Some Recent Advancements in Monotone Circuit Complexity

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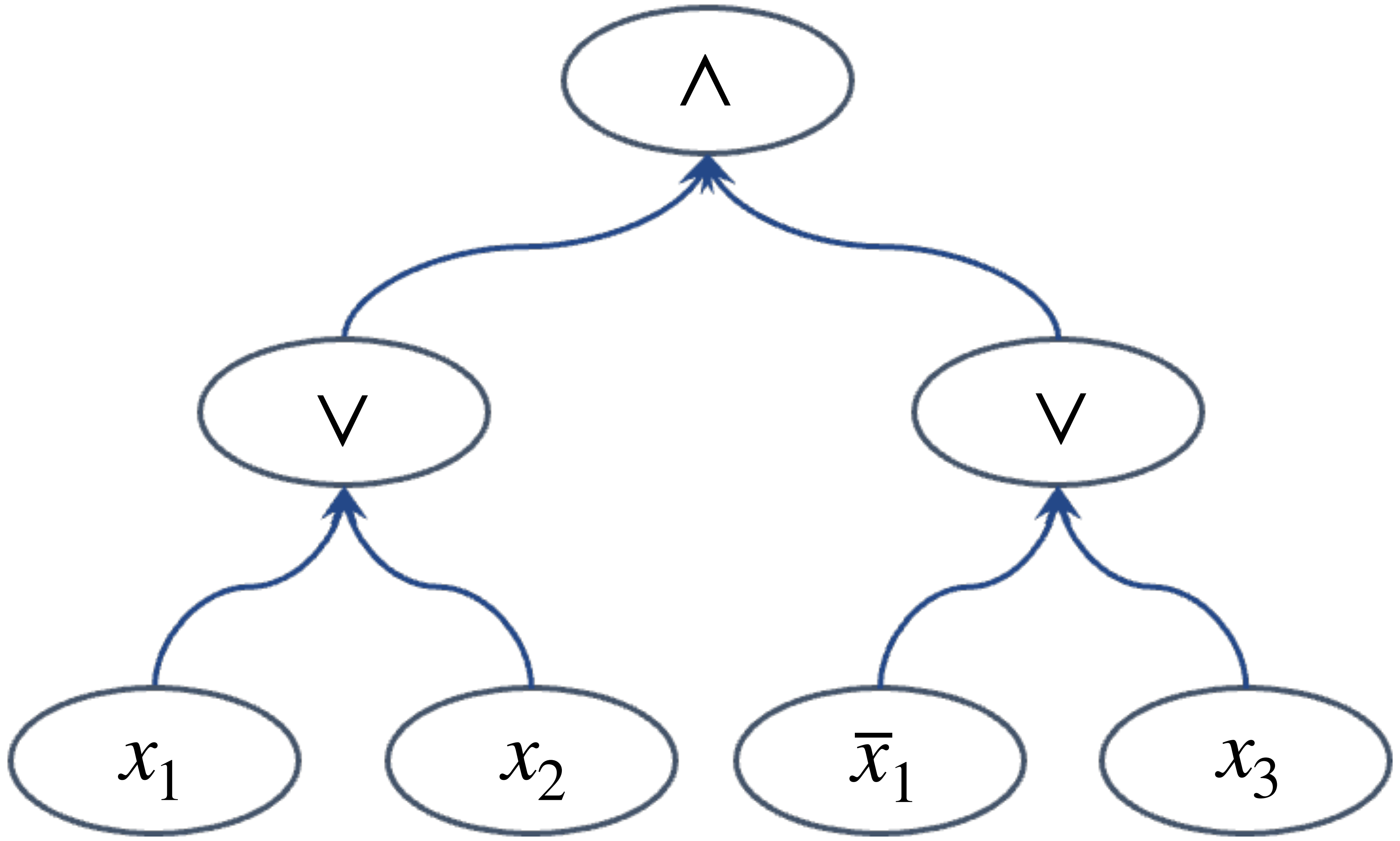
SYMPOSIUM ON THEORETICAL ASPECTS OF COMPUTER SCIENCE

Boolean Circuits

The difficulty in proving that a given boolean function has high complexity lies in the nature of our adversary: the circuit. Small circuits may work in a counterintuitive fashion, using deep, devious, and fiendishly clever ideas. How can one prove that there is no clever way to quickly compute the function? [Jukna '12]

- Input gates (Boolean literals) and $\wedge$, $\vee$ gates*
- **Size**: # of gates, **Depth**: length of longest path
- *Monotone* if only variable as inputs
- *Formula* if DAG is a tree
- Depth-d lower bound (fan-in 2) for f
  $\Rightarrow$ formula size-$2^{\Omega(d)}$ lower bound for f



* DeMorgan Boolean circuits: poly-equivalent to Boolean circuits

Boolean Circuits

- Most functions require circuits of size $2^n/n$ [Shanon '49]
- Goal: exhibit hard functions and understand why they are hard
- Best lower bound until recently was $3n$ [Blum '84]
  - Improved to $(3 + 1/86)n$ [Find, Golovnev, Hirsch, and Kulikov '16]
  - Improved to $3.1n$ [Li, Yang '22]

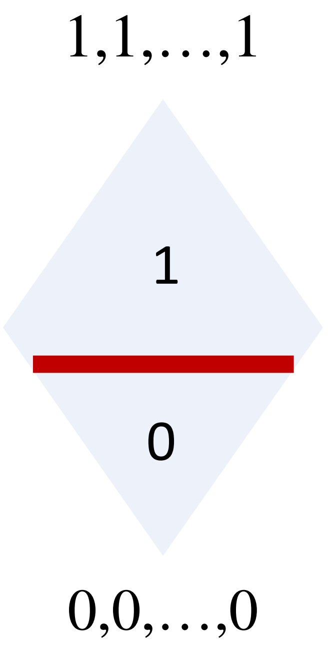
Why Study Monotone Boolean Circuits

- Natural computation model for monotone functions

> Why should one care about monotone circuits? The point is that this model has a purely "practical" importance. Namely, lower bounds for such circuits imply the same lower bounds for (min, +)-circuits, and hence, for dynamic programming. [Jukna '12]

- Connections to non-monotone: equally powerful* for slice functions

$$f(x) = \begin{cases} 1 & \text{if } |x| > k \\ g(x) & \text{if } |x| = k \\ 0 & \text{if } |x| < k \end{cases}$$



* up to constant factor and small additive factor [Berkowitz '82, Valiant '86]

Monotone Complexity of Boolean Functions

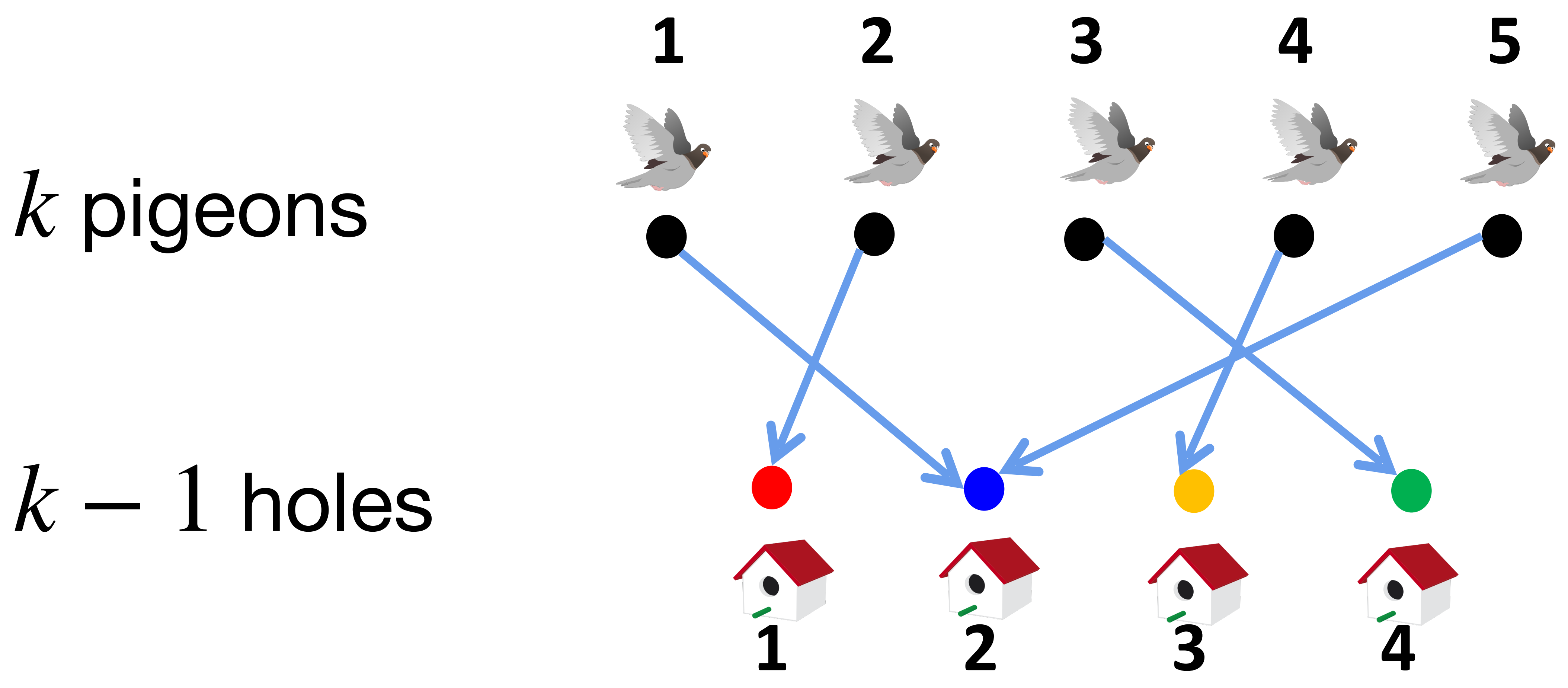
- Best lower bound for ***monotone circuits/formulas*** for f in NP? And for f in P?
  - And for f in AC^i or NC^i?
  - Or even in AC^0? [Grigni and Sipser '92]

NC^i = poly-size depth-$O(\log^i n)$ fan-in 2 circuits
AC^i = poly-size depth-$O(\log^i n)$ unbounded fan-in circuits

- Best **separations** for:
  - Monotone formulas and monotone circuits
  - Monotone AC^i and NC^i, for different levels i
- Does it make sense to consider monotone circuits of ***(supercritical) depth*** $> n$? Are they stronger than monotone circuits of depth $\leq n$?
- What is the complexity of functions like: st***-connectivity, perfect matching, clique?***

* *We will not talk about other monotone model of computations (such as monotone span programs)*

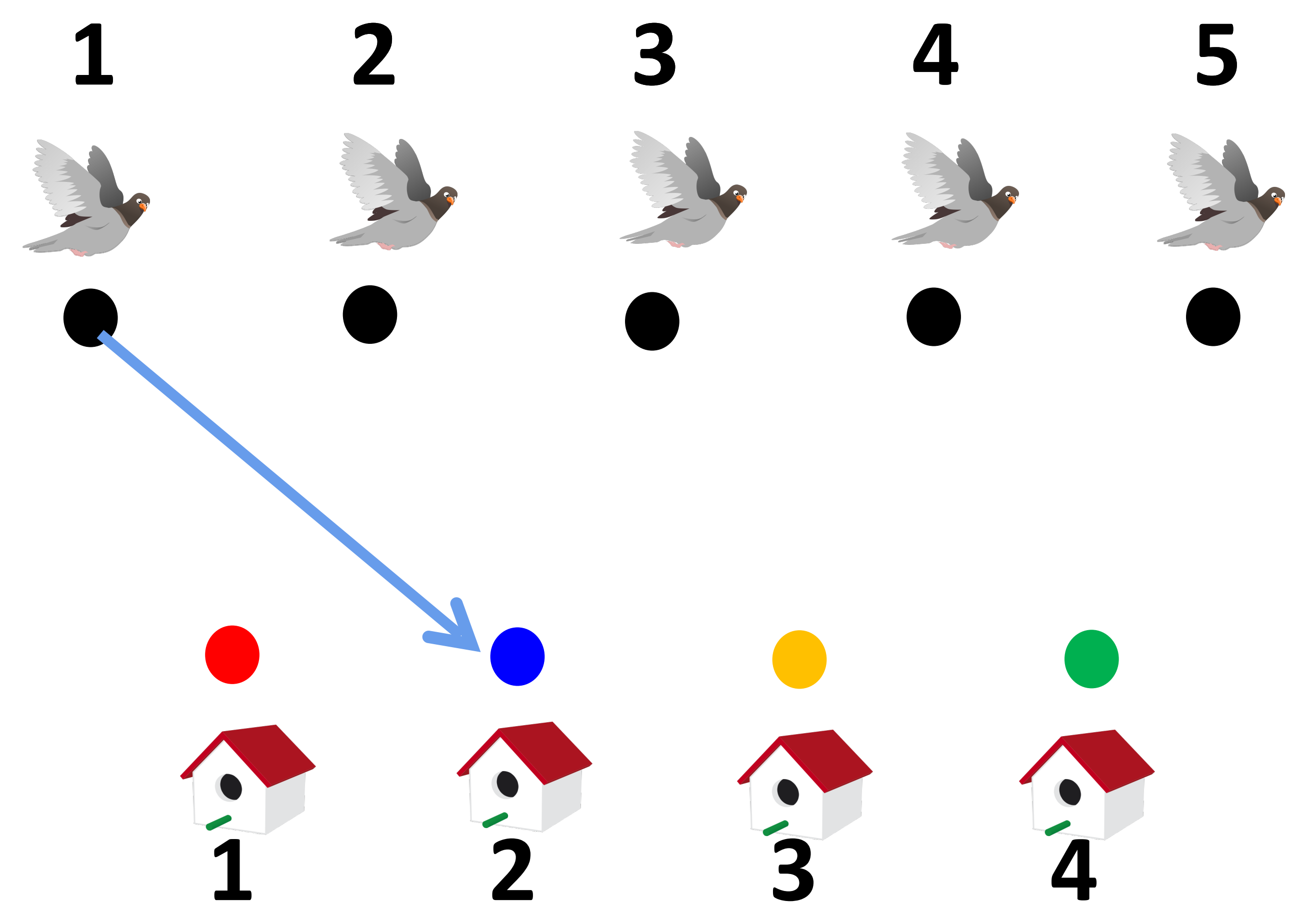
Warm Up: Find a Collision Problem



Warm Up: Find a Collision Problem

k pigeons

$k-1$ holes



- Ask where any pigeons flies and write it on the blackboard (1 line)
- How many queries until guaranteed to find collision?
- Allow you to erase: how many lines need to have (simultaneously) to find collision?

Plan

After Jukna's 2012 book: "Boolean Function Complexity: Advances and Frontiers"

Part I: Classical results

- 1985: exponential lower bounds
- 1990: Karchmer-Wigderson game for depth
- 1997: Raz-McKenzie lifting theorem for depth

Part II: Recent results

- More lower bounds
- 2017: Karchmer-Wigderson game for size
- 2018: Lifting theorem for size
- 2019-2025: Improvements and consequences

- Open problems

Part I: Classical results

Exponential Lower Bounds for Monotone Circuits

- Until 1985: only linear lower bounds for both monotone and non-monotone
- $n^{\Omega(\log n)}$-size lower bound for **clique** and **perfect matching** [Razborov '85]
- $\exp(\Omega(n^{\epsilon}))$-size lower bound for ***Andreev function*** [Andreev '85]

independent

Lower bound for f in P

- Improved above to: [Alon and Boppana '87]
  - $n^{\Omega(\sqrt{k})}$ lower bound for k-**clique** for $k \leq n^{2/3}$
  - $\exp\left(\tilde{\Omega}(n^{1/4})\right)$ lower bound for **Andreev function**

Common: approximation method

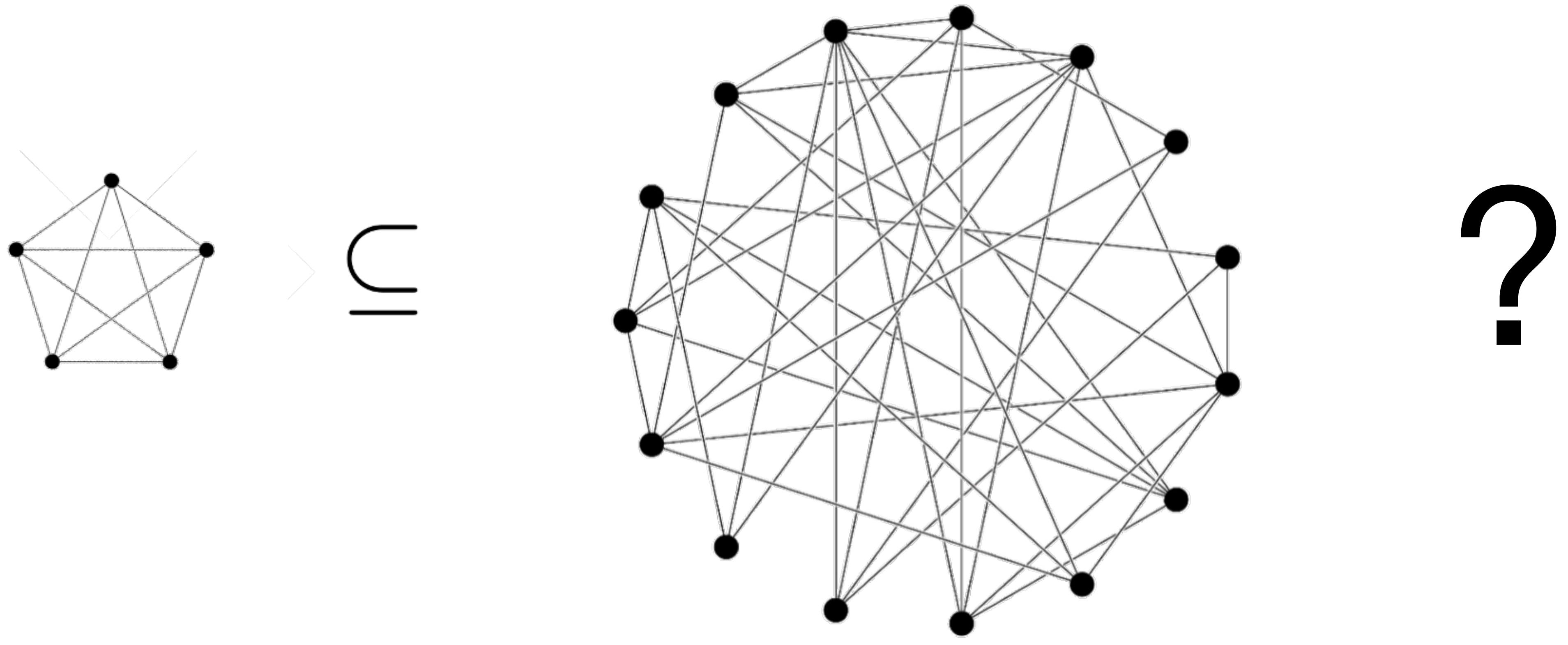
Lower bound for f in NP

- $\exp\left(\tilde{\Omega}(n^{1/3})\right)$ lower bound for **Andreev function** [Andreev '87]

Best known until 2020

A Famous NP-hard Graph Problem: Clique

- Does G have a clique of size k?



