Slightly non-linear higher-order tree transducers

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joint work with Gabriele Vanoni (IRIF, Paris)

STACS 2025 (Marseille $\xrightarrow{\text{online}}$ Jena)

- (Slightly non-)**linear**: as in *linear logic*
- **higher-order**: as in functional programming / *λ-calculus*

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Comparing the expressive power of automata-like devices:

storing λ-terms vs. more conventional

- (Slightly non-)**linear**: as in *linear logic*
- **higher-order**: as in functional programming / *λ-calculus*
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Comparing the expressive power of automata-like devices:

storing λ-terms vs. more conventional

First: conventional examples on strings

Two-way automata

Transitions: update finite state + move left/right depending on new state

Example: states $Q = \{q_1^{\rightarrow}, q_2^{\leftarrow}, q_3^{\leftarrow}\}$, initial state $q_1^{\rightarrow}$

$$q_1^{\rightarrow}, (a|b) \mapsto q_1^{\rightarrow} \qquad q_1^{\rightarrow}, c \mapsto q_2^{\leftarrow} \qquad q_2^{\leftarrow}, (a|b|c) \mapsto q_3^{\leftarrow} \qquad q_3^{\leftarrow}, b \mapsto \text{accept}$$

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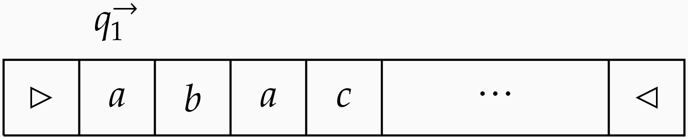

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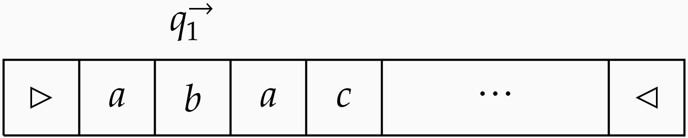

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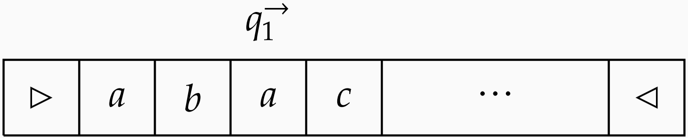

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$q_1^{\rightarrow}$

$\triangleright$	a	b	a	c	$\cdots$	$\triangleleft$

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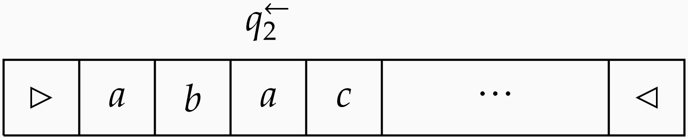

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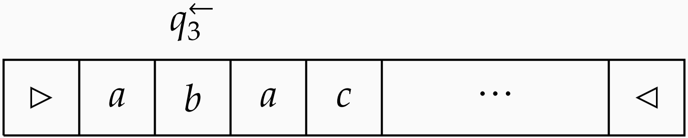

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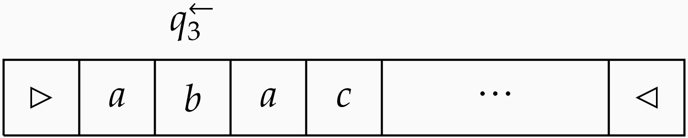

Theorem (Rabin & Scott / Shepherdson 1959)

Two-way automata ≡ one-way automata (≡ regular languages)

⟶ rightfully belong to "finite-state computation"

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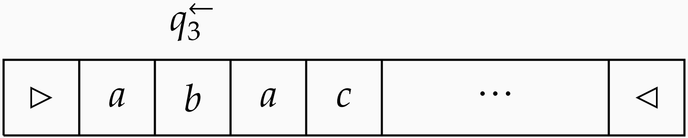

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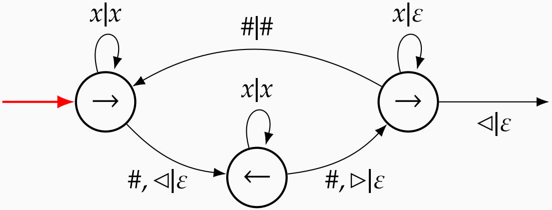
⇒ so does their extension with string outputs

Two-way transducers

Example: $w_1\#\ldots\#w_n \longmapsto w_1 \cdot \texttt{reverse}(w_1)\#\ldots\#w_n \cdot \texttt{reverse}(w_n)$



$(x \in \{a, b, c\})$

$\triangleright$	a	b	c	#	b	a	c	#	c	b	$\triangleleft$

Output:

