}(\pi) = \{\, i \in [n] : \pi(i) = i \,\}$, $\text{move}(\pi) = \{\, i \in [n] : \pi(i) \neq i \,\}$

Let $n = 4$.

$\pi = \text{id}$	$\pi = (13)$	$\pi = (234)$
$1 \to 1$	$\mathbf{1} \to \mathbf{3}$	$1 \to 1$
$2 \to 2$	$2 \to 2$	$\mathbf{2} \to \mathbf{3}$
$3 \to 3$	$\mathbf{3} \to \mathbf{1}$	$\mathbf{3} \to \mathbf{4}$
$4 \to 4$	$4 \to 4$	$\mathbf{4} \to \mathbf{2}$
$\text{move}(\pi) = \emptyset$	$\text{move}(\pi) = \{1, 3\}$	$\text{move}(\pi) = \{2, 3, 4\}$
$\lvert\text{move}(\pi)\rvert = 0$	$\lvert\text{move}(\pi)\rvert = 2$	$\lvert\text{move}(\pi)\rvert = 3$

Observation: $\lvert\text{move}(\pi)\rvert \in \{0, 2, 3, \dots, n\}$

labeled chordal graphs



permutations						
identity	1	1	$\cdots$	1	$\cdots$	1
(12)	1	1		0		0
(13)	1	1		0		0
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\ddots$	$\vdots$
(12)(6543)	1	1	$\cdots$	0	$\cdots$	0

6! (circled row: identity)

6! (circled column: complete graph)

By Burnside's lemma, the number of **1**s for each **unlabeled** graph is n!

Goal: Sample a random **1** from the matrix of permutations and labeled chordal graphs.

This gives uniform weight (i.e., weight $n!$) to all unlabeled chordal graphs.

Idea: Group permutations (rows) by their number of moved points μ

Overall strategy:

1. Choose a random value of $\mu \in \{0, 2, 3, \dots, n\}$ (weighted correctly)
2. Choose a random permutation π such that $|\text{move}(\pi)| = \mu$
3. Sample a random labeled chordal graph G with automorphism π
4. Return G without labels

Efficiently choosing a random value of μ

$A = (a_0, a_2, a_3, \dots, a_n)$ ⟵ a vector of positive real numbers

$B = (b_0, b_2, b_3, \dots, b_n)$ ⟵ a vector of upper bounds such that $b_0 = a_0, \quad b_i \geq a_i$ for all $i \geq 2$

We want to choose a random number of moved points $\mu \in \{0, 2, 3, \dots, n\}$ according to a distribution proportional to A,

i.e., $\Pr(\mu = k) = \frac{a_k}{\sum_i a_i}$ for all k.

Suppose that, given k, we can compute a_k in time 2^k.

Computing all of $a_0, a_2, a_3, \dots, a_n$ would take exponential time.

Efficiently choosing a random value of μ

$A = (a_0, a_2, a_3, \dots, a_n)$

$B = (b_0, b_2, b_3, \dots, b_n)$ $\quad b_0 = a_0, \quad b_i \geq a_i$ for all $i \geq 2$

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Suppose that, given k, we can compute a_k in time 2^k.

Algorithm for choosing μ:

1. Choose μ at random such that $\Pr(\mu = k) = \frac{b_k}{\sum_i b_i}$ for all k
2. Restart with probability $\frac{b_\mu - a_\mu}{b_\mu} = 1 - \frac{a_\mu}{b_\mu}$
3. Return μ

This algorithm can be fast if large values of μ are unlikely according to B.

Output distribution: $\Pr(\mu = k) = \frac{b_k}{\sum_i b_i} \cdot \frac{a_k}{b_k} = \frac{a_k}{\sum_i b_i}$

Goal: Sample a random **1** from the matrix of permutations / labeled chordal graphs.

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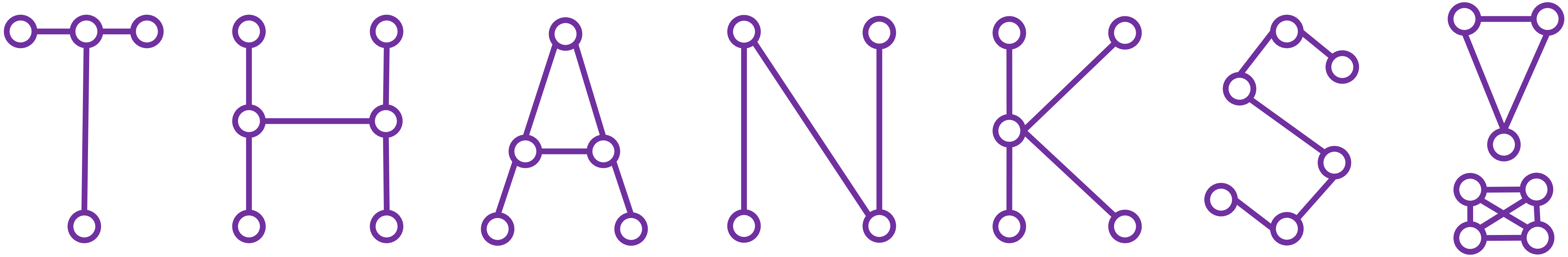
Steps 1 and 2:

Use a similar approach to the previous slide.
Need to define upper bounds b_μ and prove that
$b_\mu \geq$ (# of π with μ moved points)·(# of graphs with automorphism $\hat{\pi}$) ☆
for all $\hat{\pi}$ with μ moved points.

Step 3:

Use an algorithm based upon the DP algorithm for sampling labeled chordal graphs. ☆



a chordal graph with $n = 37$