- Brute-force: time $n^{O(k)}$
- Requires $n^{\Omega(k)}$ assuming ETH [Impagliazzo, Paturi '01, Chen, Huang, Kanj, Zia '04]

Another Famous NP-Hard Graph Problem: Colouring

- Is there a proper colouring of vertices of G with c colours?
- G cannot have a k-clique and be $(k-1)$-colourable: how hard to distinguish?

$$\text{Clique-Col}_k(G) := \begin{cases} 1 & \text{if } G \text{ has a } k\text{-clique,} \\ 0 & \text{if } G \text{ is } (k-1)\text{-colorable,} \\ * & \text{otherwise.} \end{cases}$$

- This is in P (because Lovász ϑ function is in P and distinguishes)
- [Razborov '85, Alon and Boppana '87]: also applies for clique-colouring
- $\exists$ monotone function that distinguishes [Tardos '88]

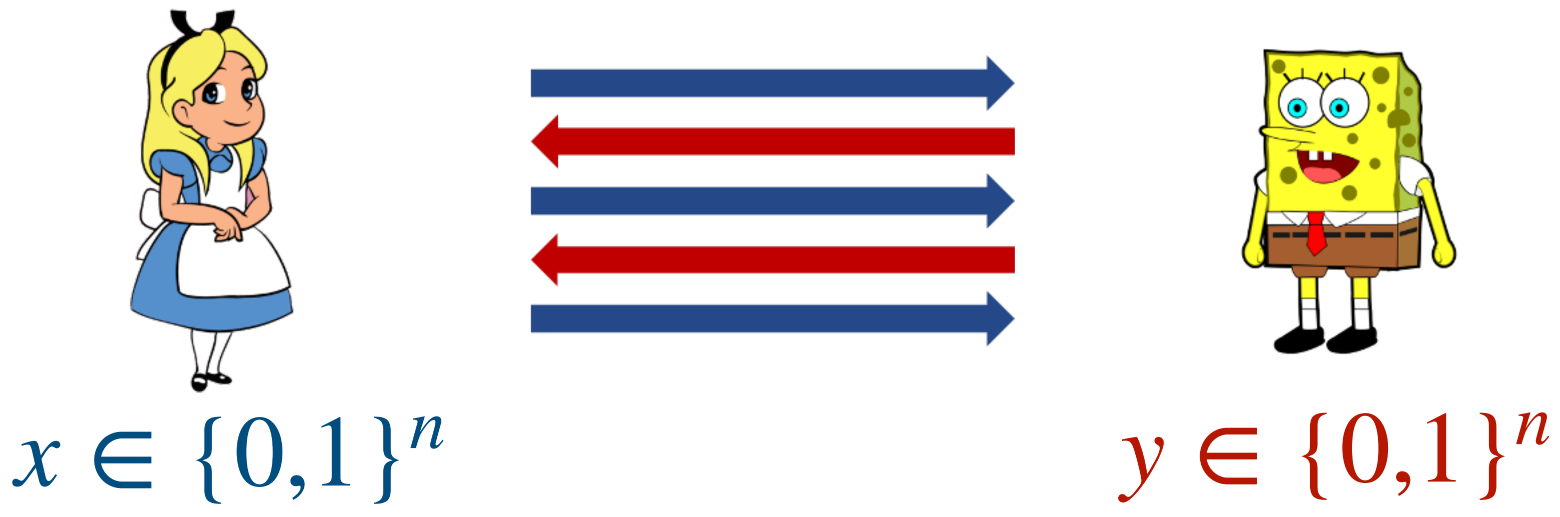
Lower bound for f in P: $\exp(\tilde{\Omega}(n^{1/6}))$

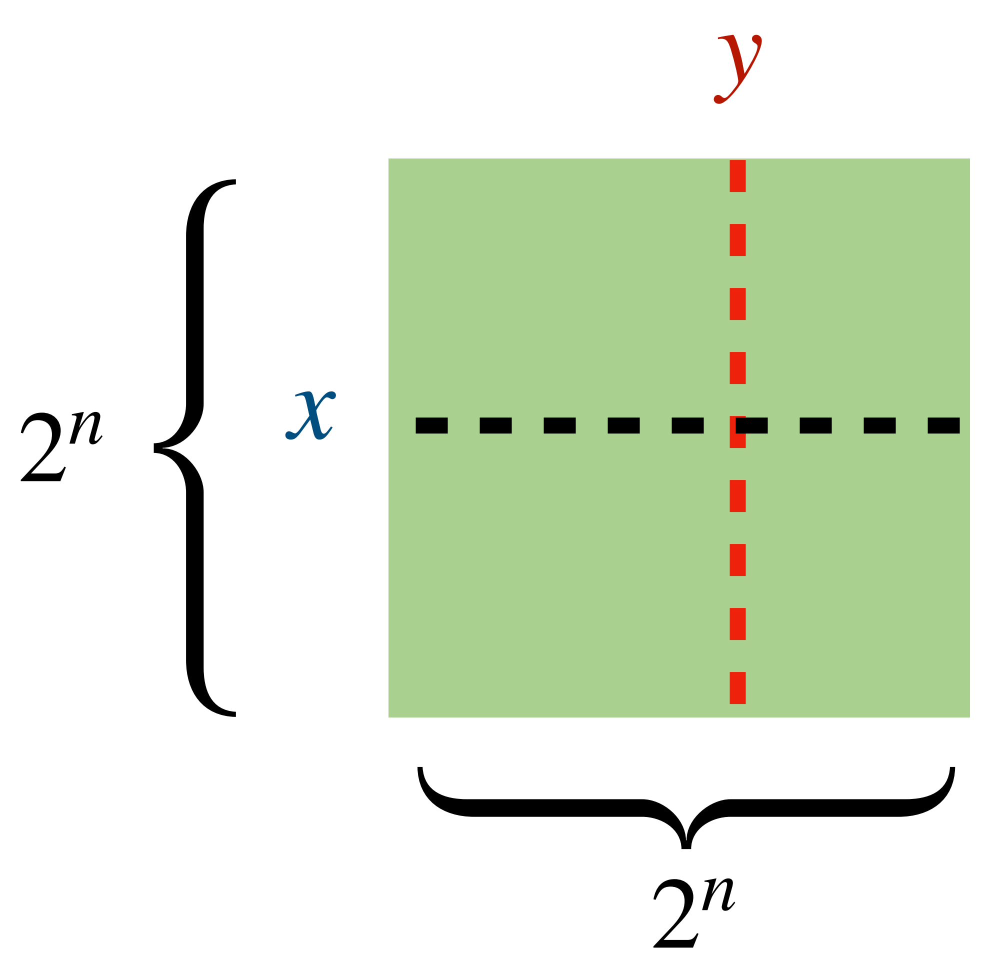
Best known until 2025

Communication complexity

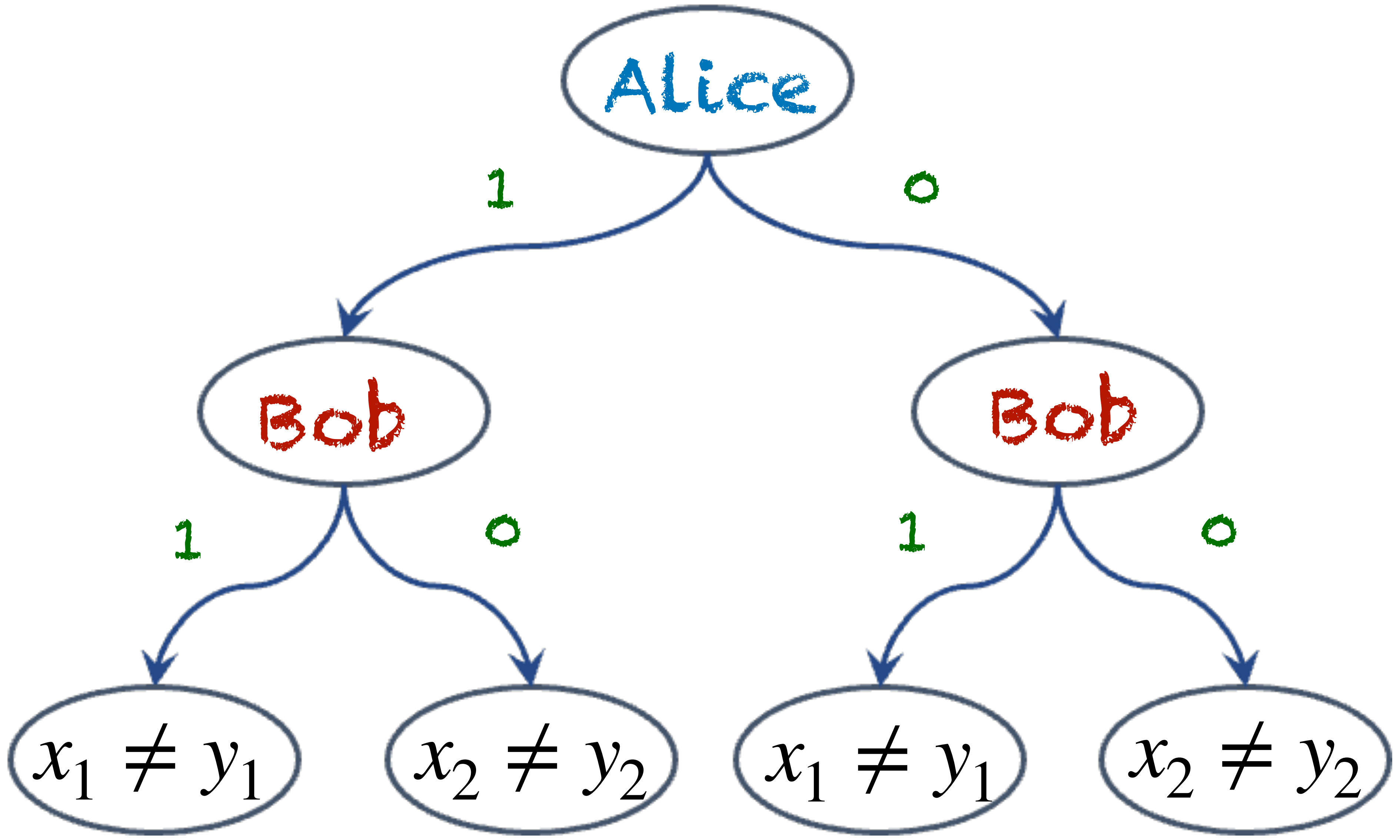
understanding circuit depth

Game: Find a Differing Bit





- Goal: communicate min # bits to find i s.t. $x_i \neq y_i$
- Before seeing input, decide communication protocol:
  - "strategy tree": who speaks when, what message means



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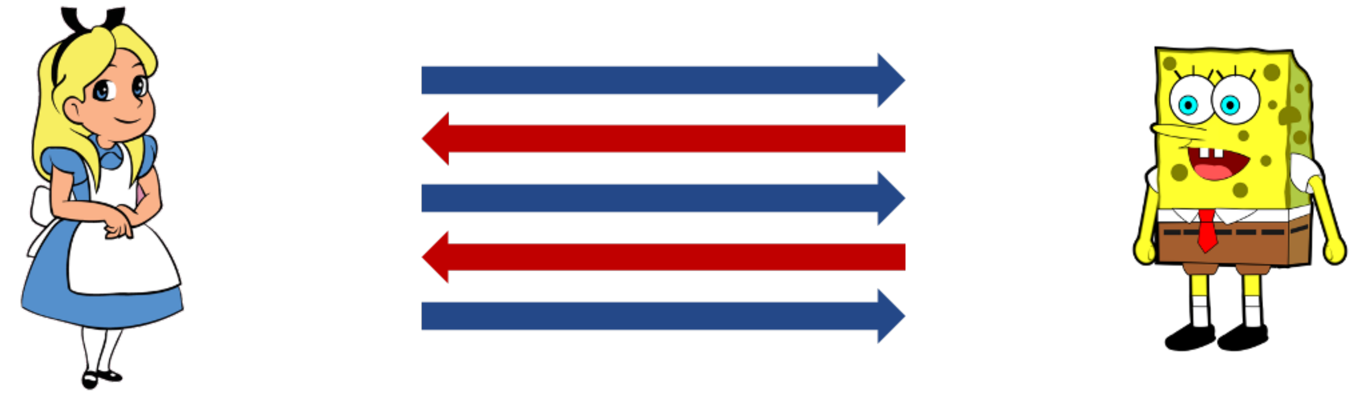
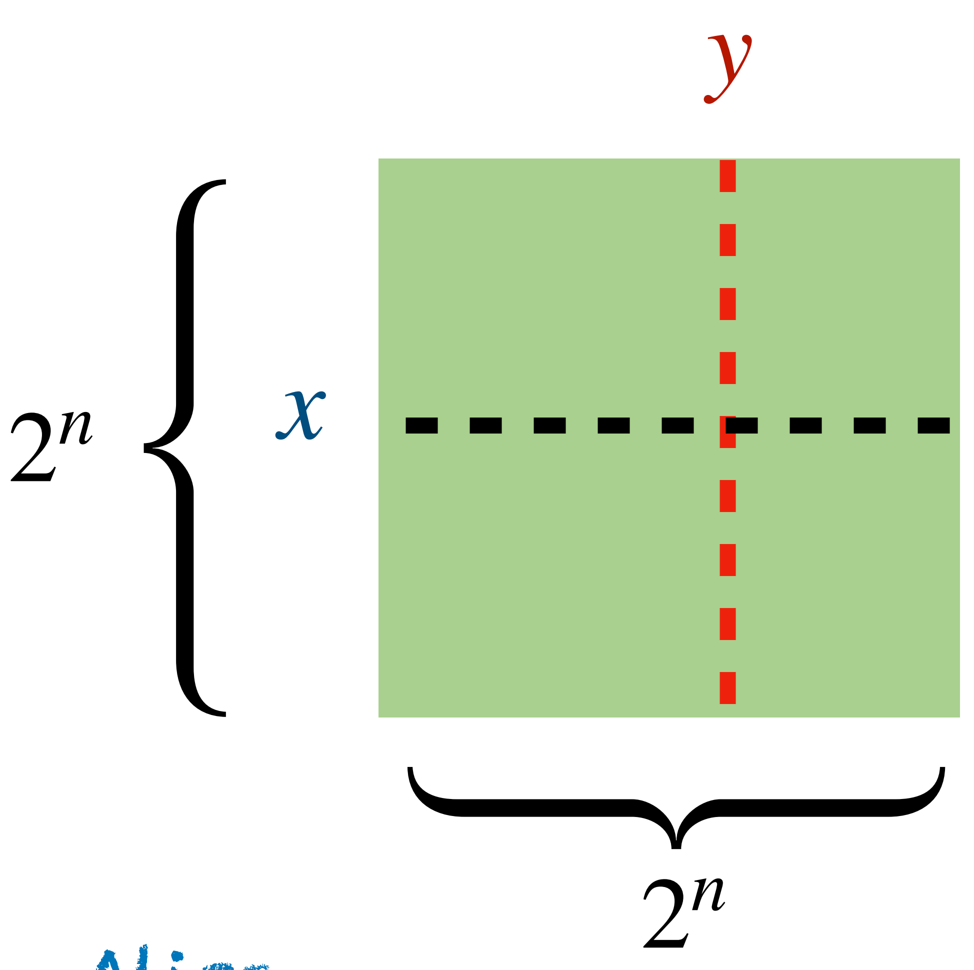


$x \in \{0,1\}^n$ $y \in \{0,1\}^n$



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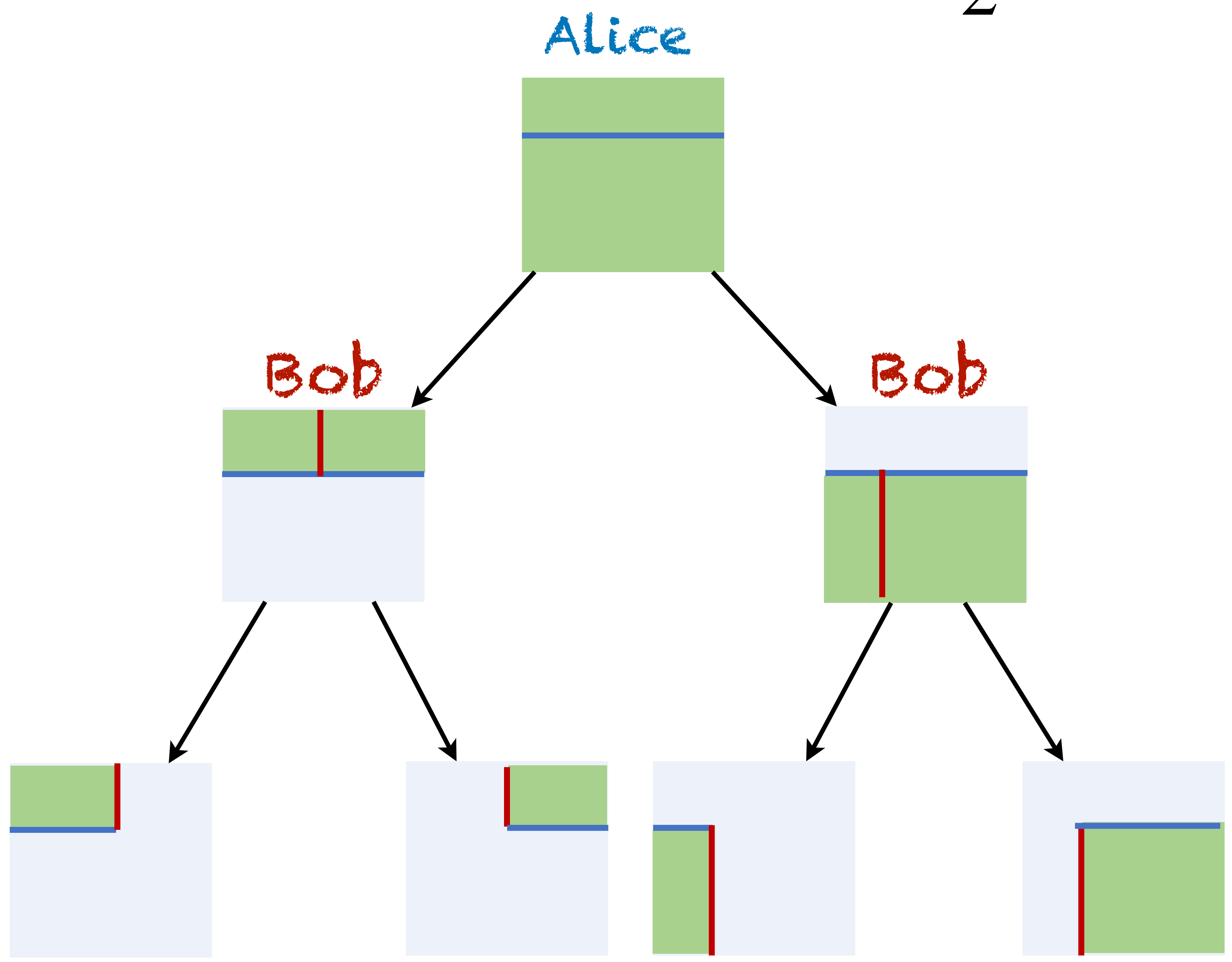
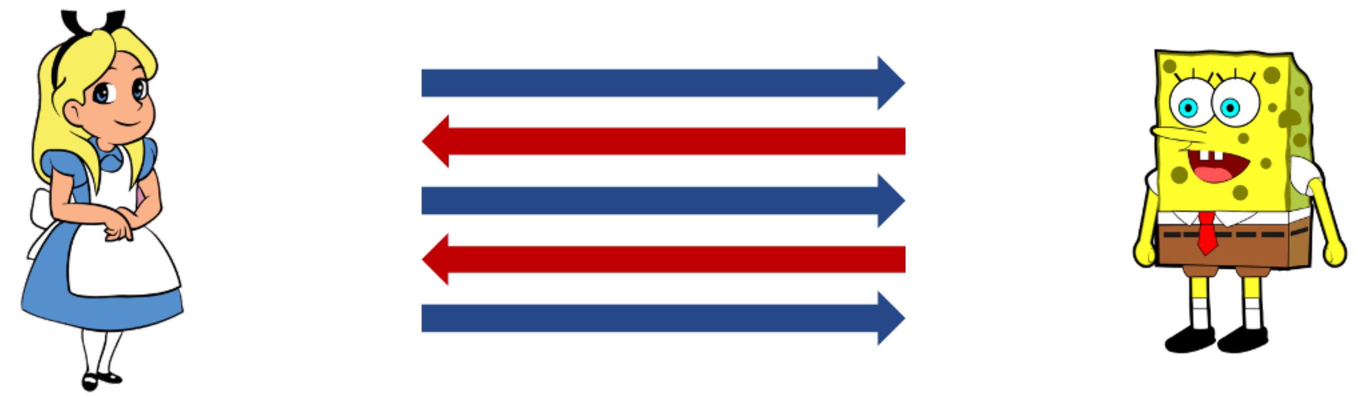


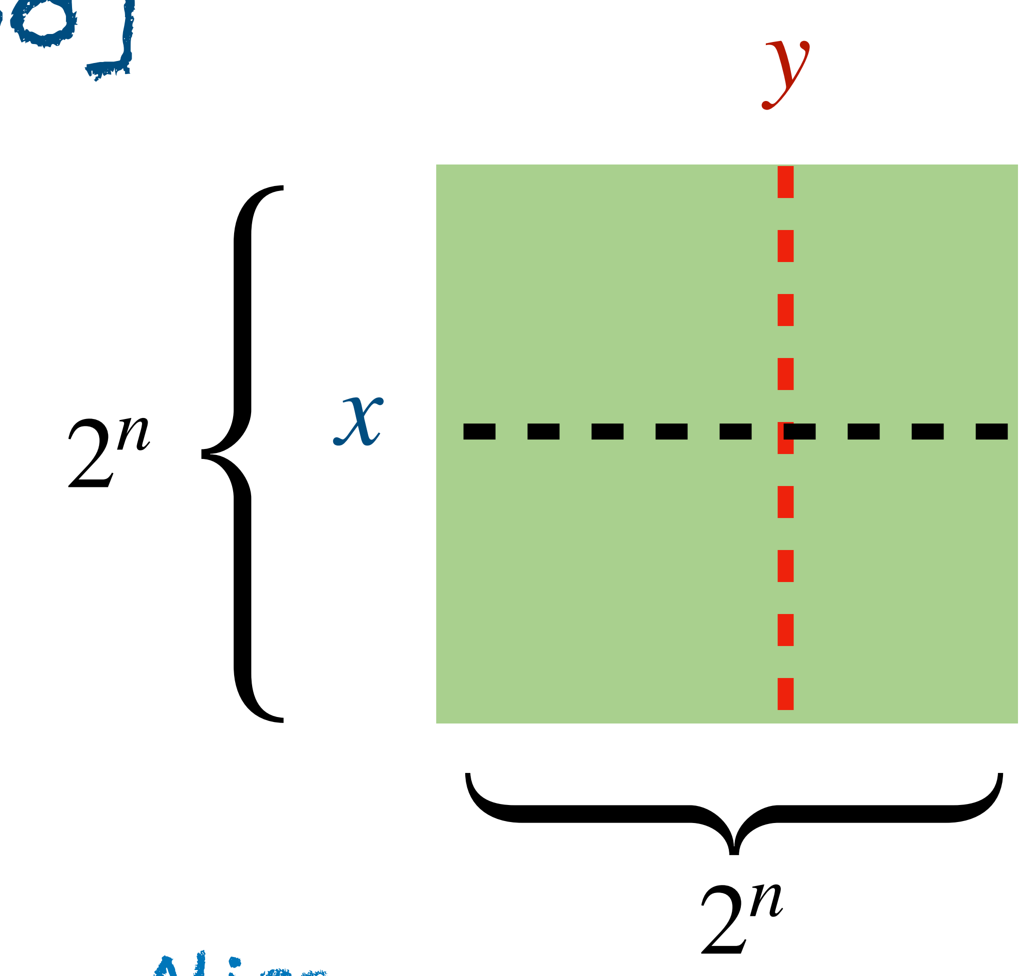
Karchmer-Wigderson game KW(f) [KW'90]