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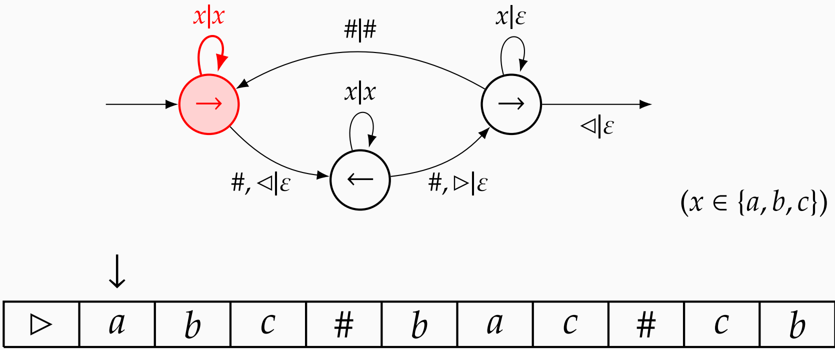

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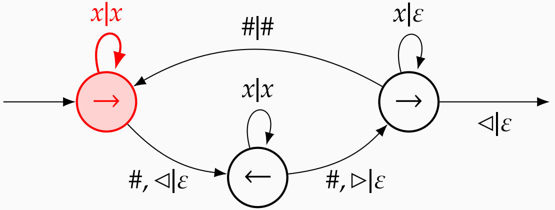
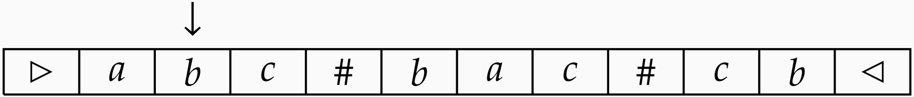


$(x \in \{a, b, c\})$



Output: a

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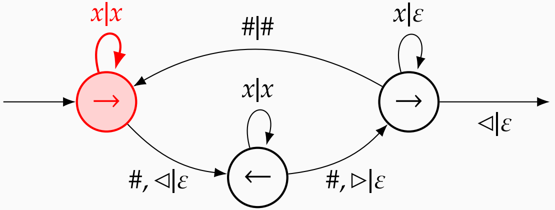

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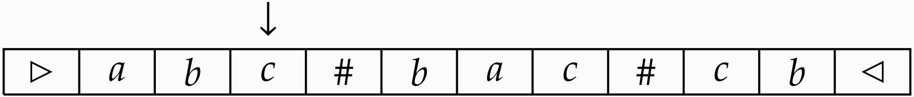

Output: *ab*

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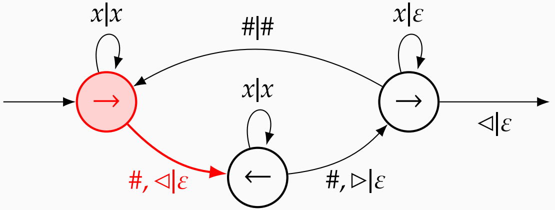

$(x \in \{a, b, c\})$

▷	a	b	c	#	b	a	c	#	c	b	◁

Output: abc

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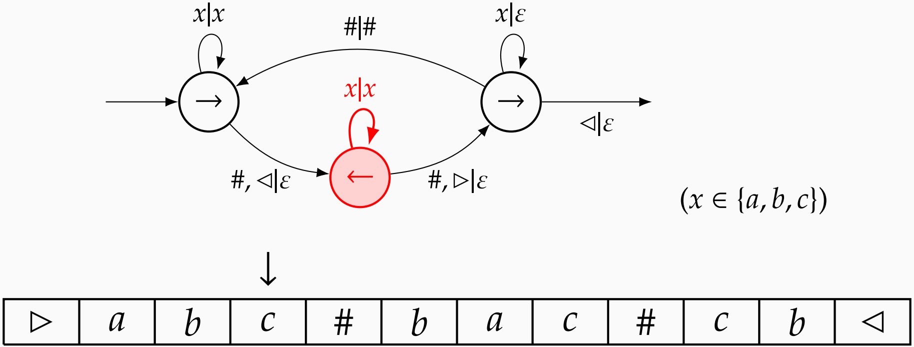

Output: *abc*

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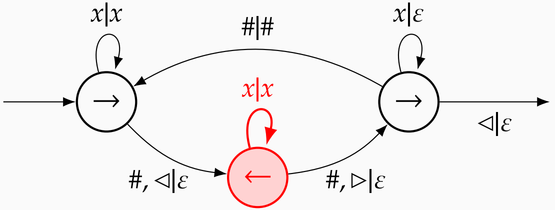
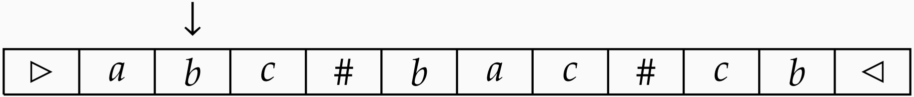