- Given f



$x \in \{0,1\}^n$ $\quad$ $y \in \{0,1\}^n$

$x \in f^{-1}(1)$ $\quad$ $y \in f^{-1}(0)$



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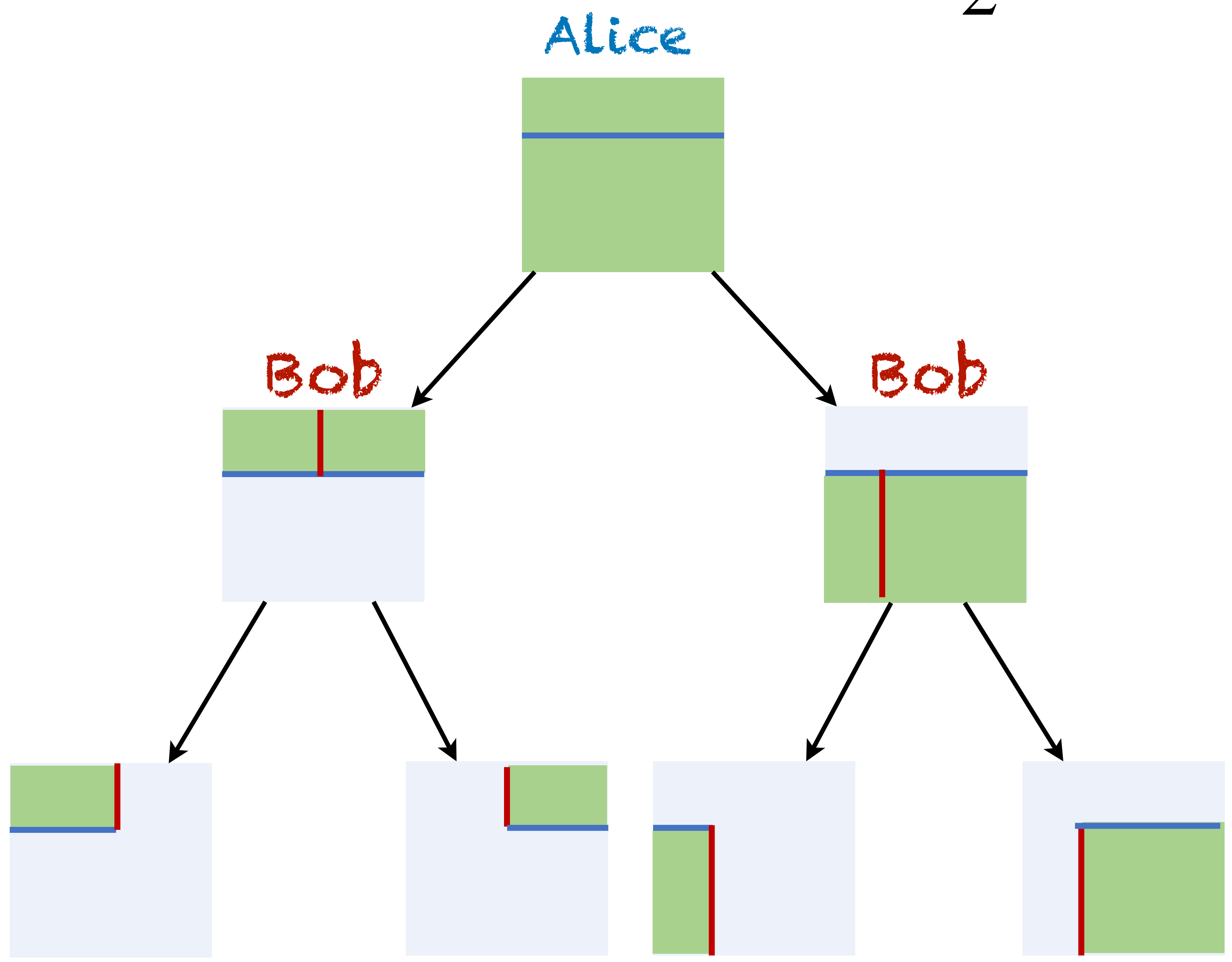
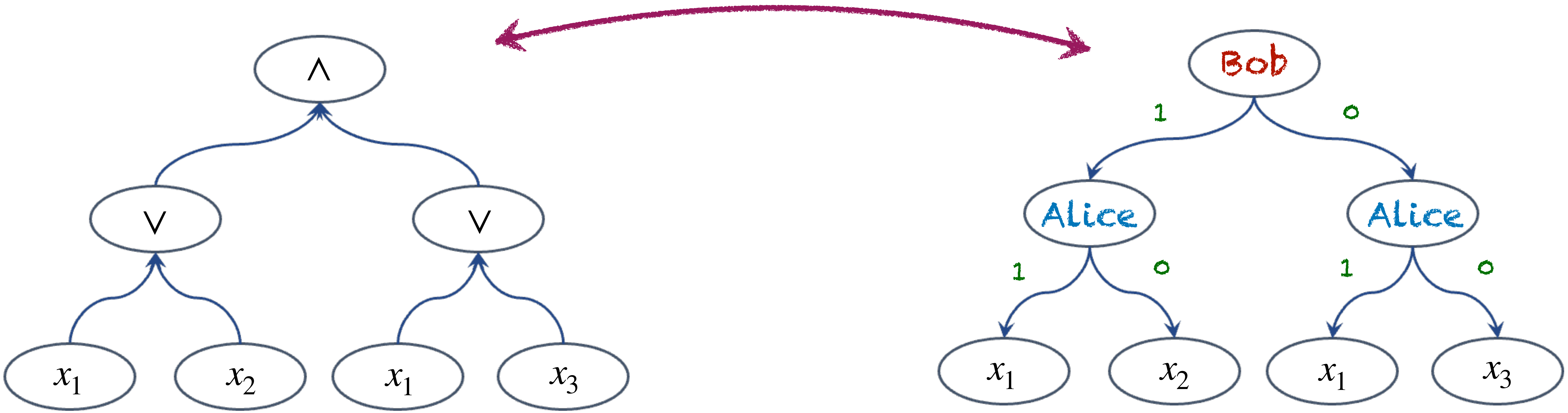


Formulas = communication protocols



∃ depth-d formula computing f ⇔
∃ depth-d communication protocol for KW(f)

[KW'90]

- Result is stronger: really the same object (even graph structure is preserved)

Monotone Karchmer-Wigderson game [KW'90]

- KW(f) : given $x \in f^{-1}(1), y \in f^{-1}(0)$ **find i s.t. $x_i \neq y_i$**

i.e. $x \geq y \Rightarrow f(x) \geq f(y)$

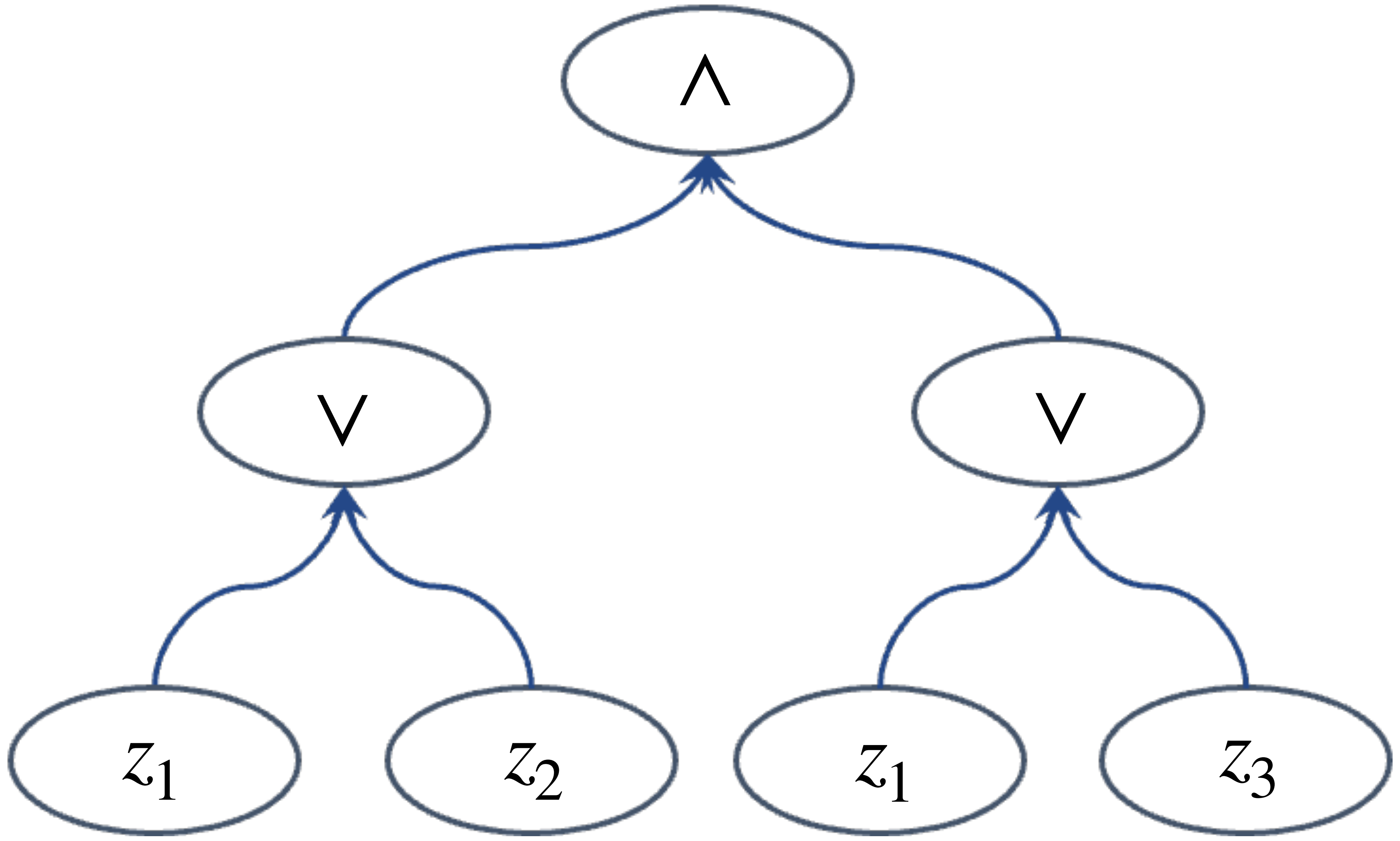
- For f monotone, mKW(f) *harder* problem: **find i s.t. $x_i > y_i$**

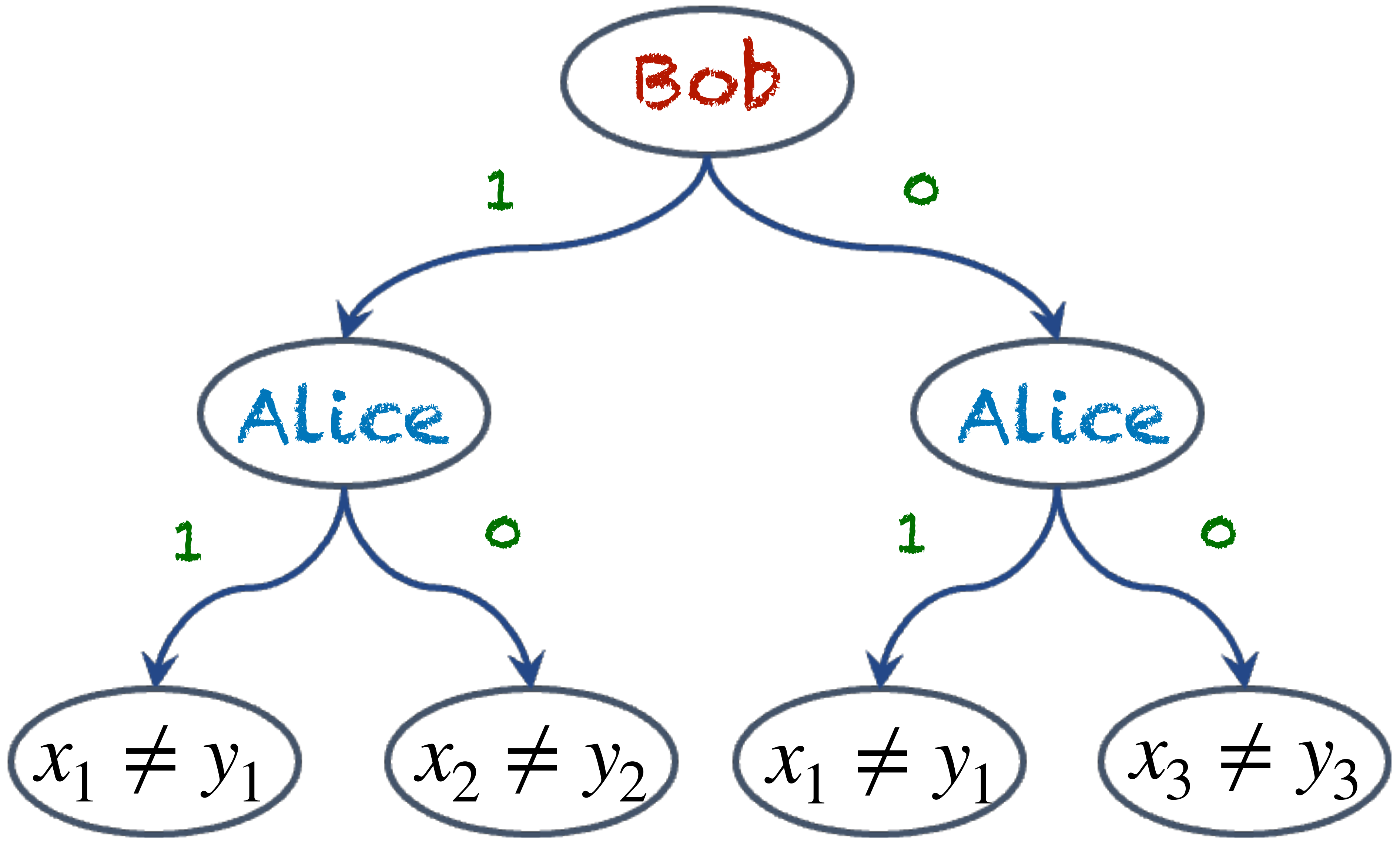
∃ depth-d monotone formula computing f ⇔
∃ depth-d communication protocol for mKW(f)

Example: f = majority

- $x = (1,1,0,1,0)$
- $y = (0,1,1,0,0)$
- 1,3,4 valid answer for KW(f)
- 1,4 valid for mKW(f) but not 3

Formulas = communication protocols

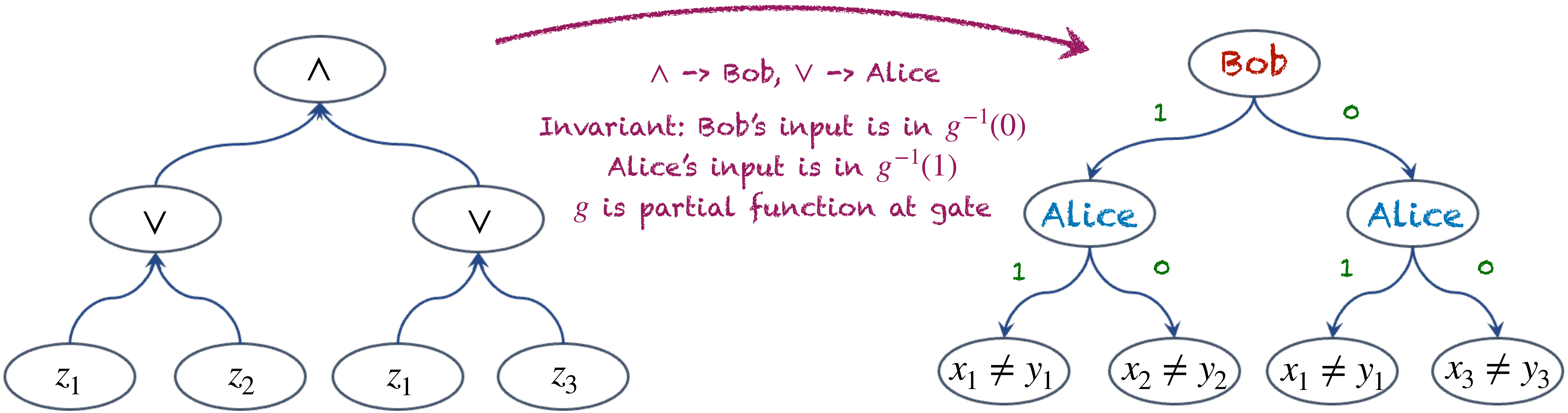




$\exists$ depth-d (monotone) formula computing $f \Leftrightarrow$
$\exists$ depth-d protocol for (monotone) KW(f)

[KW'90]

Formulas = communication protocols



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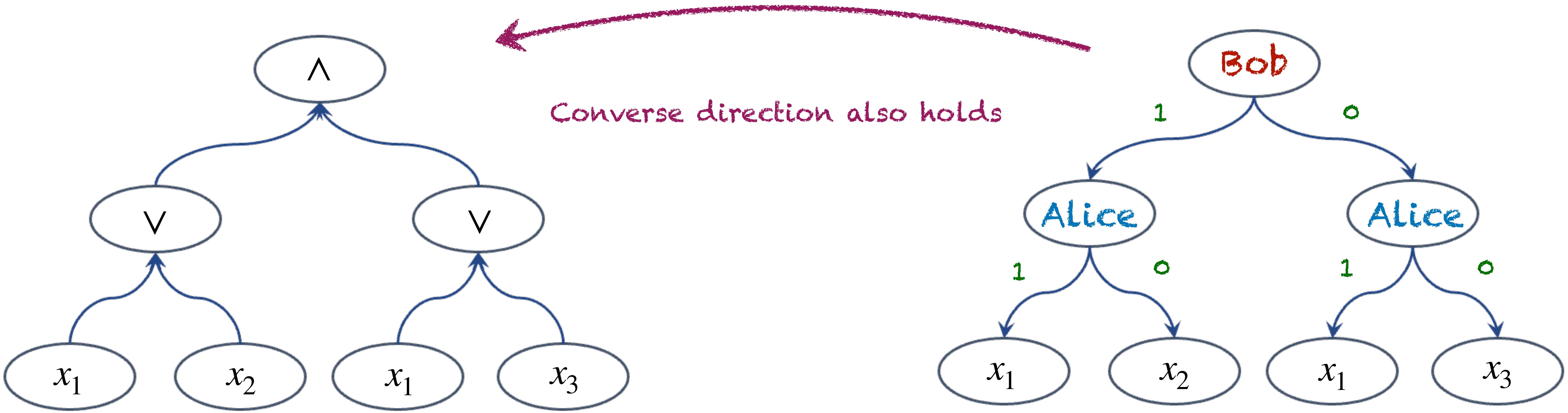

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[KW'90]

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Depth lower bound for st-connectivity [KW'90]

- Framework used for $\Omega(\log^2 n)$ depth lower bound for st-connectivity
- Separates mon-NC^1 and mon-NC^2
- KW(f) is a *total search problem*
  - Total search problem S $\subseteq I \times O$ s.t. $\forall z \in I \;\; \exists o \in O : (z, o) \in$ S
  - KW$(f) \subseteq \left(f^{-1}(1) \times f^{-1}(0)\right) \times [n]$

Raz-McKenzie: Lifting Theorem [RM '97]

- Idea: sometimes structured protocols (communicates bits of input) are best possible
  - $\exists$ special gadget $g : X \times Y \to \{0,1\}$
  - $\forall$ total search problem $\mathsf{S} \subseteq \{0,1\}^n \times O$

given $x \in X^n$, $y \in Y^n$ find $o \in O$
s.t. $(z, o) \in \mathsf{S}$ for $z_i = g(x_i, y_i)$

If $\mathsf{S} \circ g$ requires structured protocol of depth $\geq c \Rightarrow$
any protocol for $\mathsf{S} \circ g$ has depth $\Omega(c)$

- Structured protocol for $\mathsf{S} \circ g$ can only simulate decision trees for S

depth-d decision tree lower bound for $\mathsf{S} \Rightarrow$
depth-$\Omega(d \log n)$ communication protocol lower bound for $\mathsf{S} \circ g$

Raz-McKenzie: Lifting Theorem [RM '97]

- What does this have to do with circuits? mKW is universal for total search problems

depth-d decision tree lower bound for S $\Rightarrow$
depth-$\Omega(d \log n)$ monotone circuit lower bound for f_{S}

S	f_{S}
Induction Principle	st-connectivity
Pebbling	Generation
Find collision	Clique

Depth lower bound for f in mP
$\exp(\Omega(n^{\epsilon}))$

- reproved depth-$\Omega(\log^2 n)$ for st-connectivity
- separated mon-NC^i and mon-NC^{i+1}
- depth-$\Omega(k \log n)$ for k-clique for $k \leq n^{\epsilon}$

Part II: Recent developments

Monotone depth lower bounds

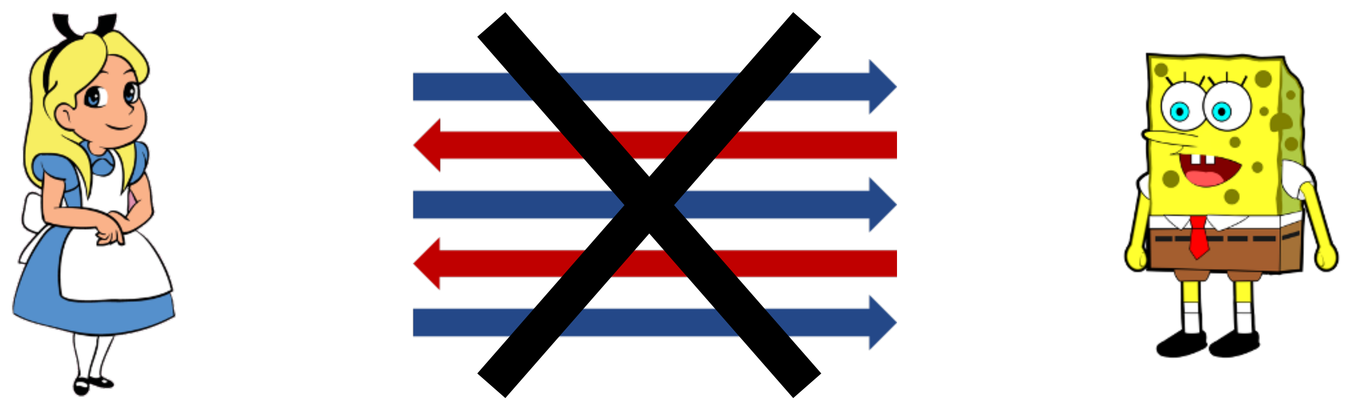
- For function in NP: $\Omega(n)$ [Pitassi, Robere '17]
  - $\Omega(n/\log n)$ [Göös, Pitassi '14], previously $\Omega(\sqrt{n})$ [Raz, Wigderson '90]
- For function in mP: $\Omega(n/(\log^{O(1)} n))$ [dR, Meir, Nordström, Pitassi, Robere, Vinyals '20]
  - $\Omega(\sqrt{n})$ [Göös, Pitassi '14], previously $\Omega(n^{\epsilon})$ [Raz, McKenzie '97]
- Bringing Raz-McKenzie to light again [Göös, Pitassi, Watson '14]
- Separated mon-AC^{i} and mon-NC^{i+1} [dR, Nordström, Vinyals '16]

DAG-like communication complexity

understanding circuit size

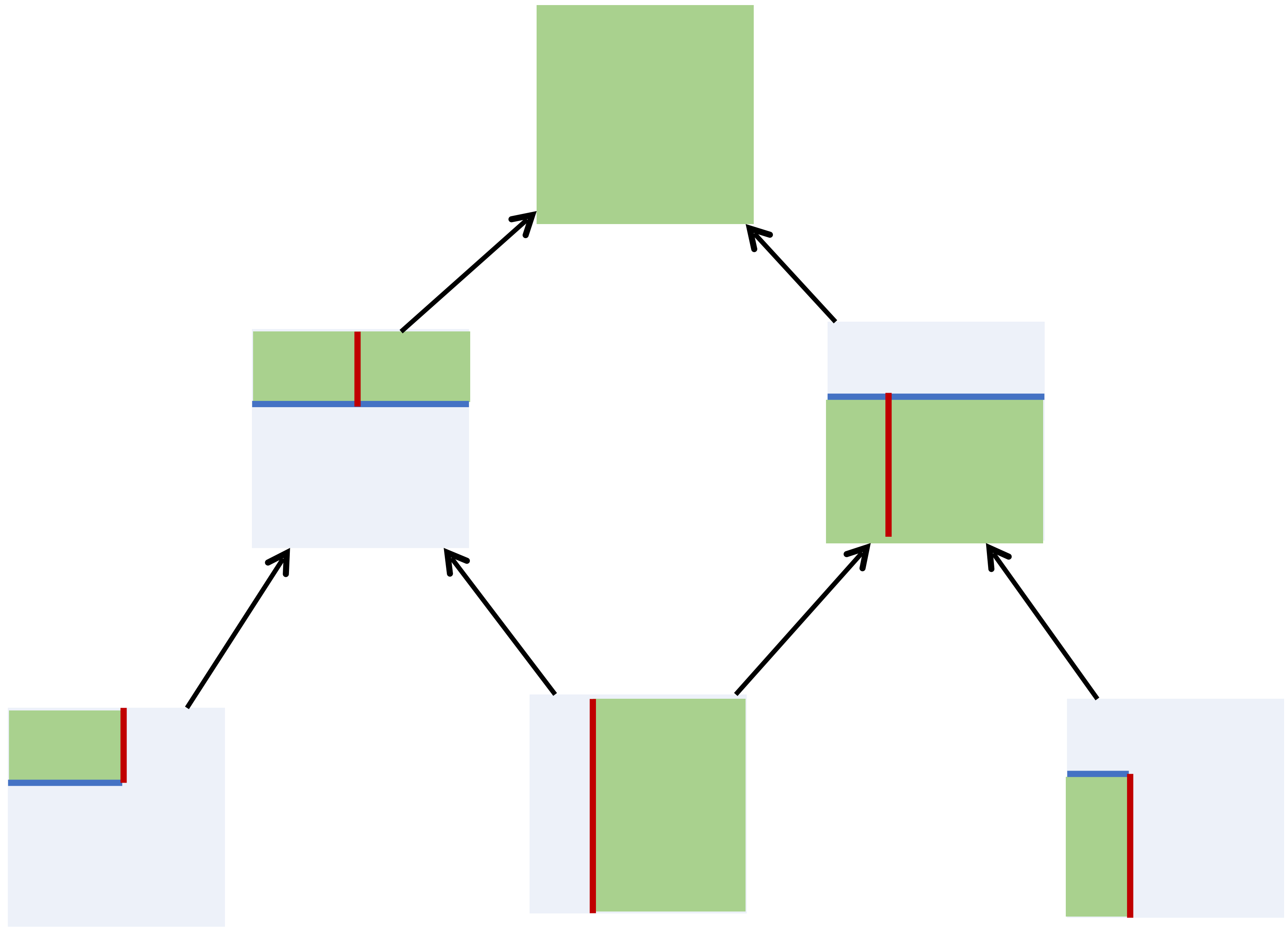
Karchmer-Wigderson for Circuits [Razborov '95, Sokolov '17]

- As before, given f



$x \in f^{-1}(1)$ $y \in f^{-1}(0)$

- Goal: find i s.t. $x_i \neq y_i$
  - Monotone case: find i s.t. $x_i > y_i$
- DAG-like protocols: *rectangle-DAGs*



Circuits = rectangle-DAGs [Razborov '95, Sokolov '17]

$\exists$ size-s (monotone) circuit computing $f \Leftrightarrow$
$\exists$ size-s rectangle-DAG for (monotone) KW(f)

- Result is stronger: really the same object (even graph structure is preserved)

Raz-McKenzie: Lifting Theorem **for Circuits**

[Garg, Göös, Kamath, Sokolov '18]

Remember "Find Collision Problem"? # lines needed when allowed to forget

width-d decision DAG lower bound for S $\Rightarrow$
size-$n^{\Omega(d)}$ monotone circuit lower bound for f_S

- Compare with [Raz, McKenzie '18]

depth-d decision tree lower bound for S $\Rightarrow$
size-$n^{\Omega(d)}$ monotone *formula* lower bound for f_S

Results for monotone circuits

- $\exp(\Omega(n^\epsilon))$-size lower bound for f in NC^2 [Göös, Kamath, Robere, Sokolov '19]
  - Follows from lifting theorem [Garg, Göös, Kamath, Sokolov '18]
- $n^{\Omega(k)}$-size lower bound for k-clique for $k \leq n^{1/2-o(1)}$
- $\exp(\tilde{\Omega}(n^{1/3}))$-size lower bound for f in P

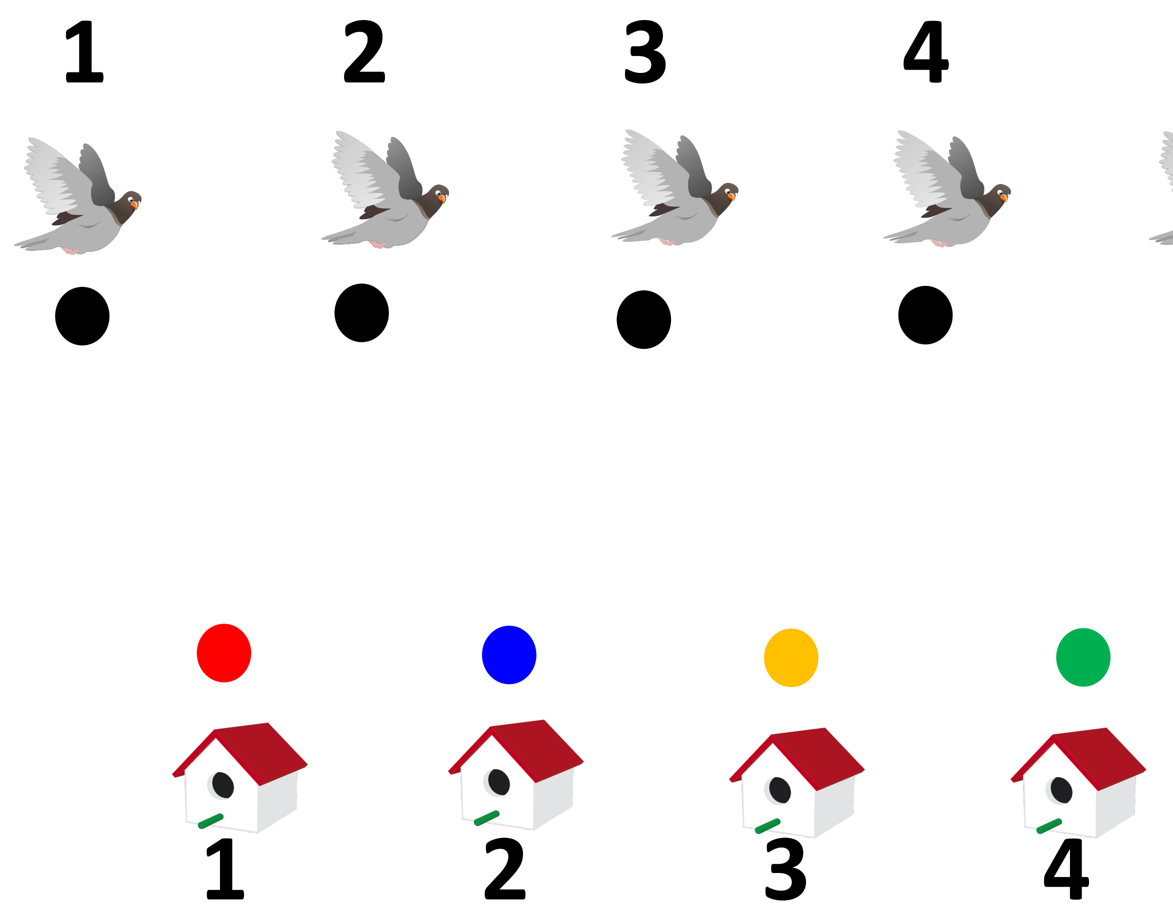
[dR, Vinyals '25] (for both results above)

  - Follows from improvement of lifting theorem [Lovett, Meka, Mertz, Pitassi, Zhang, Jiapeng '21]
  - Use simplification from [dR, Fleming, Janett, Nordström, Pang '25]

for $S \subseteq \Sigma^n \times O$, $m \gg |\Sigma| \cdot d \log(n)$

width-d decision DAG lower bound for S $\Rightarrow$
size-$m^{\Omega(d)}$ monotone circuit lower bound for S $\circ$ Ind_m

$\text{Find-collision}_k \circ \text{Ind}_m \preccurlyeq \text{mKW}(\text{Clique-Col}_{k,n=mk})$



$z_i \in [k-1], \ \forall i \in n$

find $i \neq j$ s.t. $z_i = z_j$

Alice: $x_1, x_2, x_3, x_4, x_5 \in [m]$

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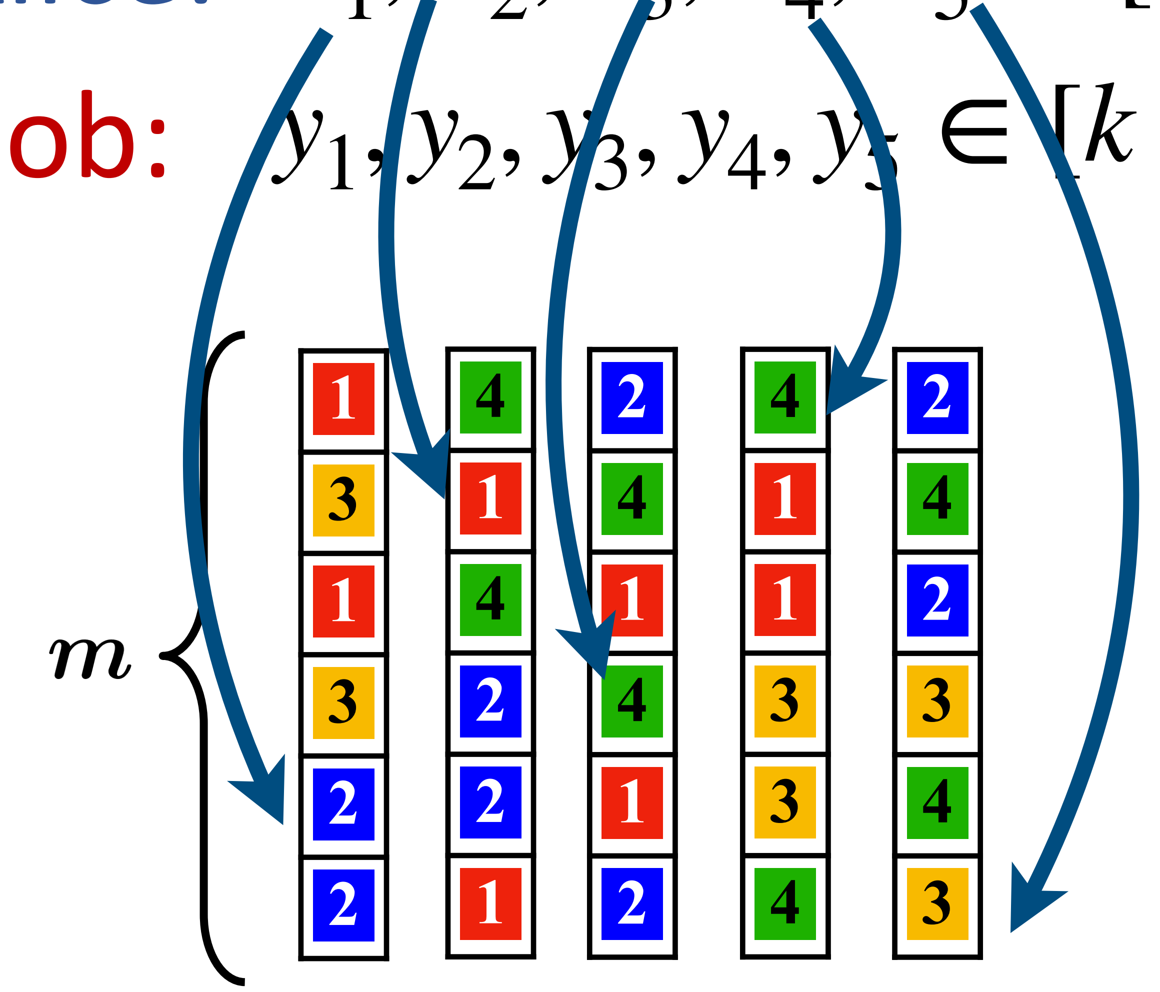

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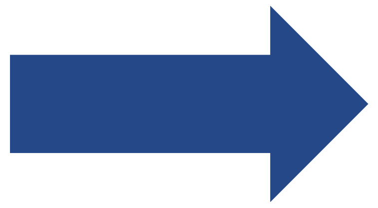
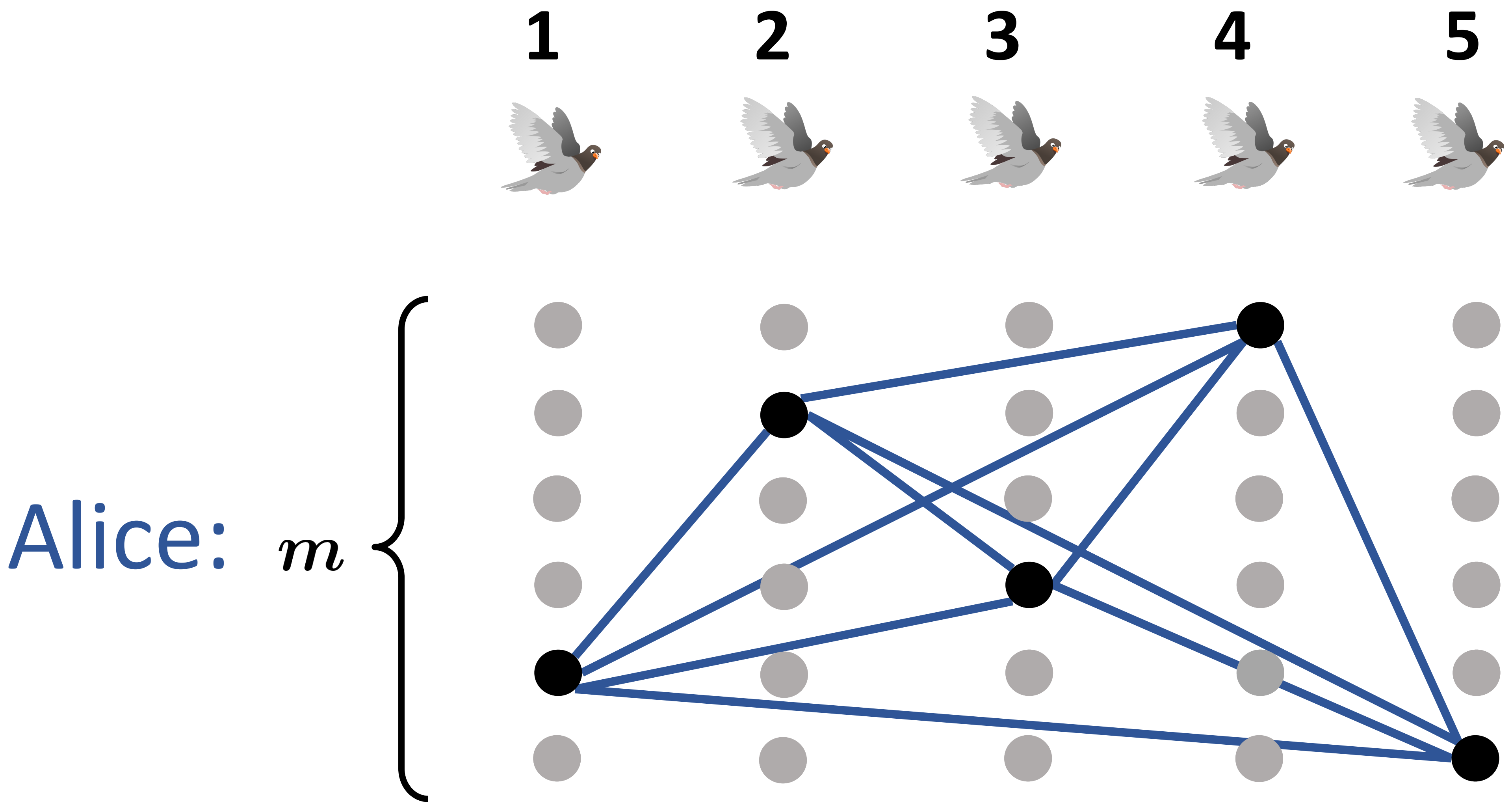
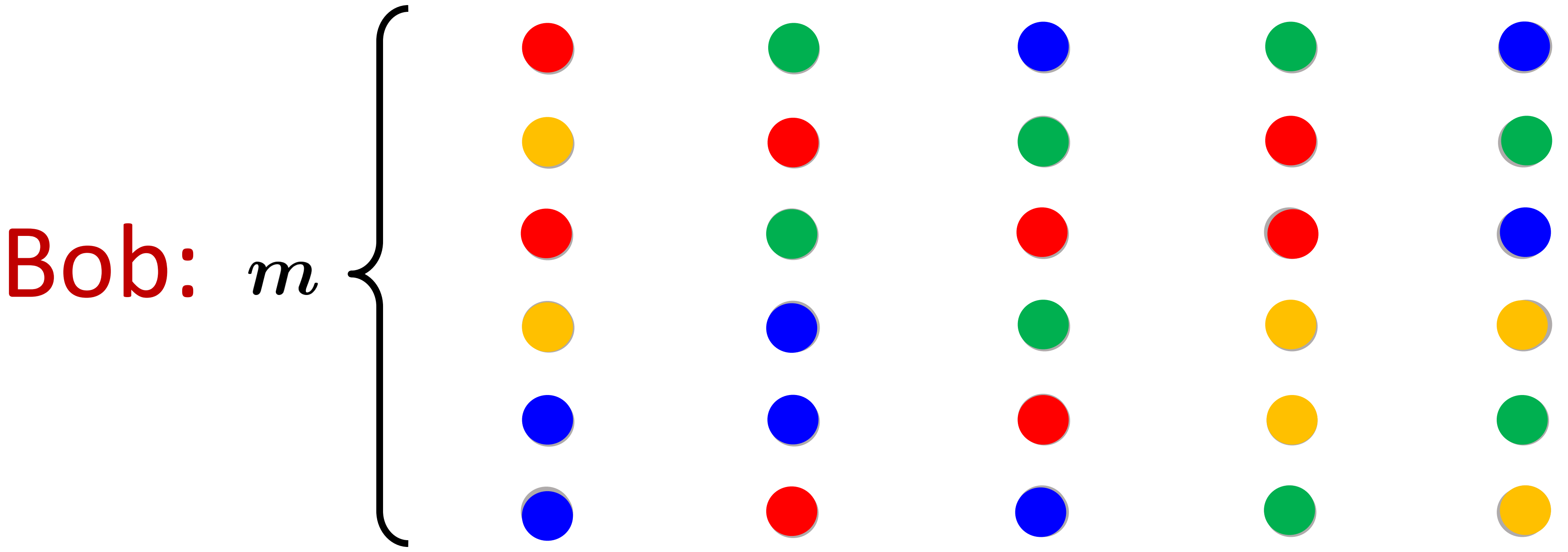
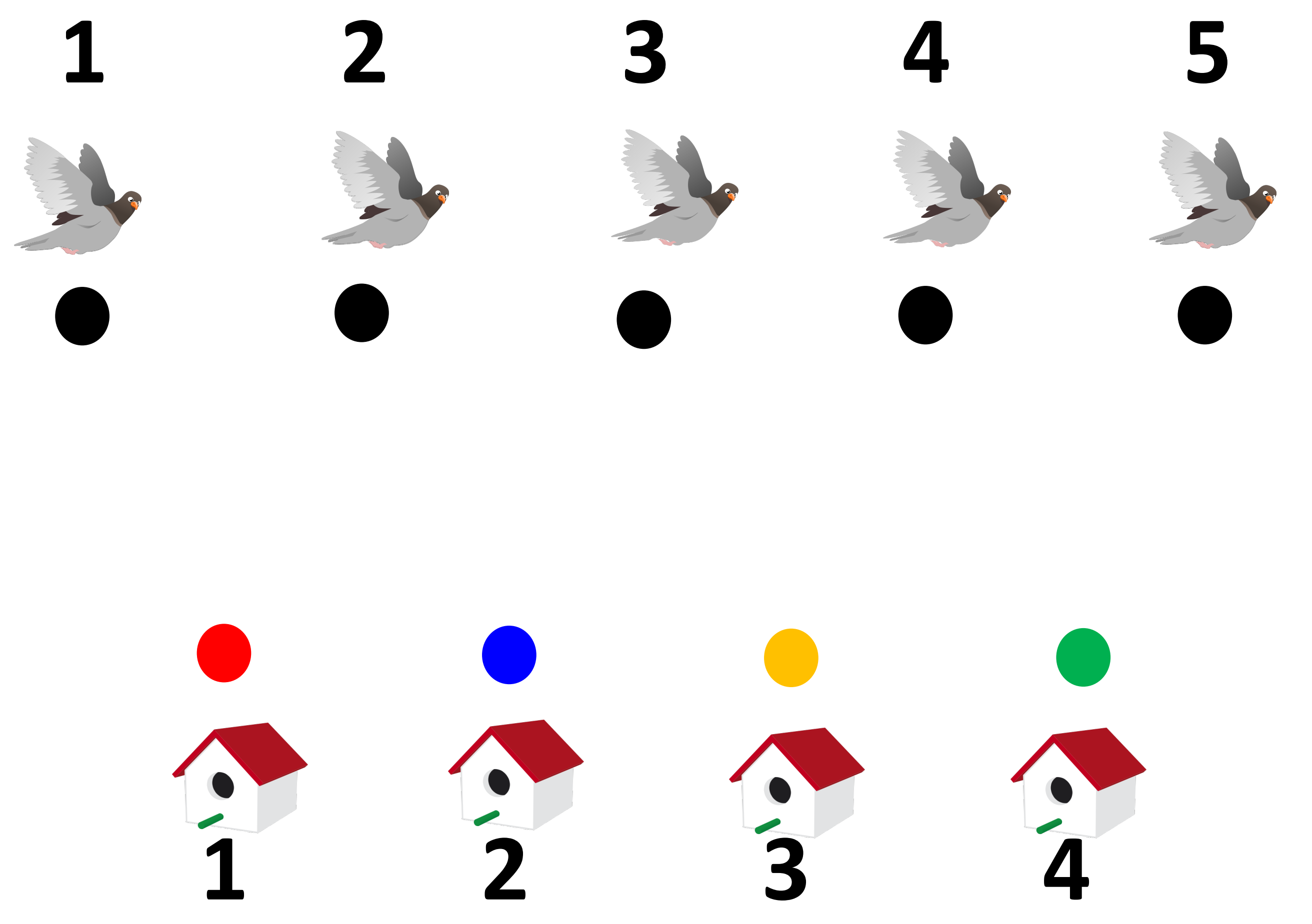