$(x \in \{a, b, c\})$



Output: *abcc*

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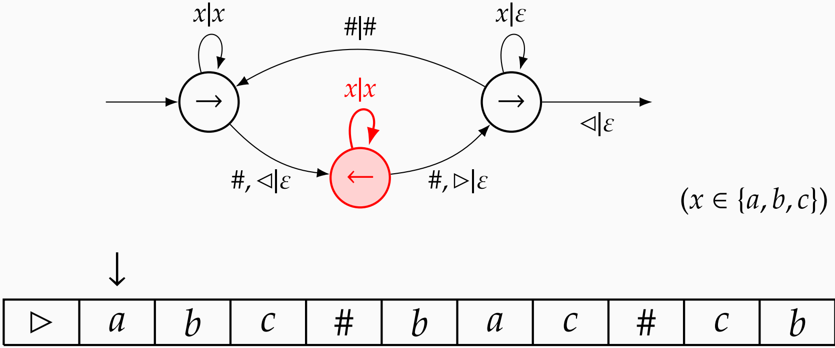

Output: $abccb$

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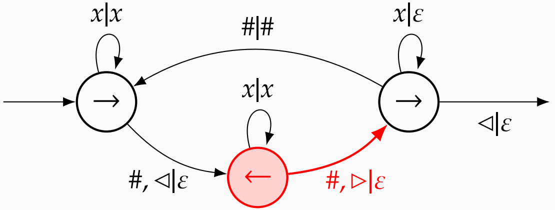

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↓

▷	a	b	c	#	b	a	c	#	c	b	◁

Output: $abccba$

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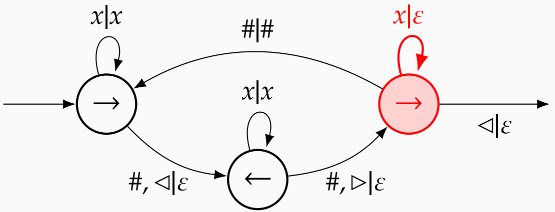

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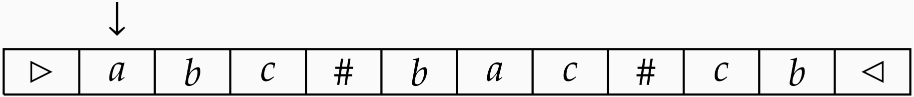

Output: *abccba*

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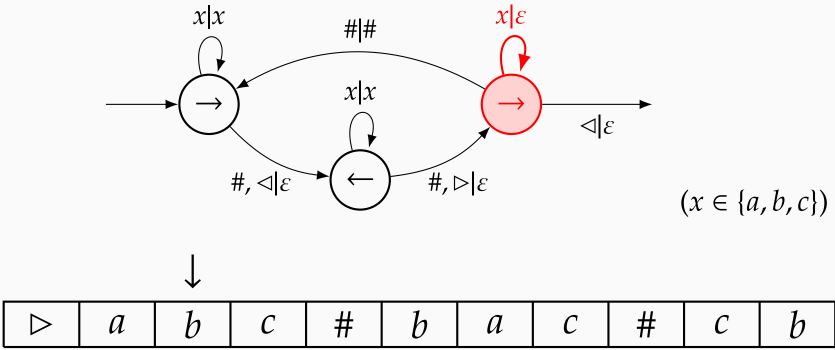

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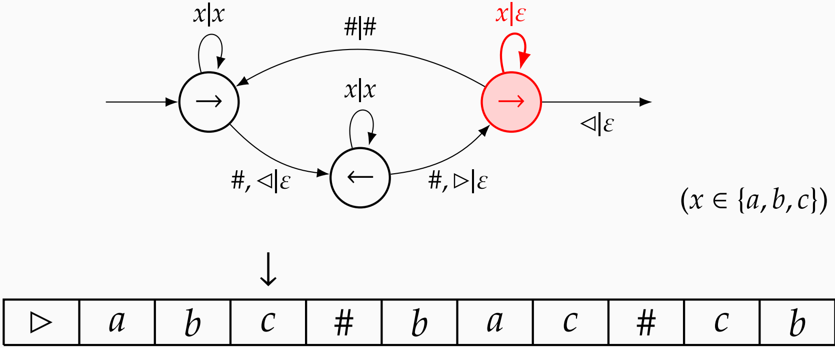

Output: *abccba*

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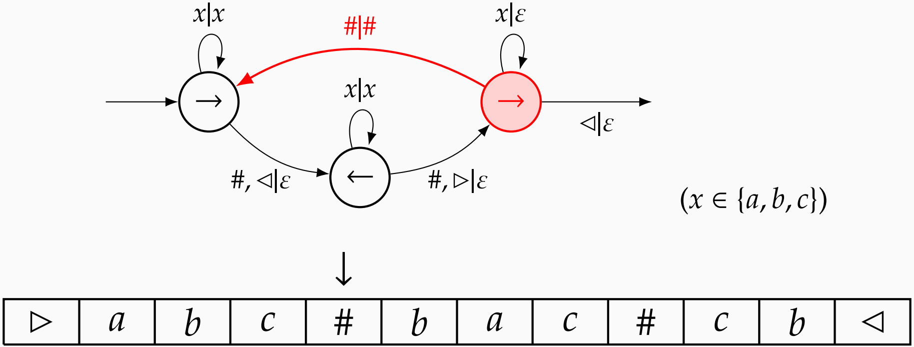