Find-collision$_k \circ \mathrm{Ind}_m \preccurlyeq$ mKW(Clique-Col$_{k,n=mk}$)



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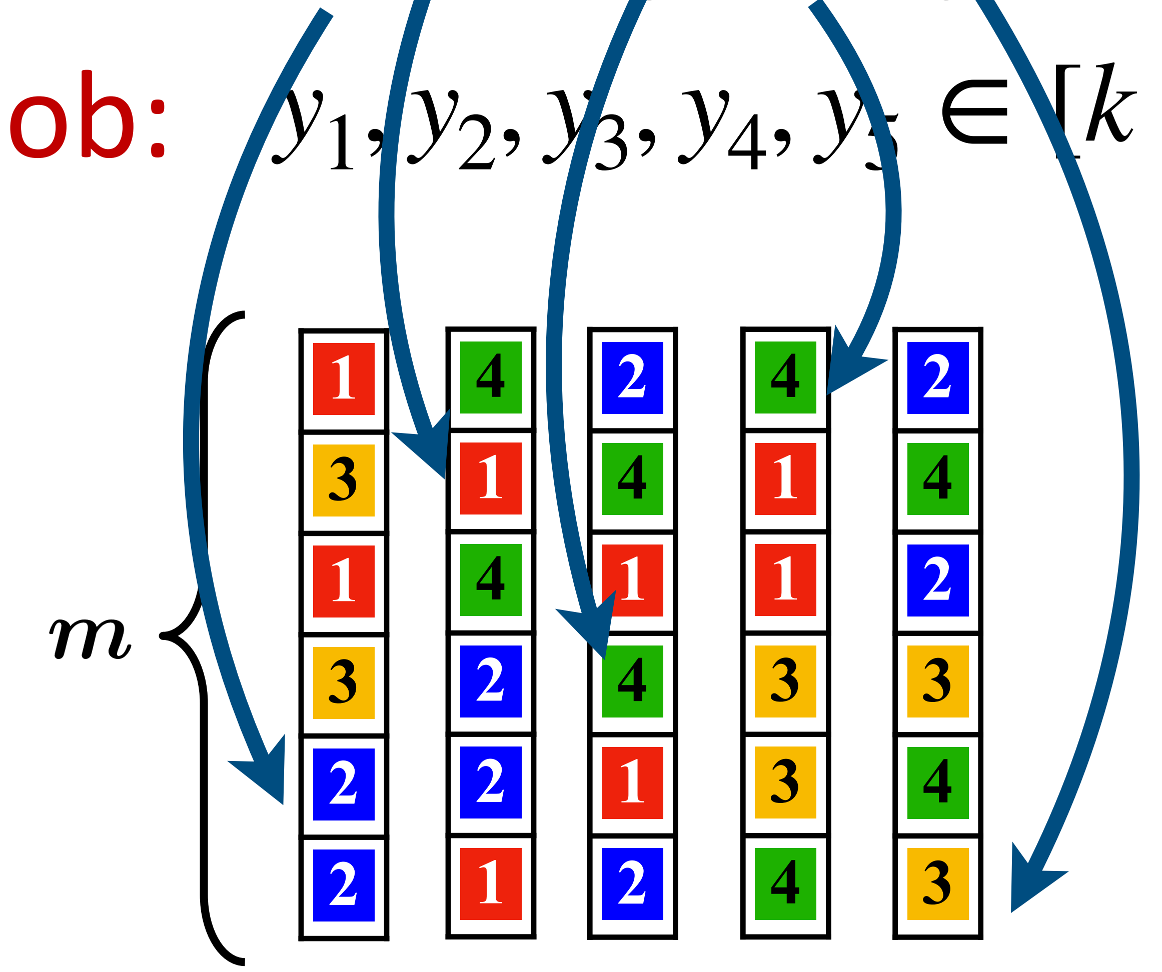

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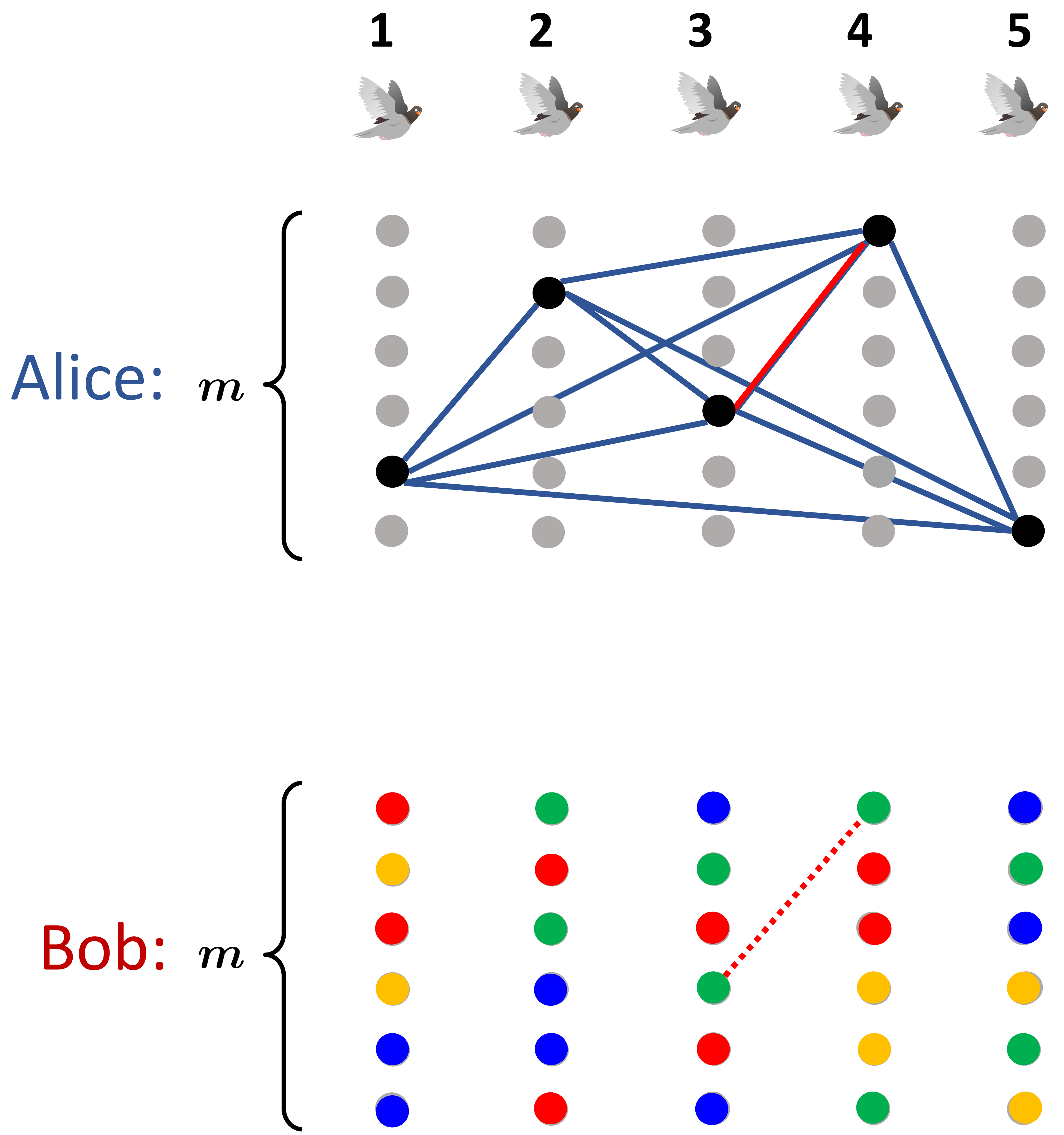

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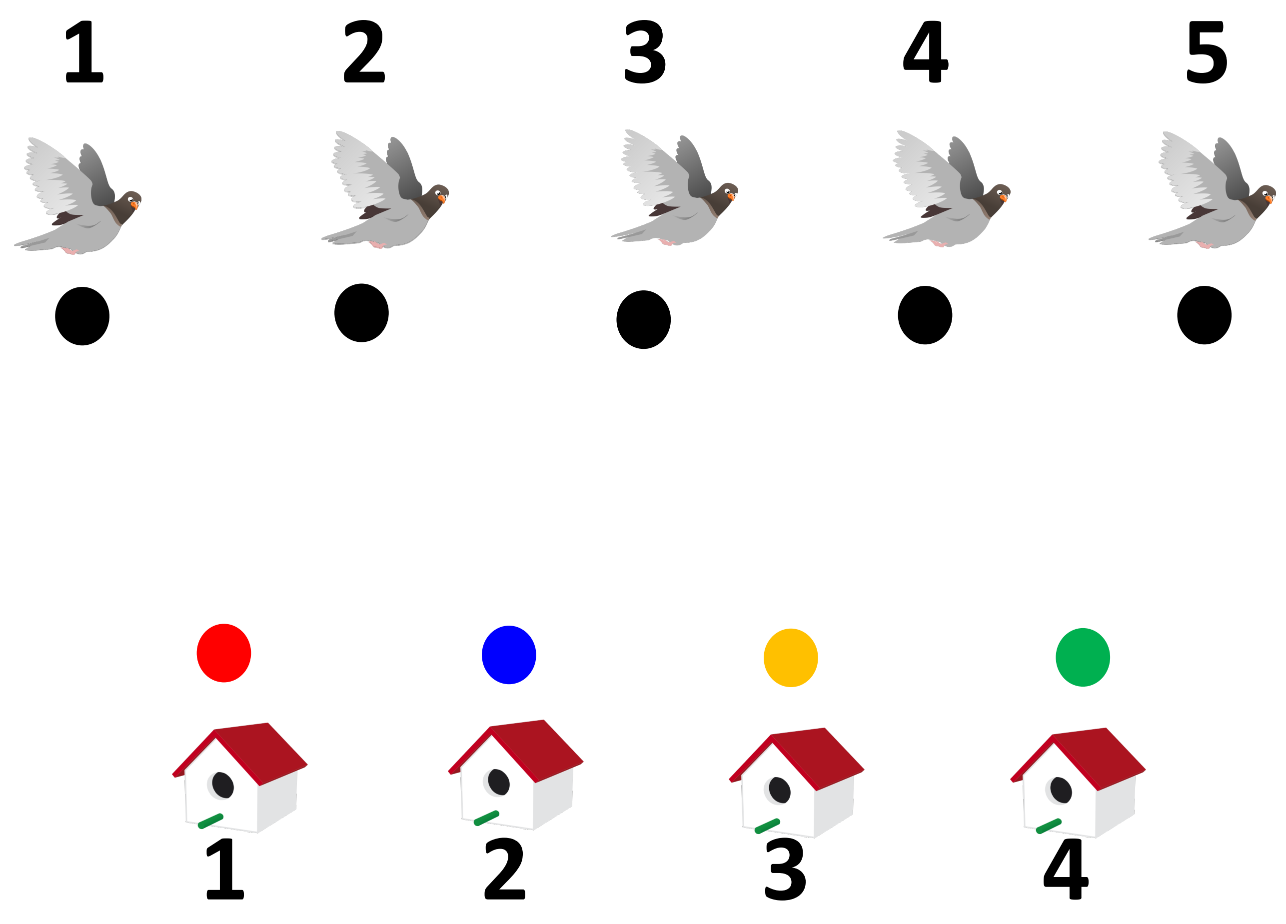

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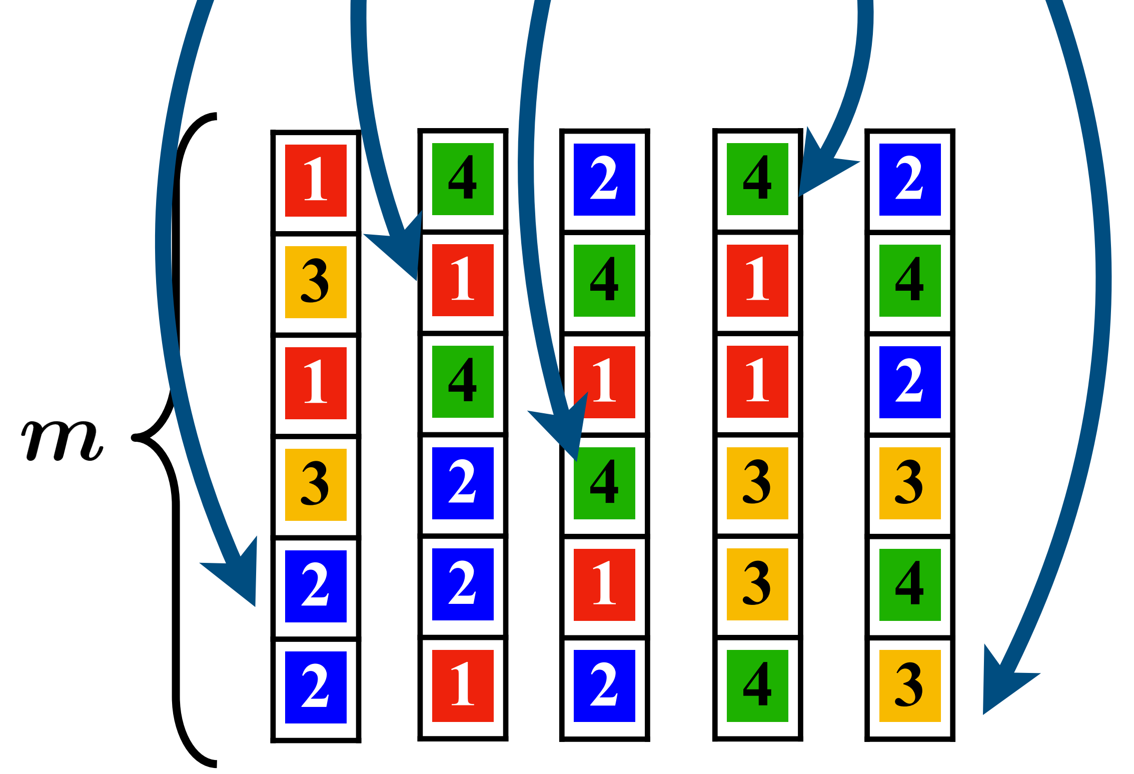

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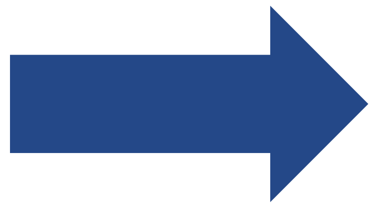
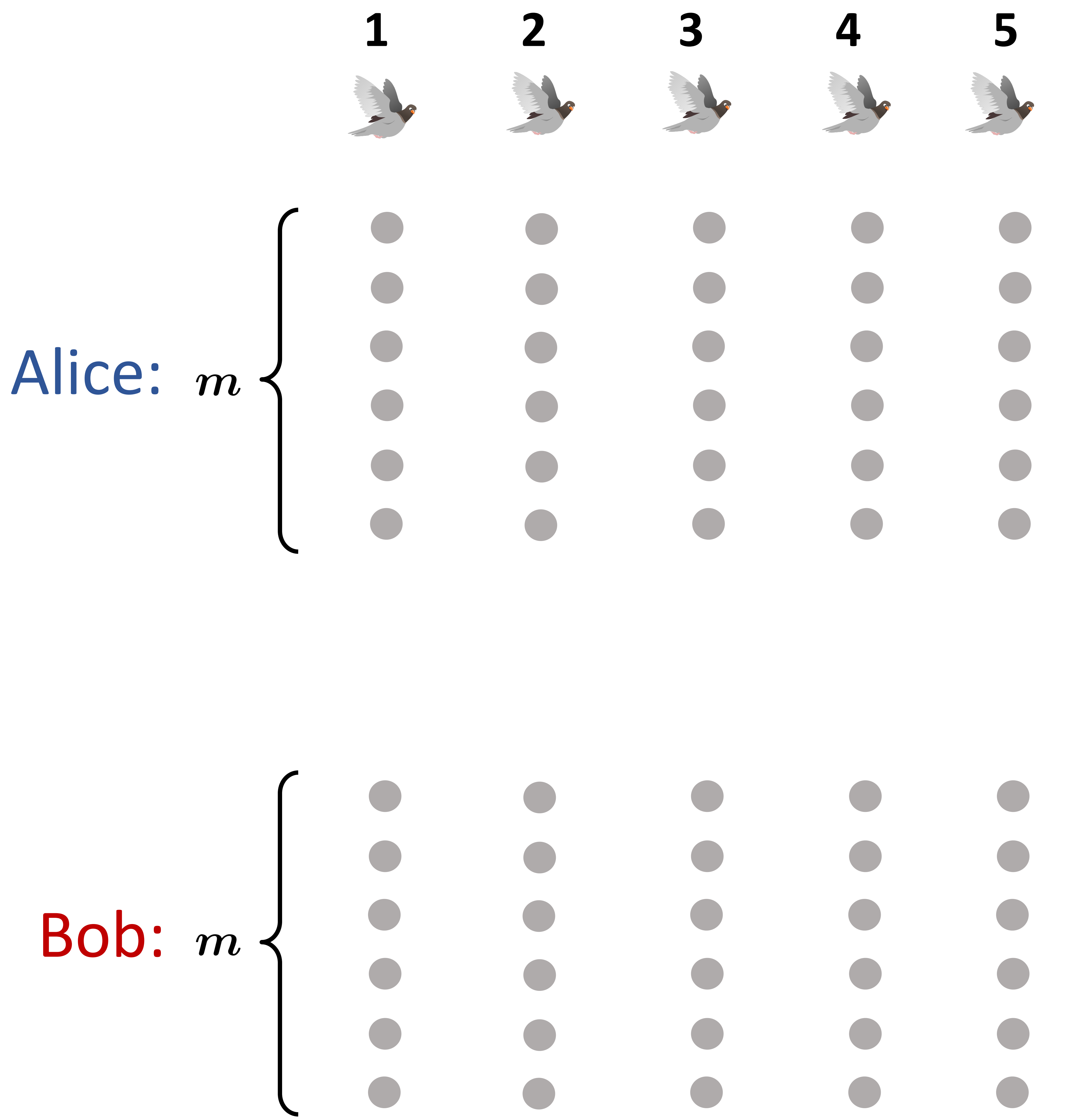
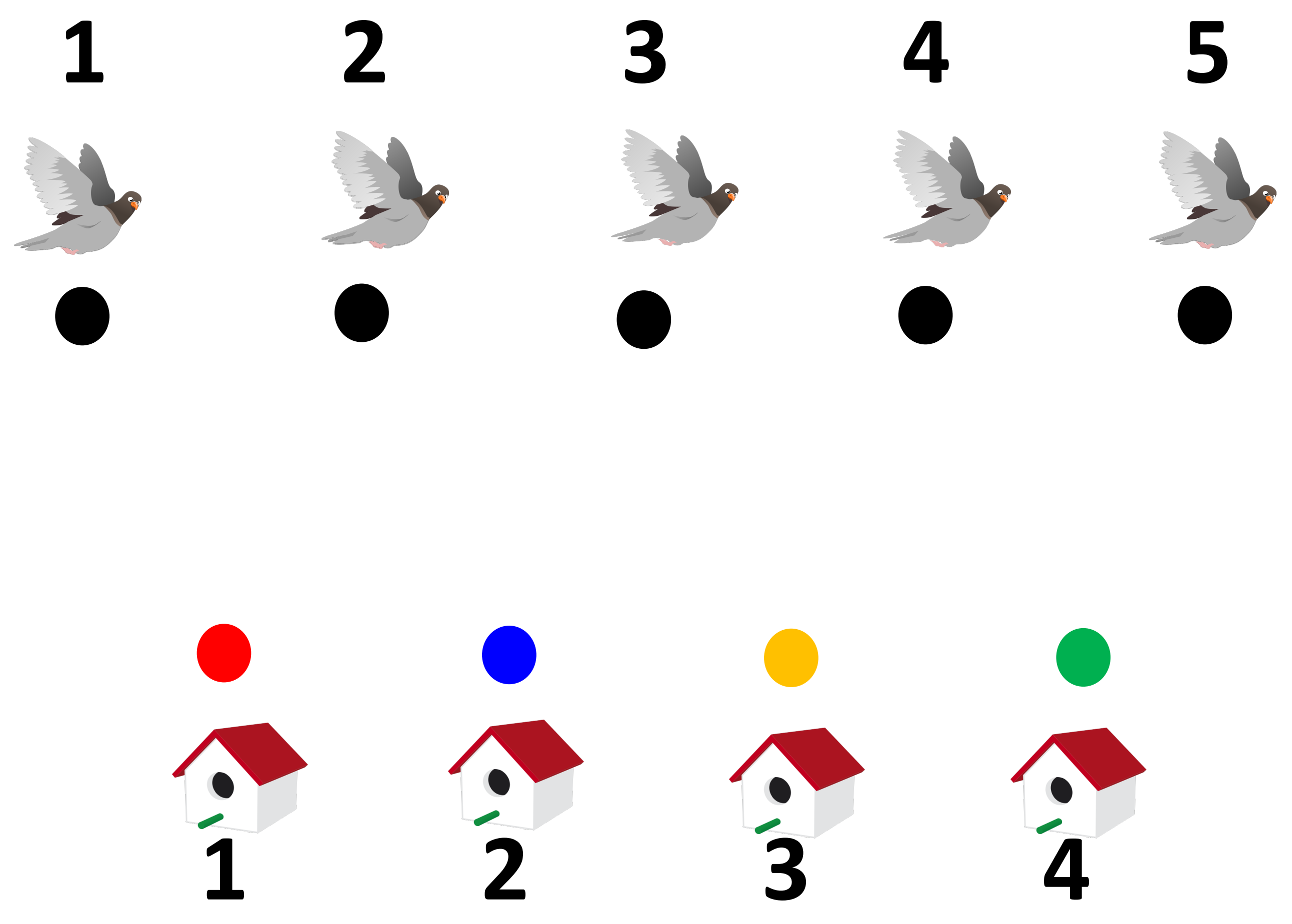




Find-collision$_k$ ∘ $\mathrm{Ind}_m \preccurlyeq$ mKW(Clique-Col$_{k,n=mk}$)



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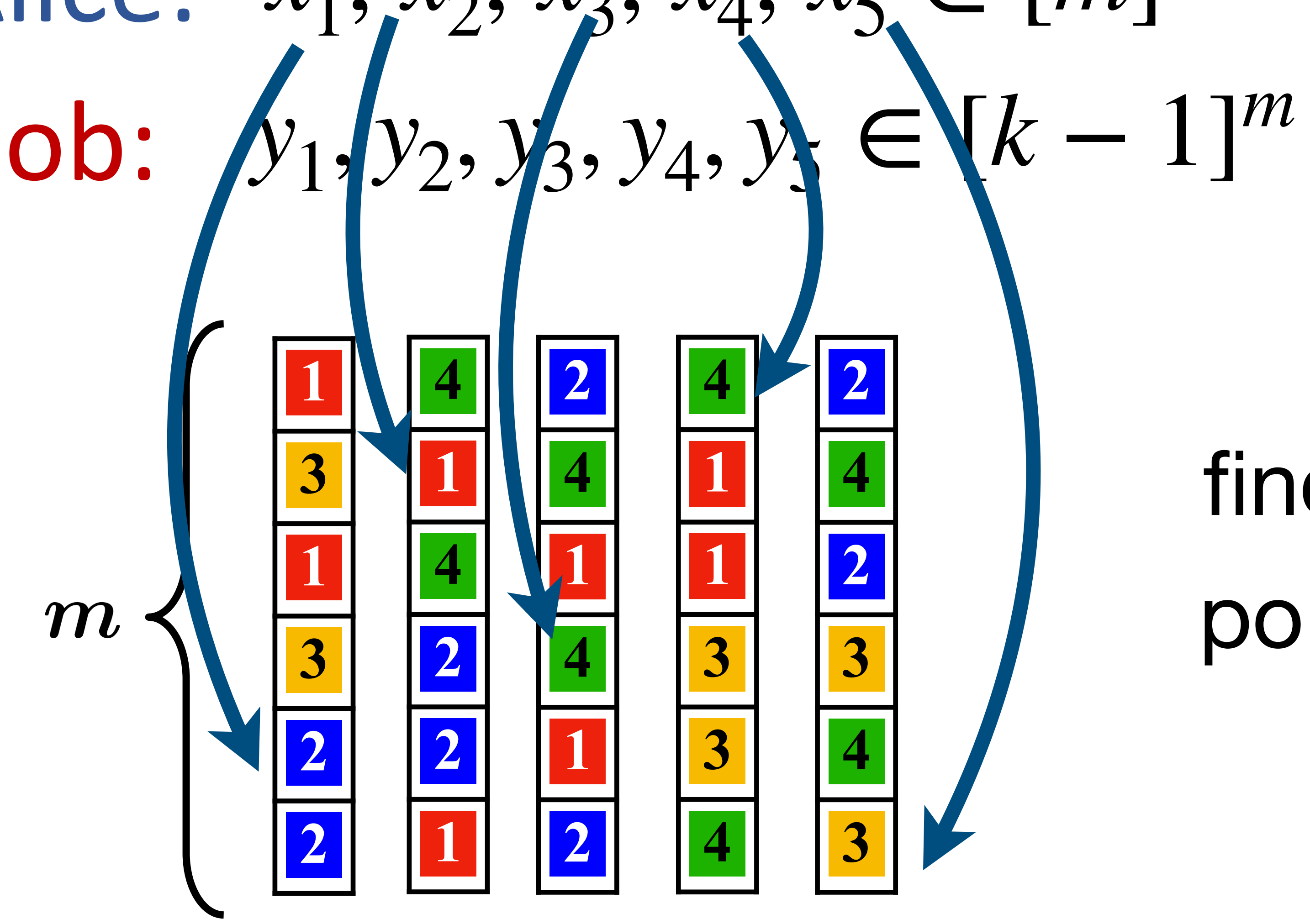

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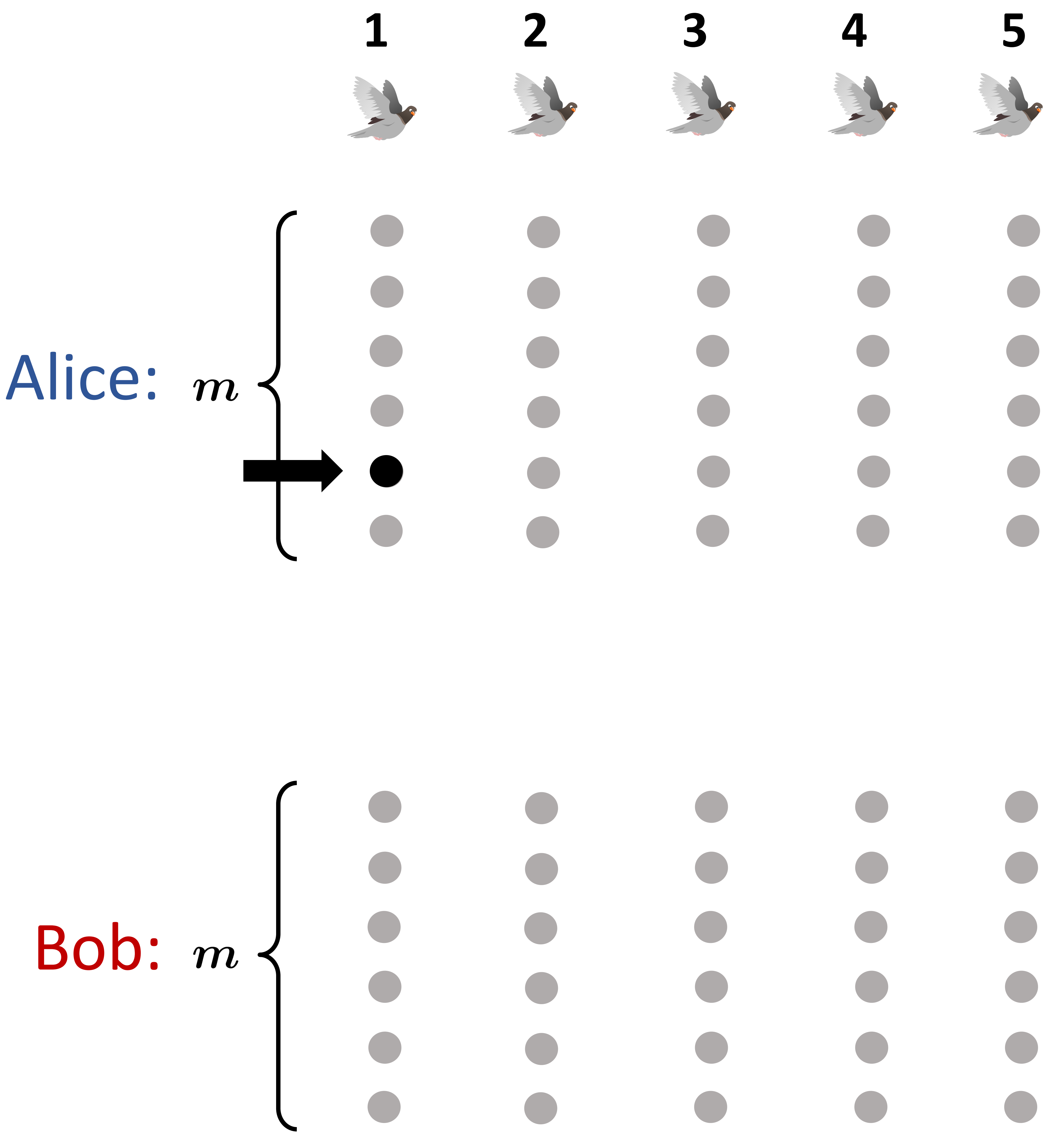
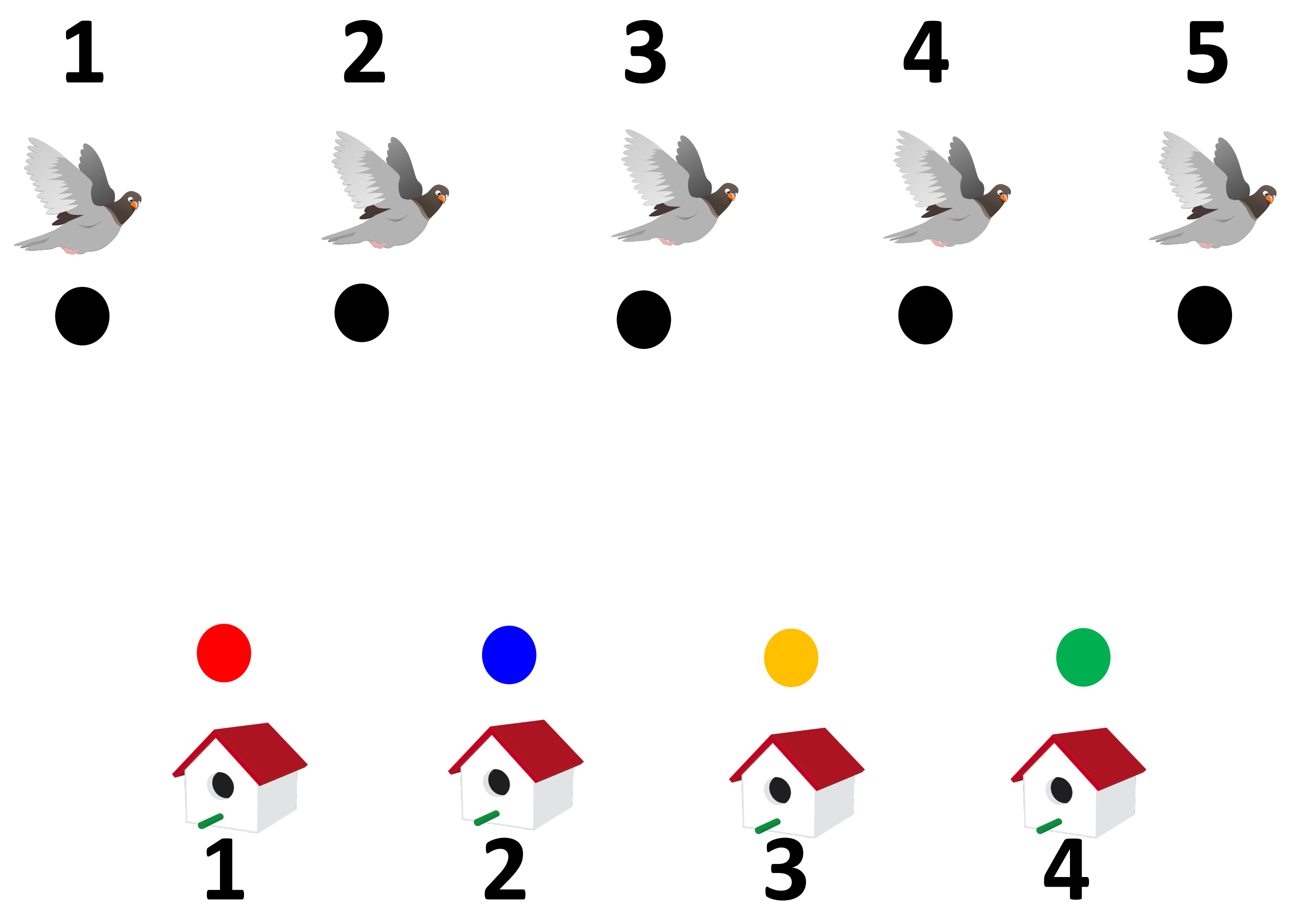


Find-collision$_k \circ \mathrm{Ind}_m \preccurlyeq$ mKW(Clique-Col$_{k,n=mk}$)



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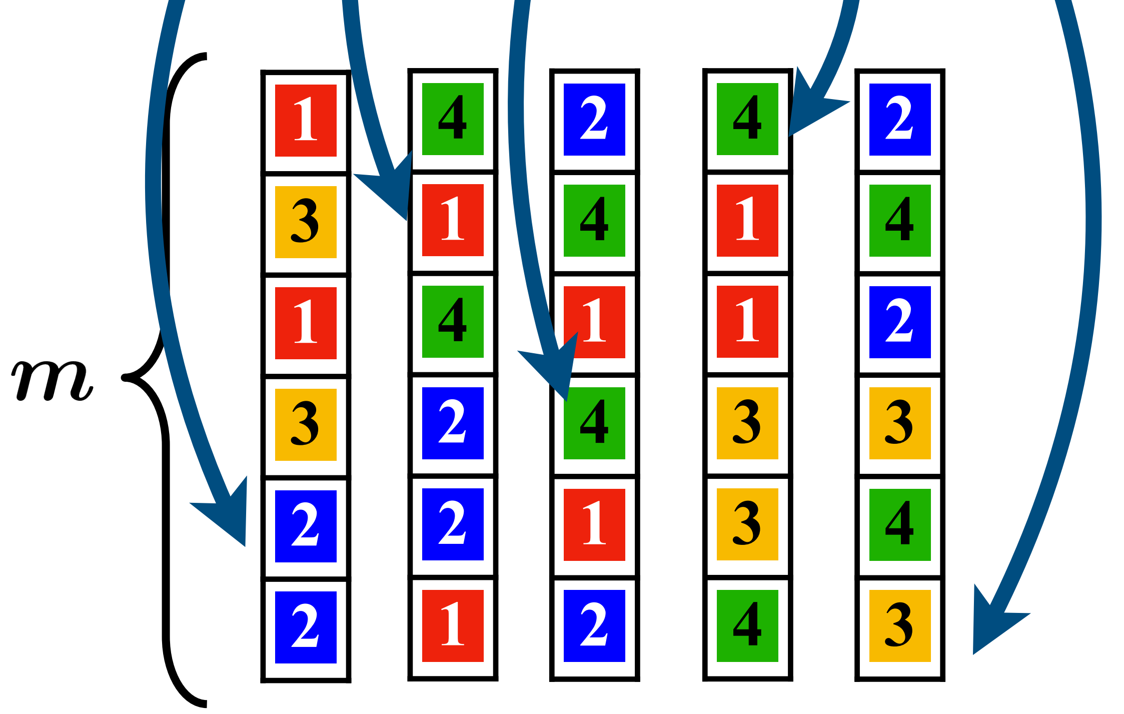

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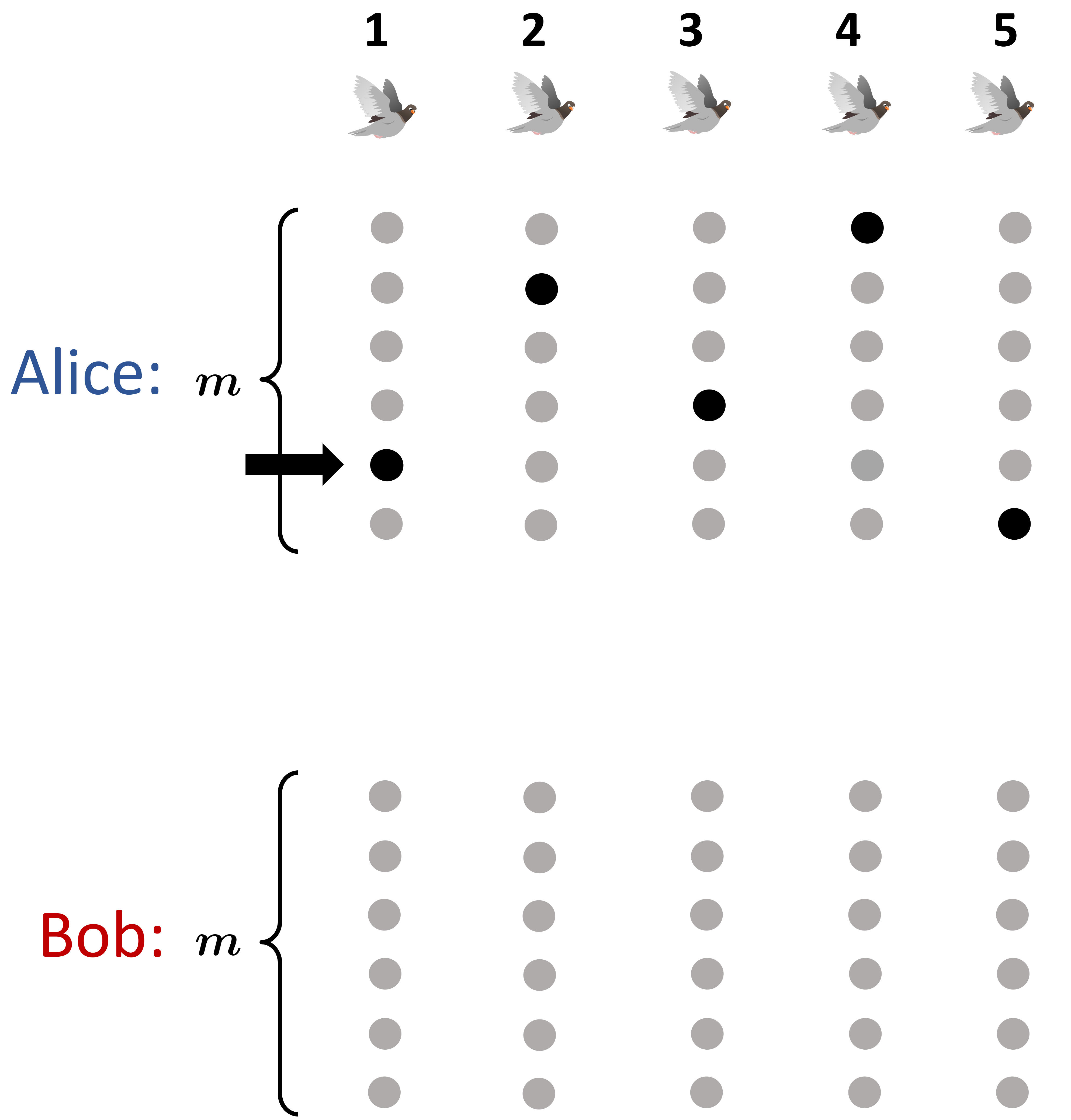

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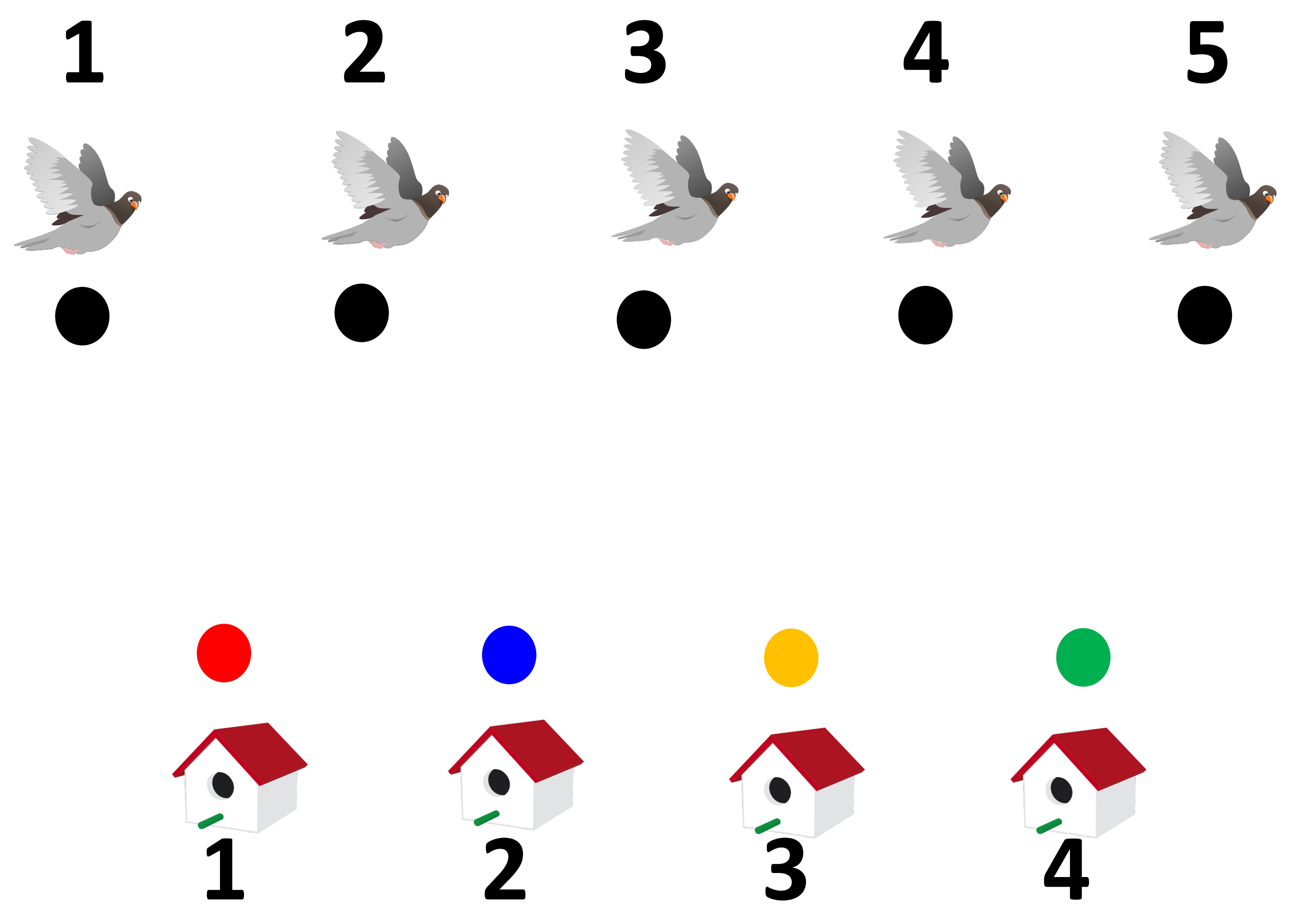