Output: *abccba*

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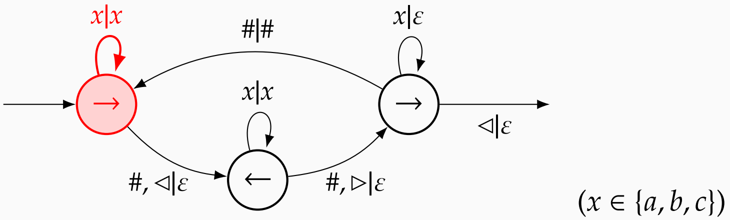

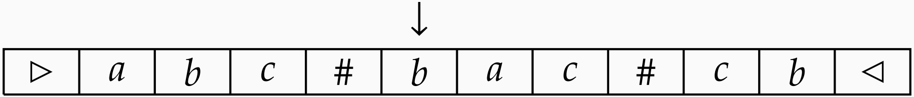

Output: $abccba\#$

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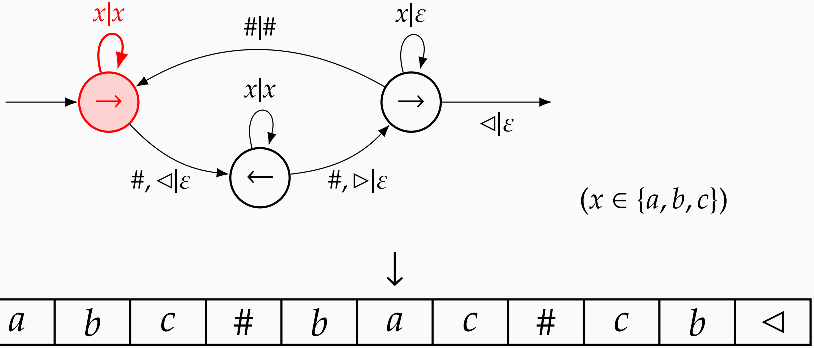

Output: *abccba#b*

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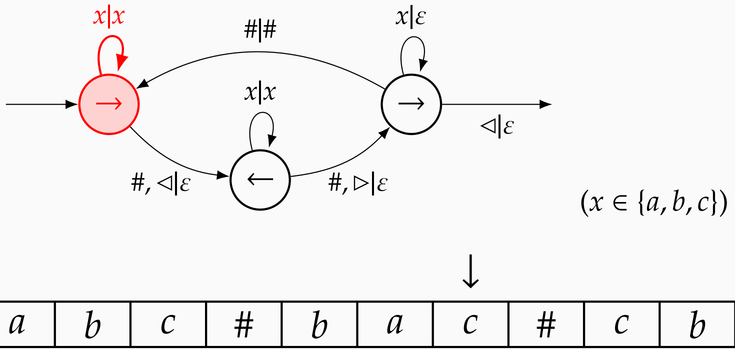

Output: *abccba#ba*

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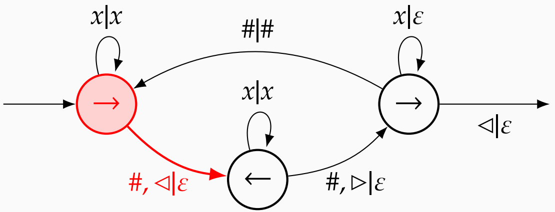

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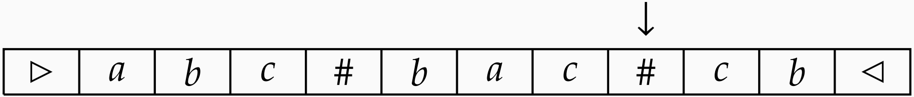

Output: *abccba#bac*

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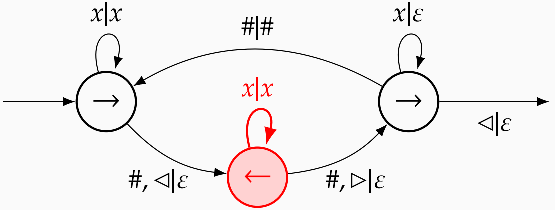

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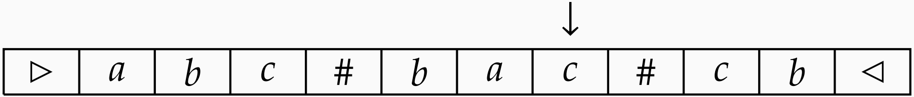

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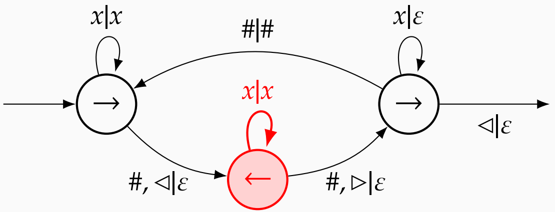