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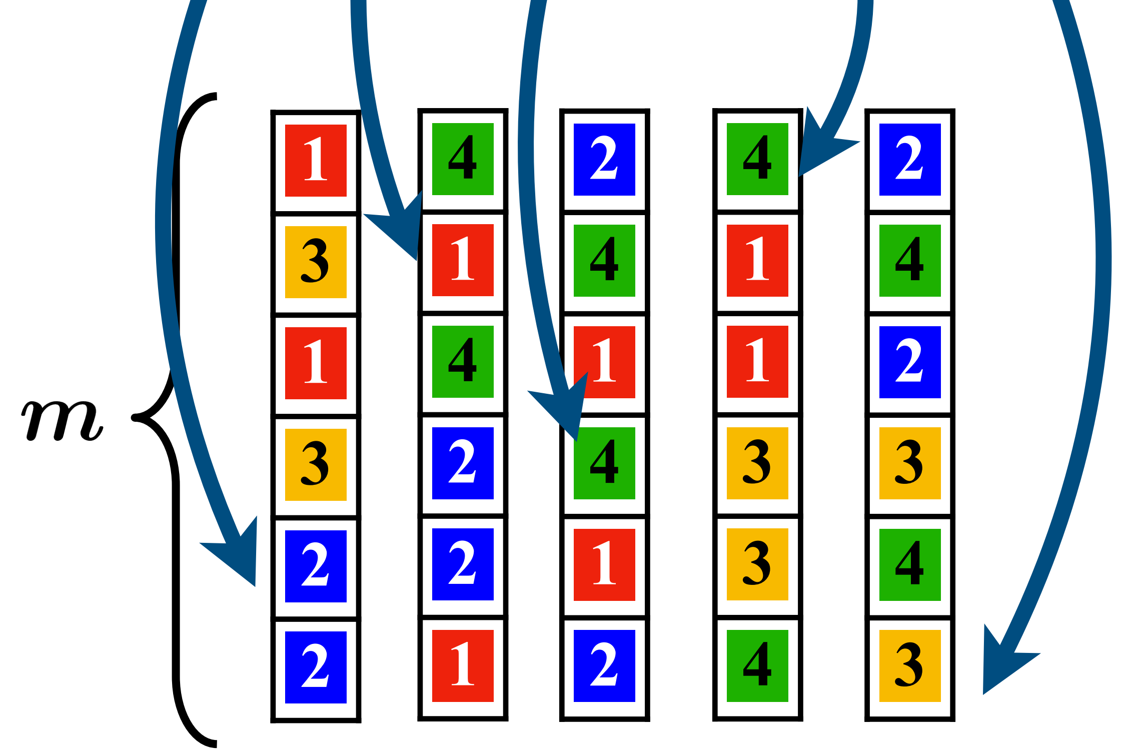

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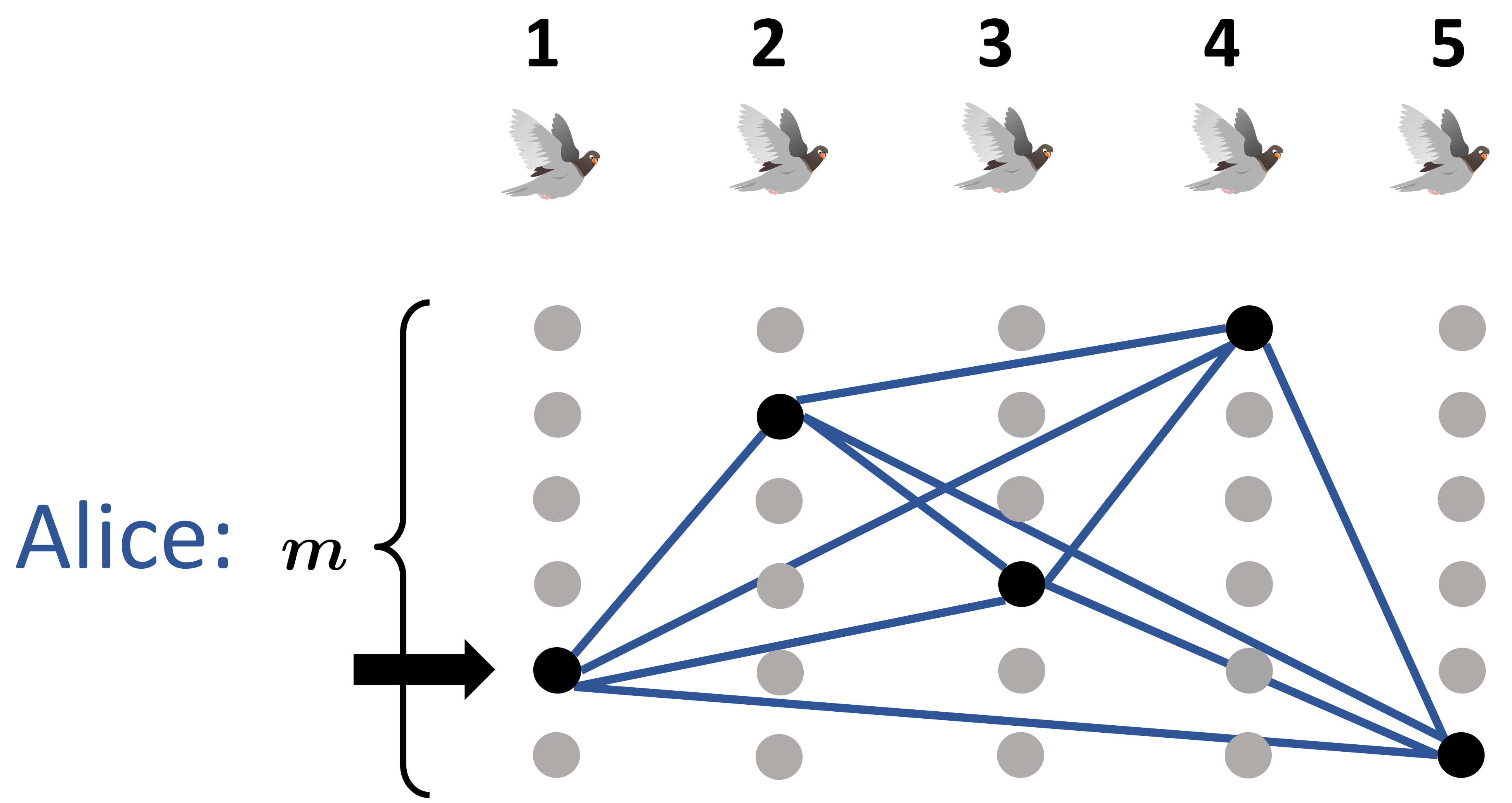

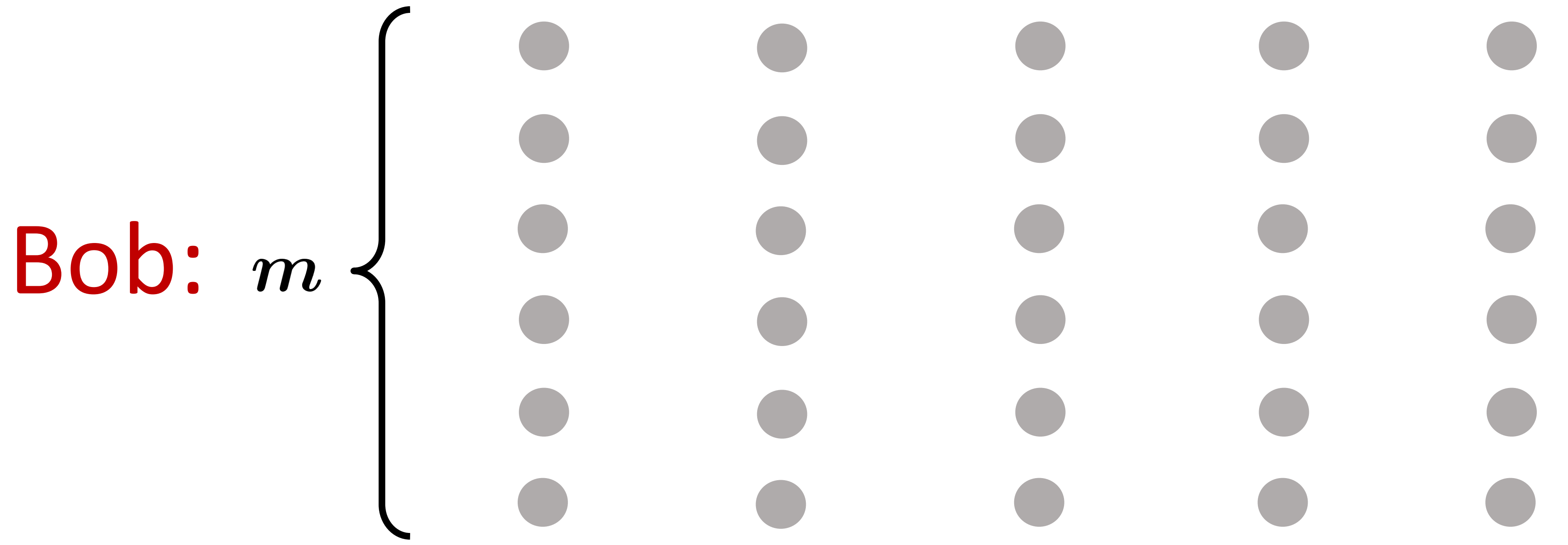

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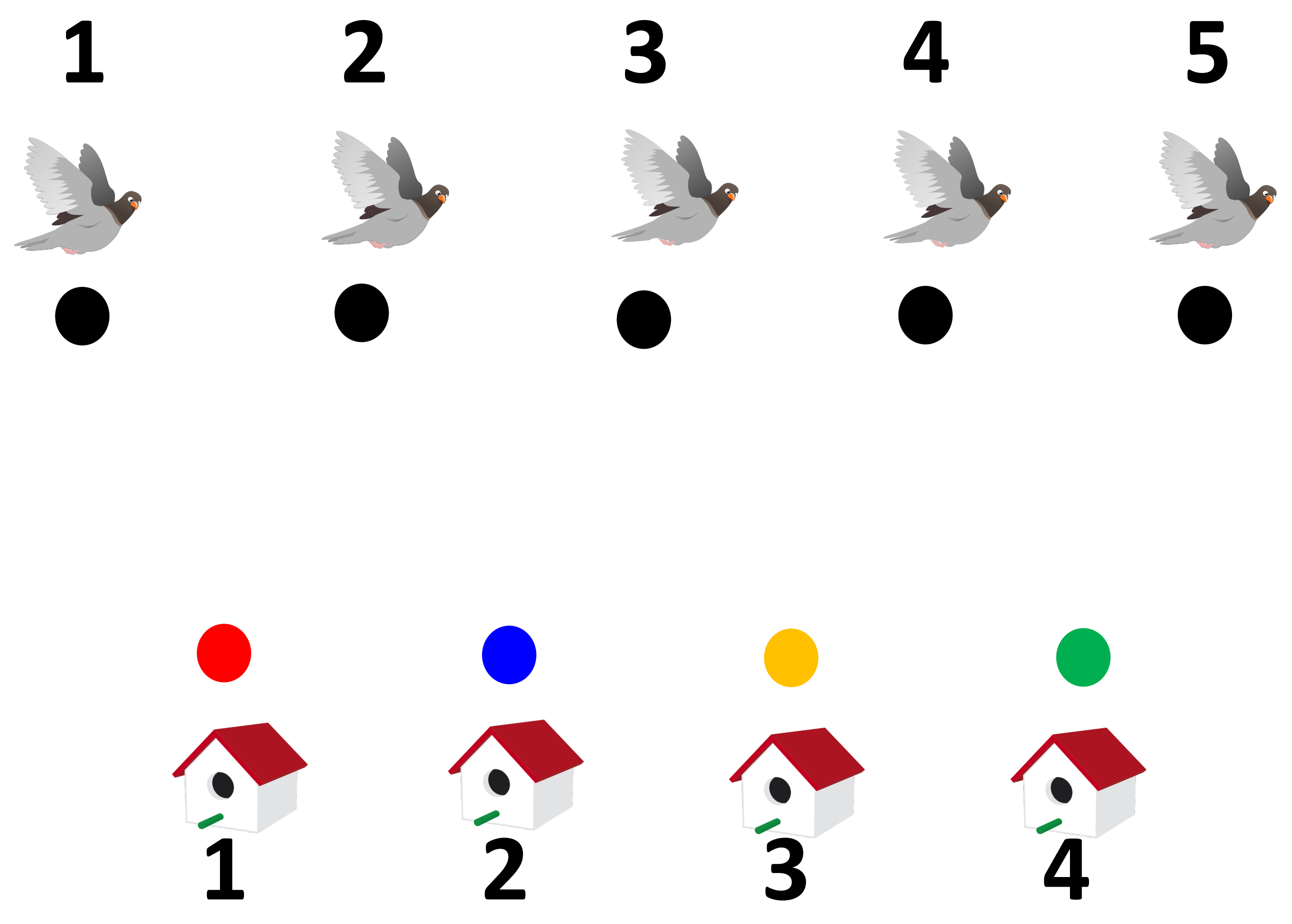

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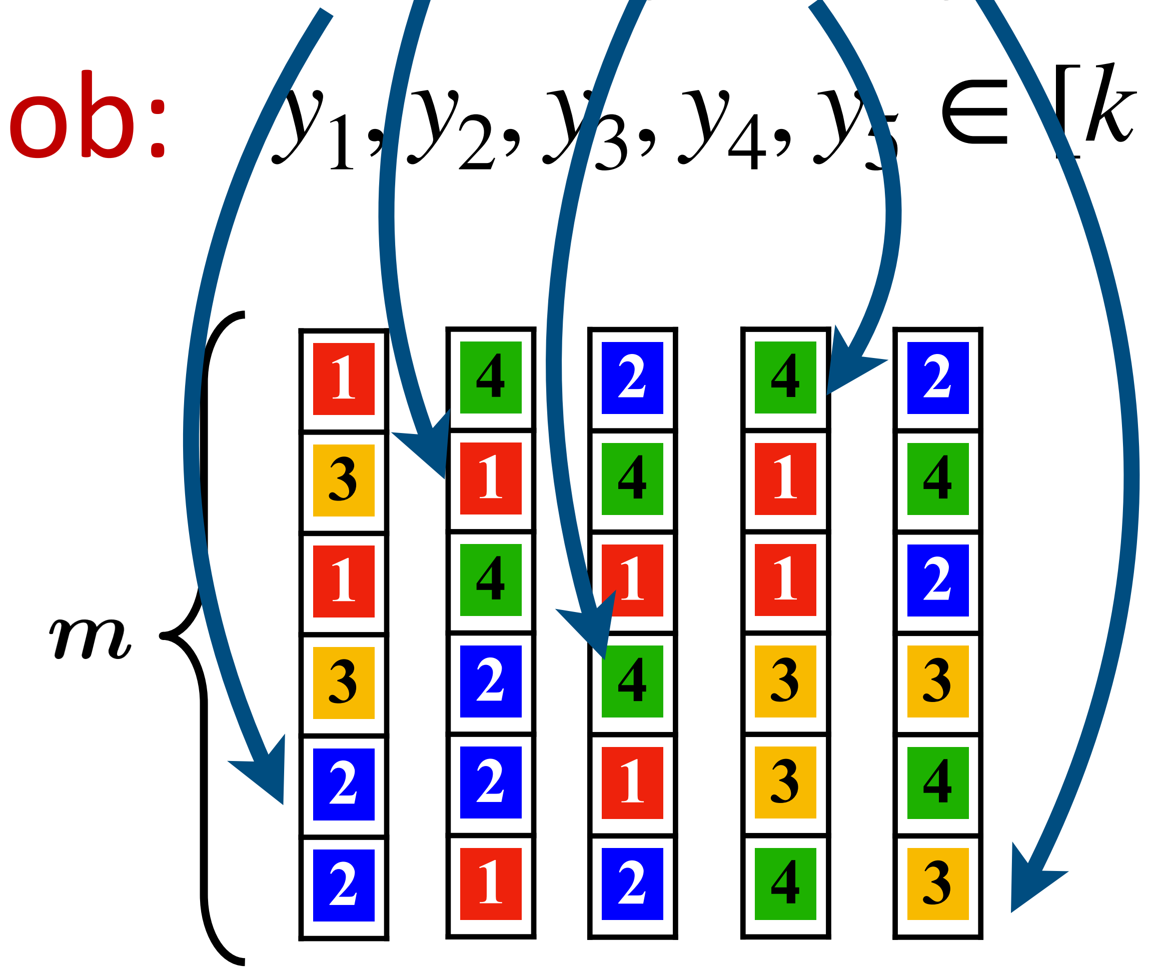

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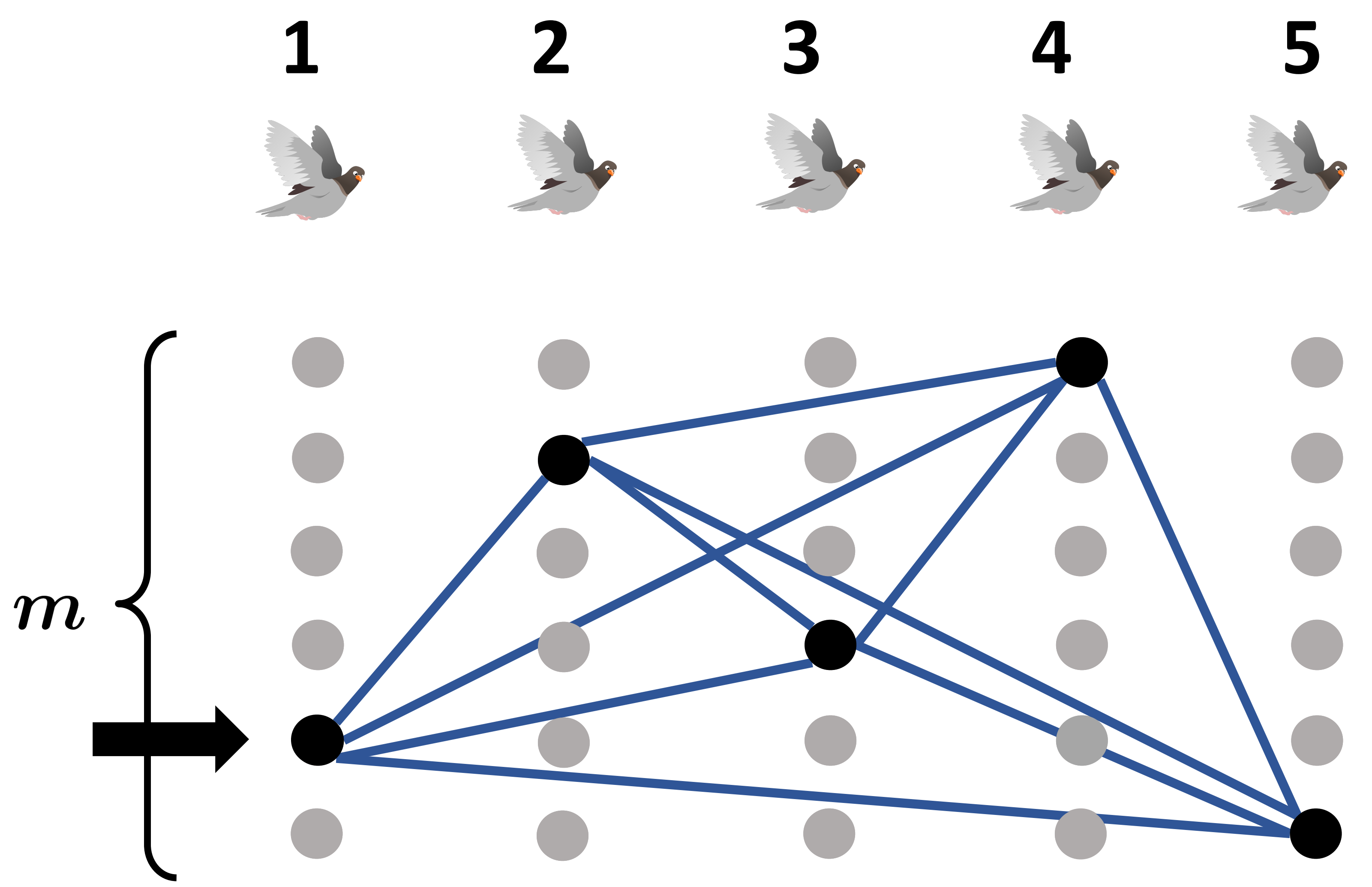
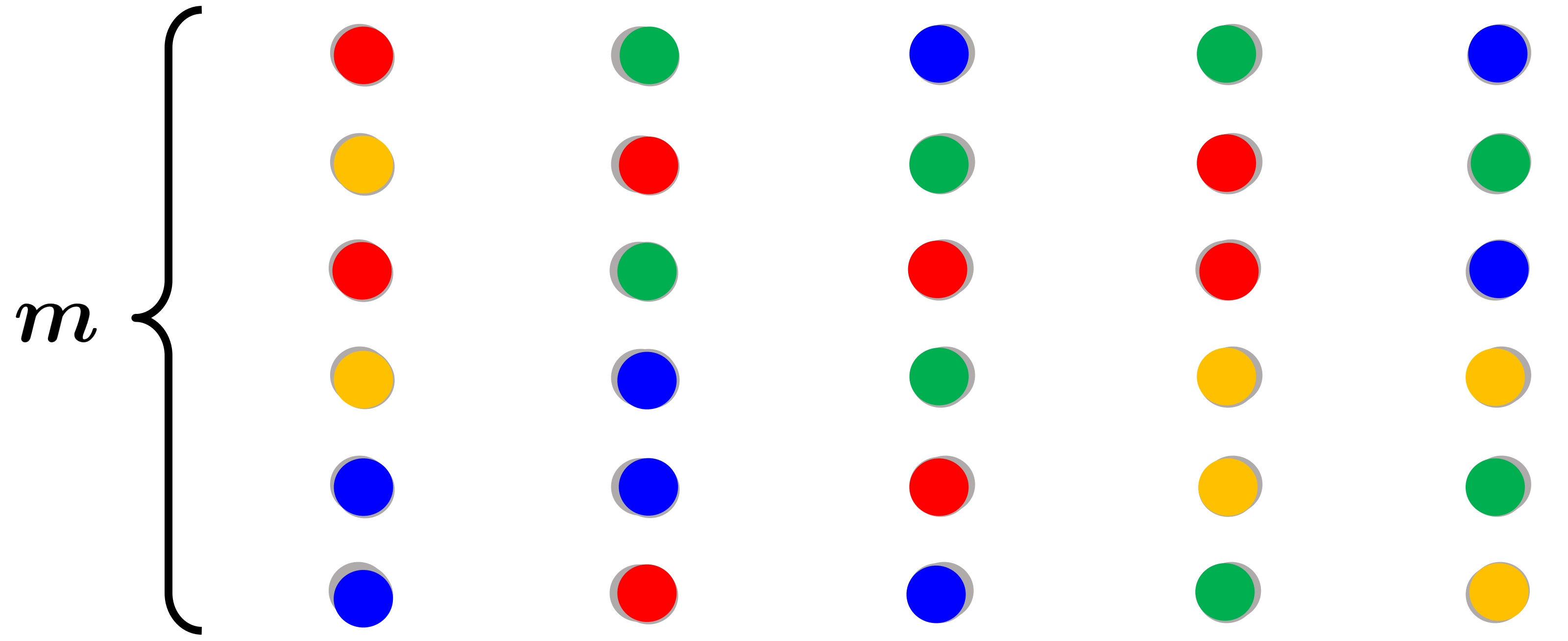
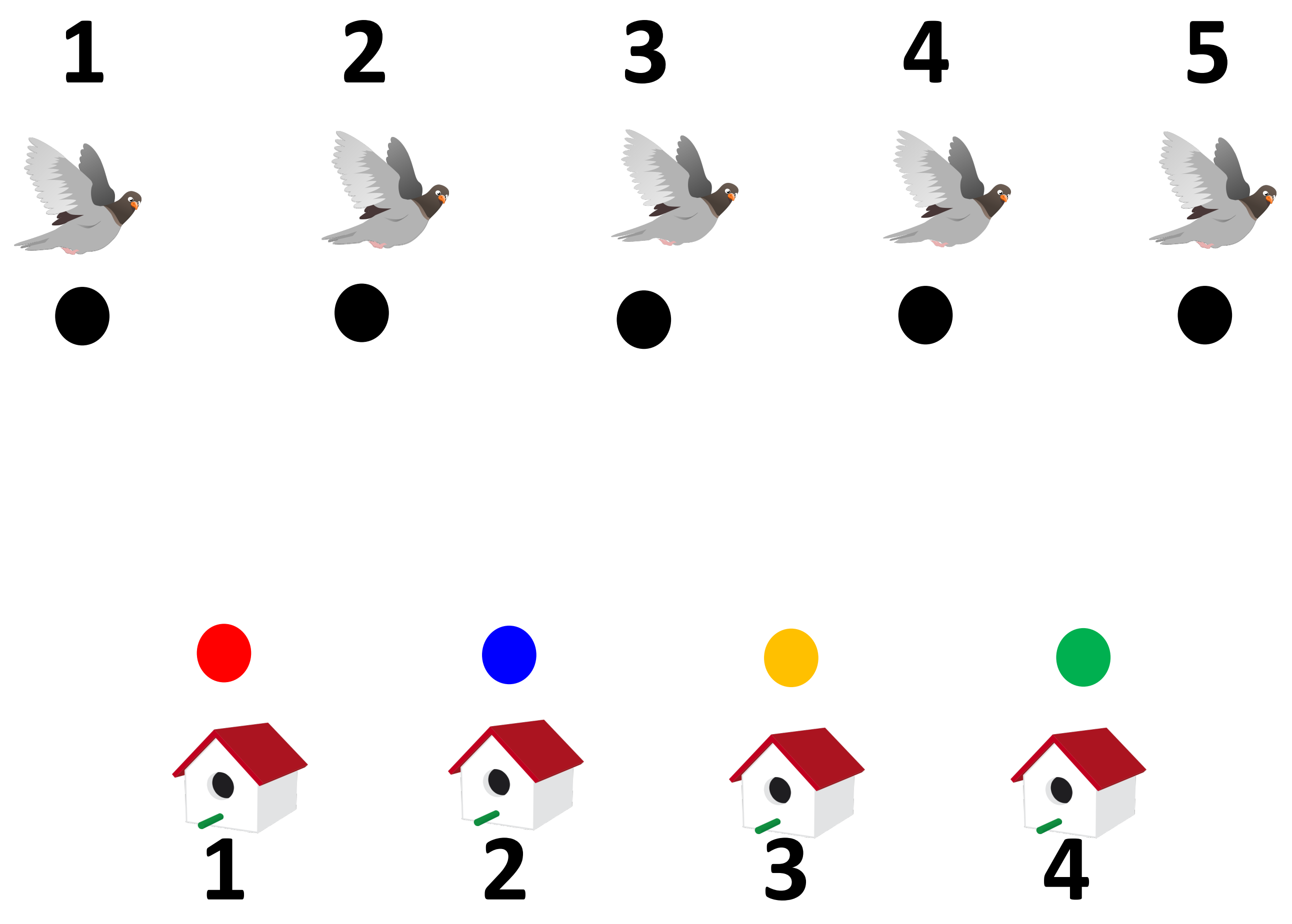