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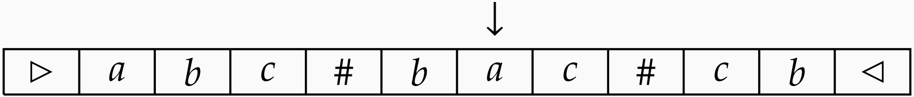

Output: *abccba#bacc*

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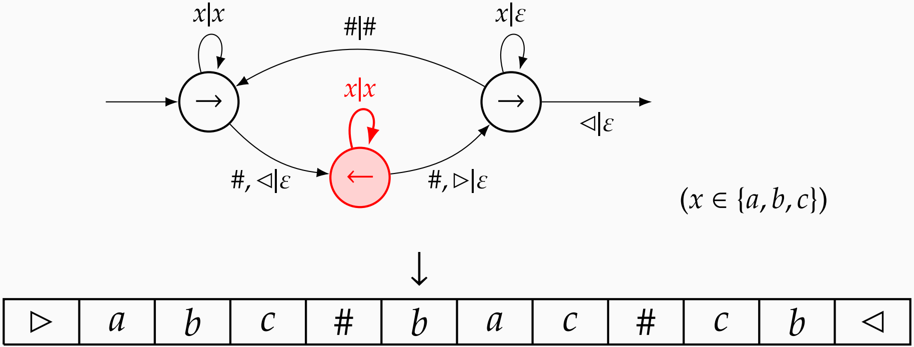

Output: $abccba\#bacca$

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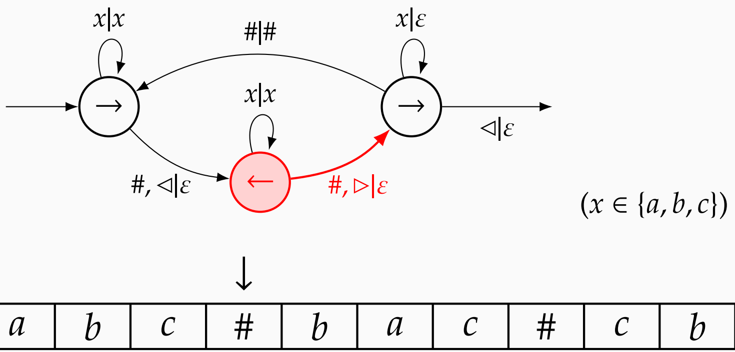

Output: *abccba#baccab*

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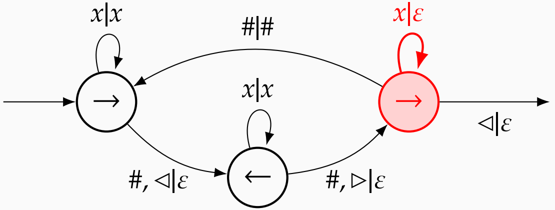

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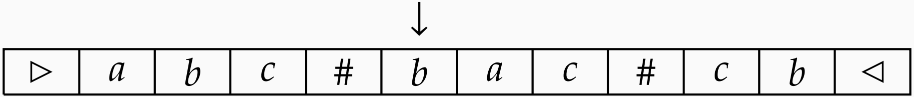

Output: *abccba#baccab*

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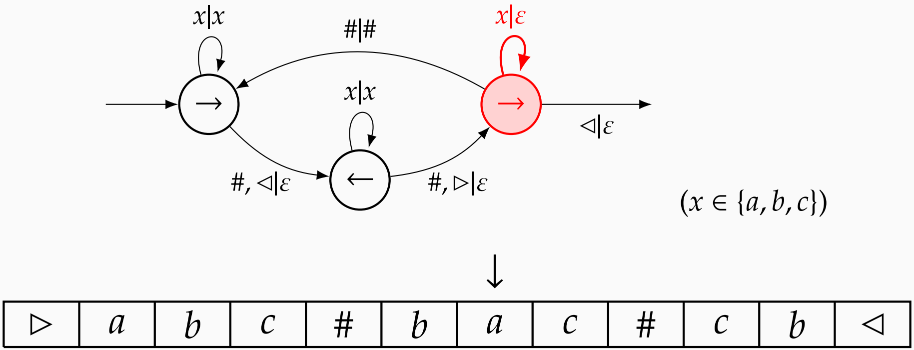

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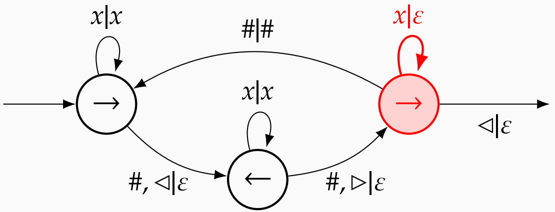

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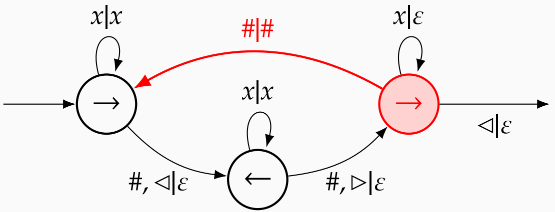

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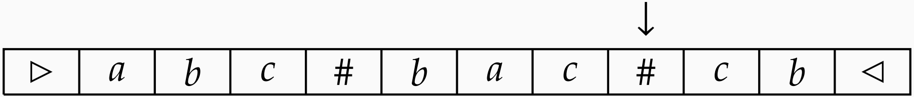

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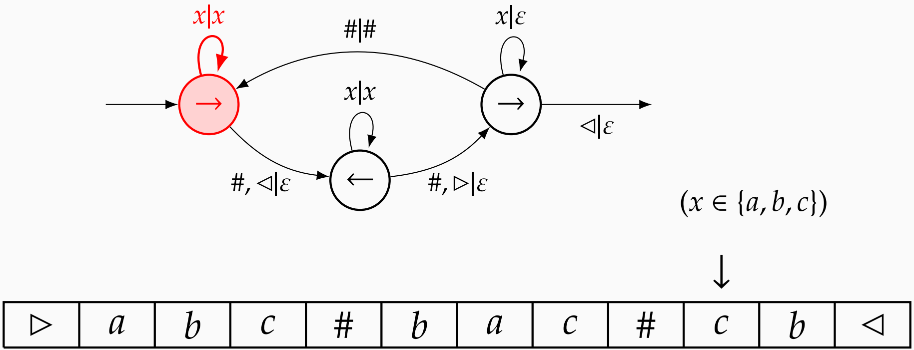

Output: *abccba#baccab#*

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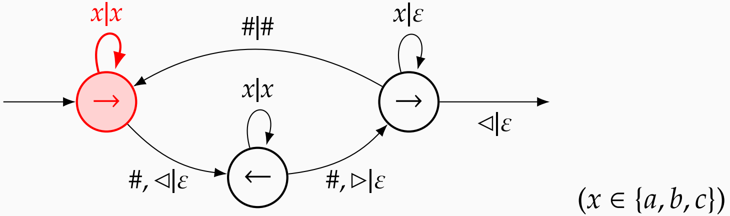

↓

▷	a	b	c	#	b	a	c	#	c	b	◁

Output: $abccba\#baccab\#c$

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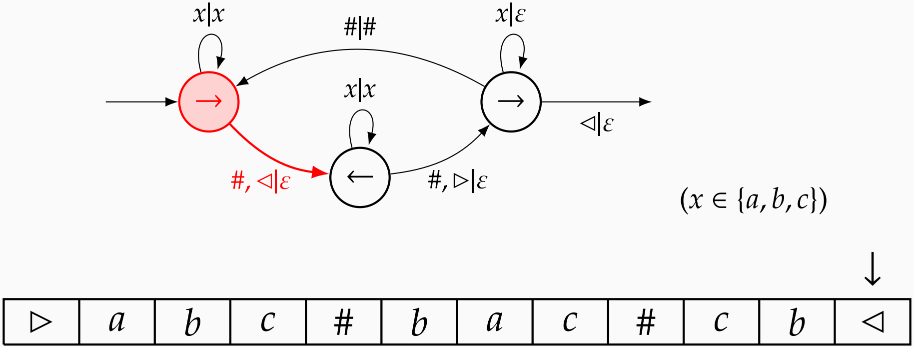

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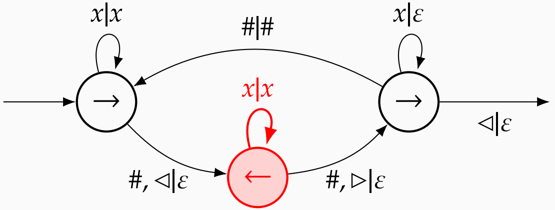

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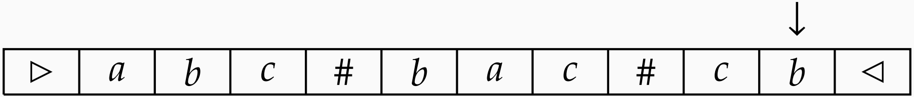

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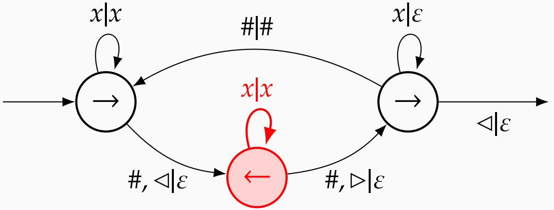

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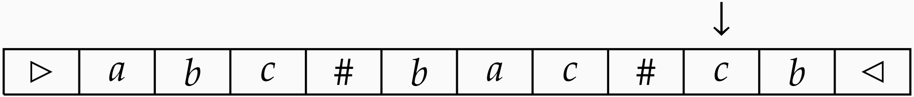

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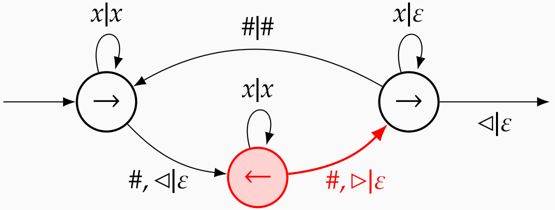

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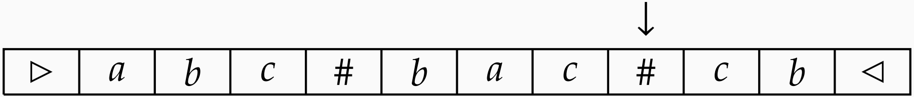

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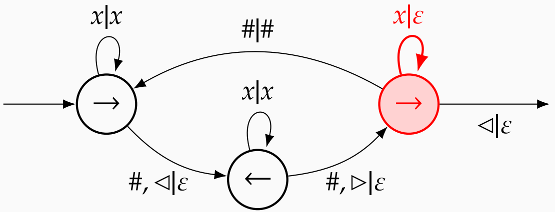

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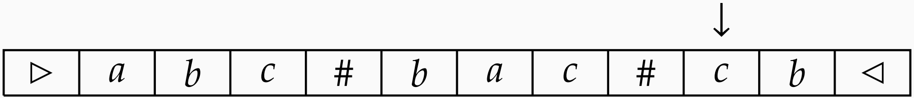

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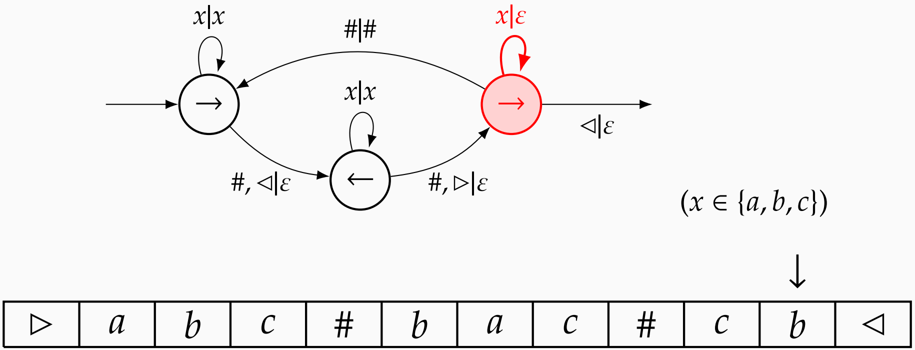

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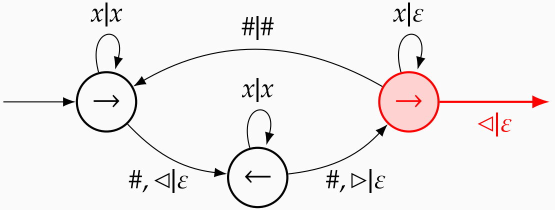

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▷	a	b	c	#	b	a	c	#	c	b	◁

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Two-way transducers are more powerful than one-way

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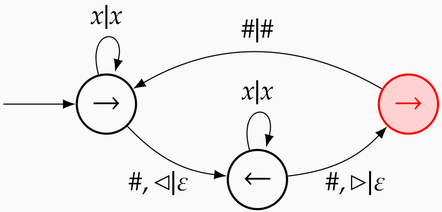

$(x \in \{a, b, c\})$

▷	a	b	c	#	b	a	c	#	c	b	◁

Output: *abccba#baccab#cbbc*

How to realize this with a one-way device?

$$\begin{aligned} \texttt{mapReverse}: \{a,b,c,\#\}^* &\to \{a,b,c,\#\}^* \\ w_1\# \dots \# w_n &\mapsto \texttt{reverse}(w_1)\# \dots \#\texttt{reverse}(w_n) \end{aligned}$$

a	c	a	b	#	b	c	#	c	a

$X = \varepsilon \qquad Y = \varepsilon$

How to realize this with a one-way device?

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$\downarrow$

a	c	a	b	$\#$	b	c	$\#$	c	a

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	↓								
a	c	a	b	#	b	c	#	c	a

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		↓							
a	c	a	b	#	b	c	#	c	a

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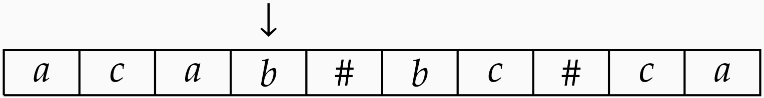

$X = baca \qquad Y = \varepsilon$

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$\downarrow$

a	c	a	b	$\#$	b	c	$\#$	c	a

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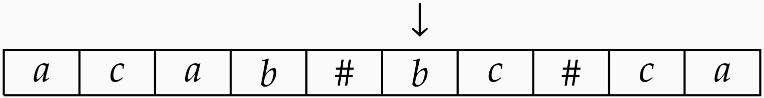

$X = b \qquad Y = baca\#$

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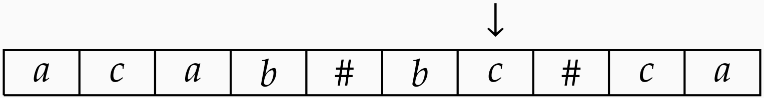

$X = cb \qquad Y = baca\#$

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a	c	a	b	$\#$	b	c	$\#$ ↓	c	a

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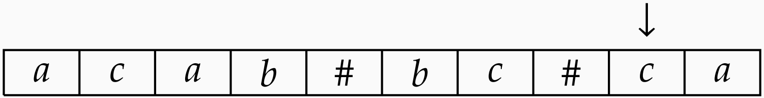

$X = c \qquad Y = baca\#cb\#$

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$\downarrow$

a	c	a	b	#	b	c	#	c	a

$X = ac \qquad Y = baca\#cb\#$

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Important point

X, Y concatenable, but **not inspectable** (“if $X[k] = a$ then…”)

$\longrightarrow$ control flow stays finite-state

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What can you “reasonably” put in memory?

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What can you “reasonably” put in memory? **LAMBDA: THE ULTIMATE**

Higher-order tree automata / transducers: simply typed λ-calculus

$$\text{Bottom-up tree aut.:} \quad a(b(c), c) \mapsto \texttt{accept?}(\delta_a(\delta_b(\delta_c), \delta_c)) \text{ with } \delta_a \colon Q^2 \to Q, \ldots$$

$$\text{Higher-order tree aut.:} \quad a(b(c), c) \mapsto \texttt{accept?}\ (t_a\ (t_b\ t_c)\ t_c) \text{ with } t_a : A^2 \Rightarrow A, \ldots$$

$$Q \text{ finite set} \qquad \text{vs.} \qquad A, B ::= o \mid A \times B \mid A \Rightarrow B$$

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Q finite set vs. $A, B ::= o \mid A \times B \mid A \Rightarrow B$

Theorem (≃ Hillebrand & Kanellakis 1996 (Damm 1982?))

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$A = (o^{\ell_1} \Rightarrow o) \times \cdots \times (o^{\ell_k} \Rightarrow o)$ ▶ k-state *macro tree transducer*, e.g. previous slide!

[Engelfriet & Vogler 1986], staple of "old-school" transducer theory

Higher-order tree automata / transducers: affine types

Problem (feature?): HO tree transducers can express *a lot* of functions
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Theorem (claimed in my PhD; details: [Pradic & Price, MFPS'24])

Affine HO string transducers $\equiv$ *two-way transducers* ($\equiv$ *MSO transductions*)

i.e. replace $A \Rightarrow B$ by *affine* $A \multimap B$ which *can only use* A *once to produce* B
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Conjecture (N. & Pradic, ICALP'21)

Affine HO tree automata $\subsetneq$ regular tree languages

Results of the paper (1)

Tree-walking: generalization of two-way automata

1 reading head moving around the tree in any direction

Theorem (N. & Vanoni, this paper)

Affine HO tree automata/transducers $\subseteq$ *reversible tree-walking aut./trans.*

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Theorem (N. & Vanoni, this paper)

Affine HO tree automata/transducers	$\subseteq$	*reversible tree-walking aut./trans.*
Almost affine HO tree automata/transducers	$\subseteq$	*tree-walking aut./trans.*

Inexpressivity conjecture from last slide follows from:

Theorem (Bojańczyk & Colcombet 2005)

Tree-walking automata $\subsetneq$ *regular tree languages*

Almost affine [Kanazawa]: the base type o can be duplicated, but not the others
$\leftrightarrow$ "sharing" in the configuration graph of a tree-walking transducer

Results of the paper (2)

Lookaround = can inspect regular information at each node
= preprocessing by very simple transducers / MSO relabeling

Corollary (new proof of [Kanazawa 2008; Gallot, Lemay & Salvati 2020])

Affine HO tree transducers with lookaround	$\equiv$	*MSO transductions*
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Other way to overcome inexpressivity [N. & Pradic]: add &/$\oplus$ types

$A \otimes B$ ("multiplicative") vs. $A \& B$ ("additive")

(better suited to "implicit automata" POV)

Results of the paper (3)

Exponential modality $!A$ makes A duplicable

$A, B ::= o \mid A \multimap B \mid !A \qquad (A \Rightarrow B = !A \multimap B)$

Affine = !-free

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Theorem (N. & Vanoni, this paper)

Almost !-depth 1 HO tree trans. w/ lookaround ≡ invisible pebble tree transducers

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Thanks for your attention!

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