Includes all edges between vertices of $\neq$ colours

$\text{Find-collision}_k \circ \text{Ind}_m \preccurlyeq \text{mKW}(\text{Clique-Col}_{k,n=mk})$



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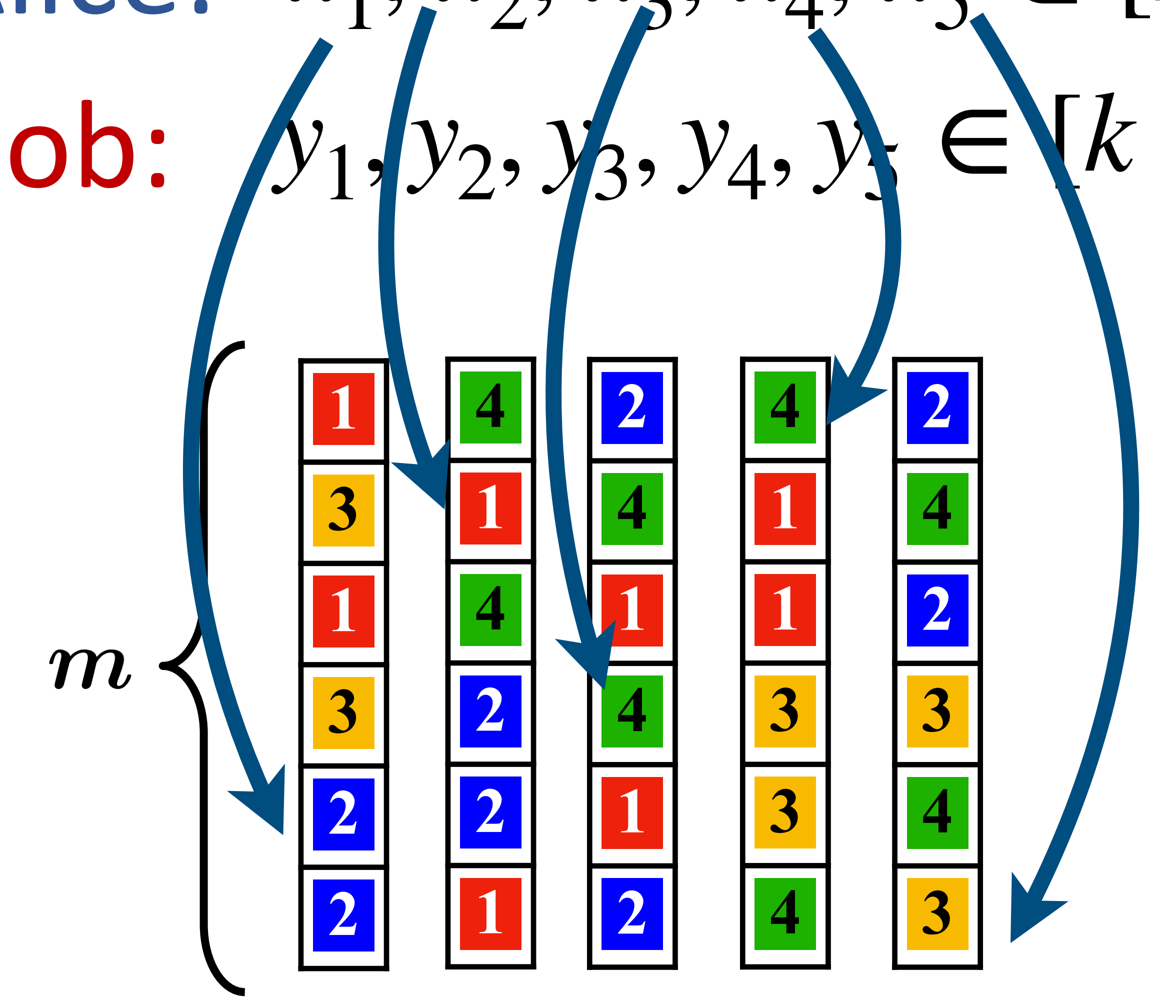

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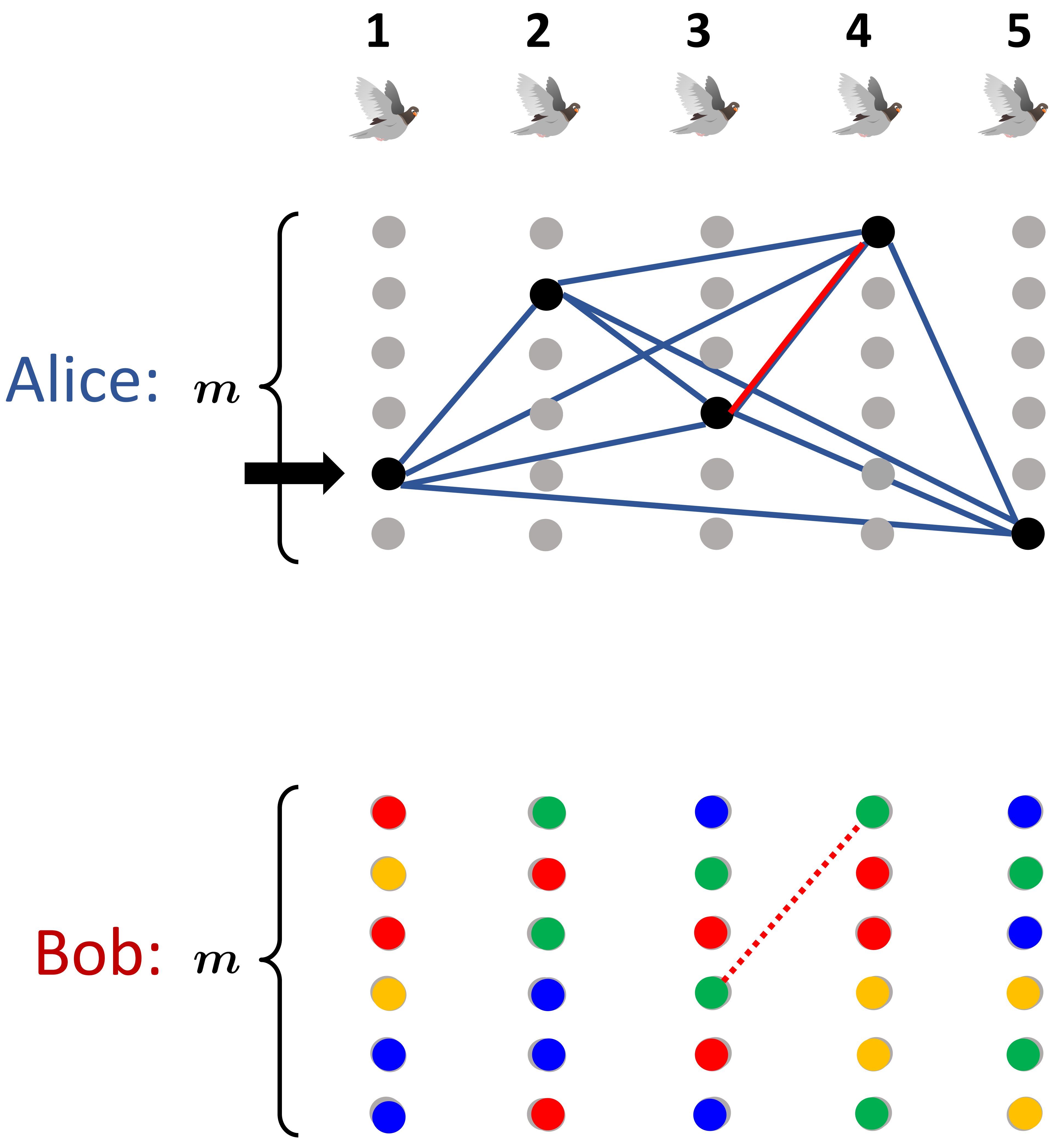

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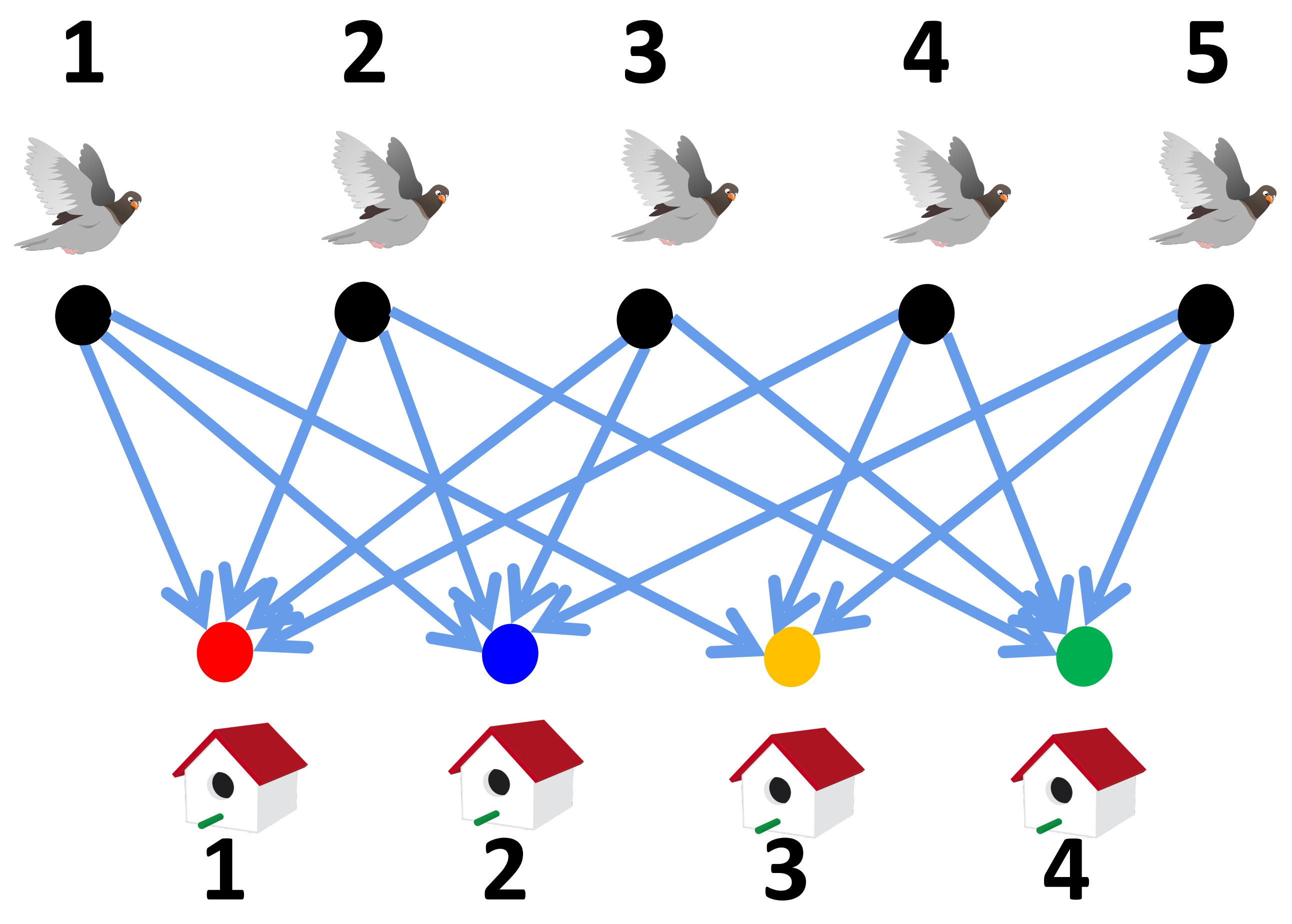

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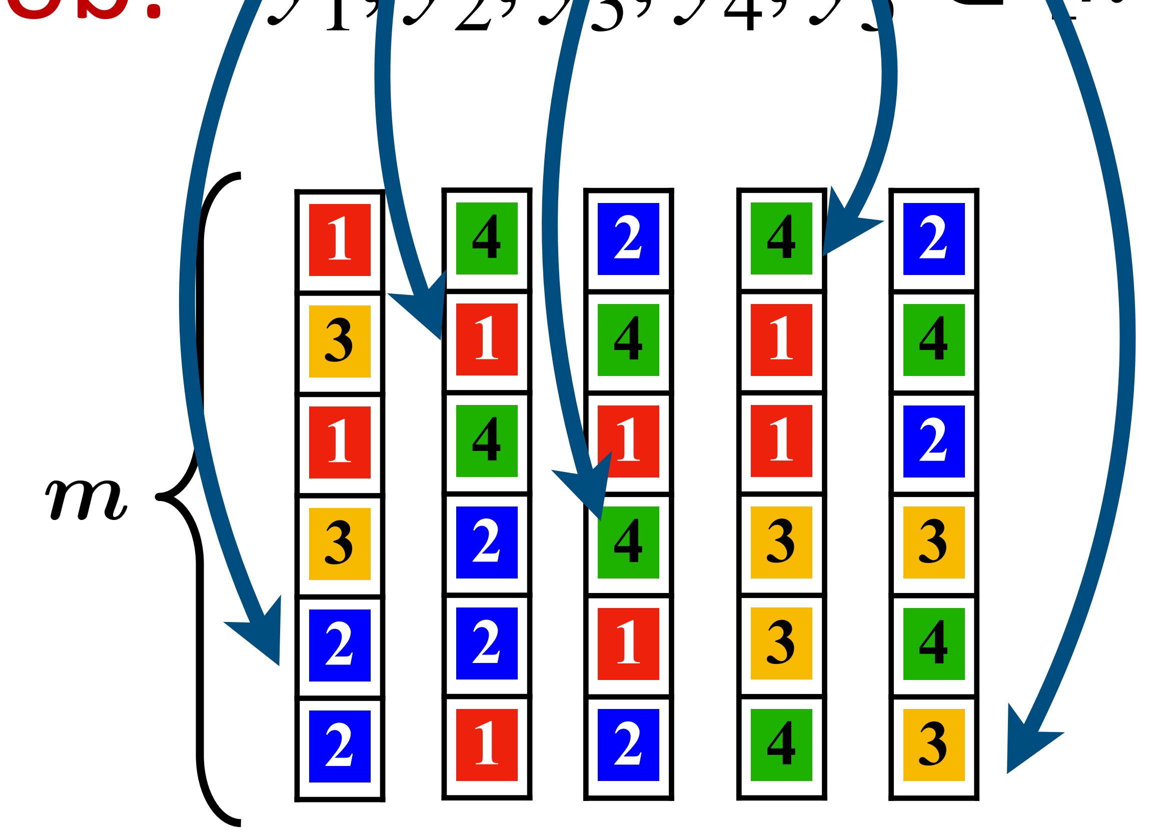

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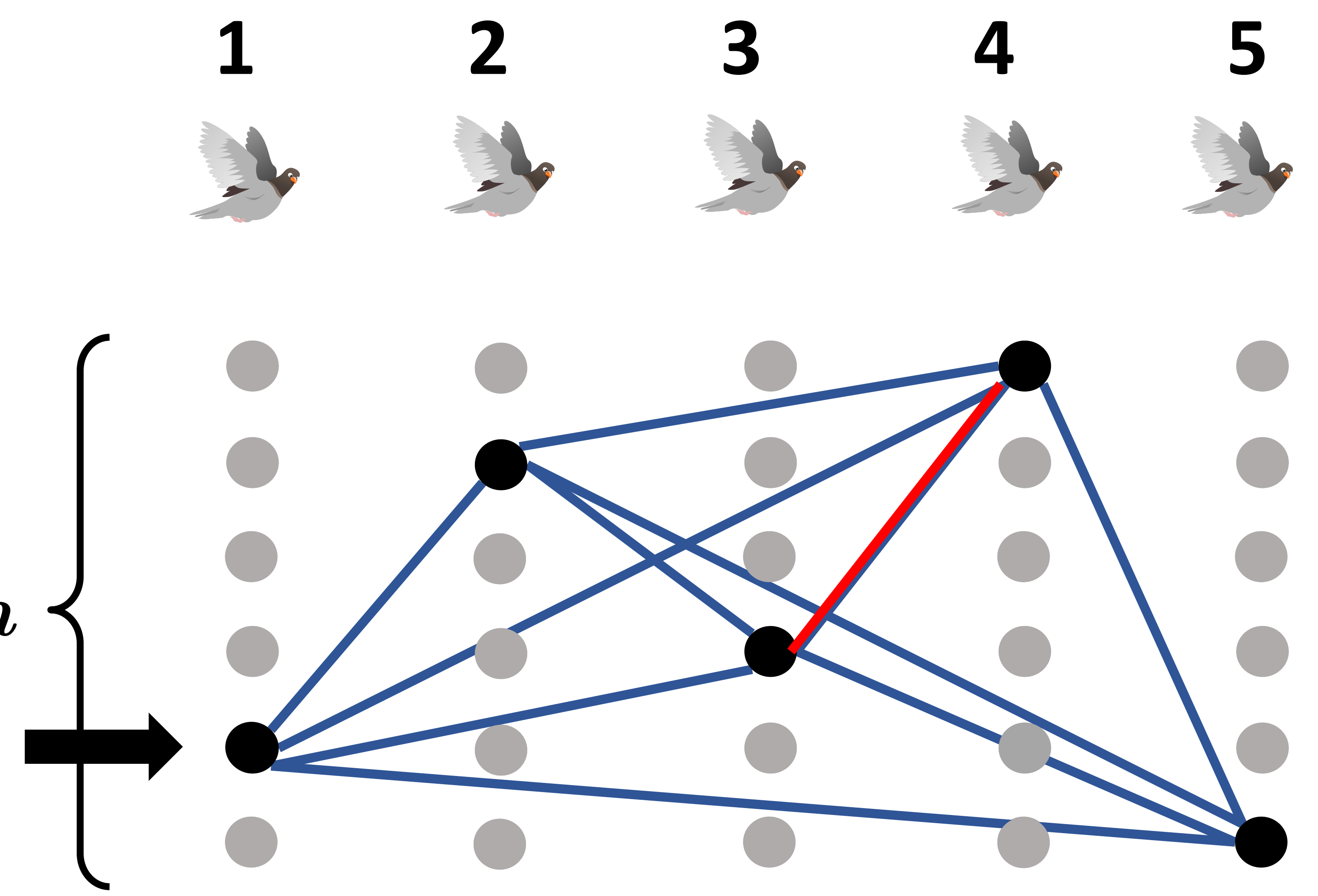
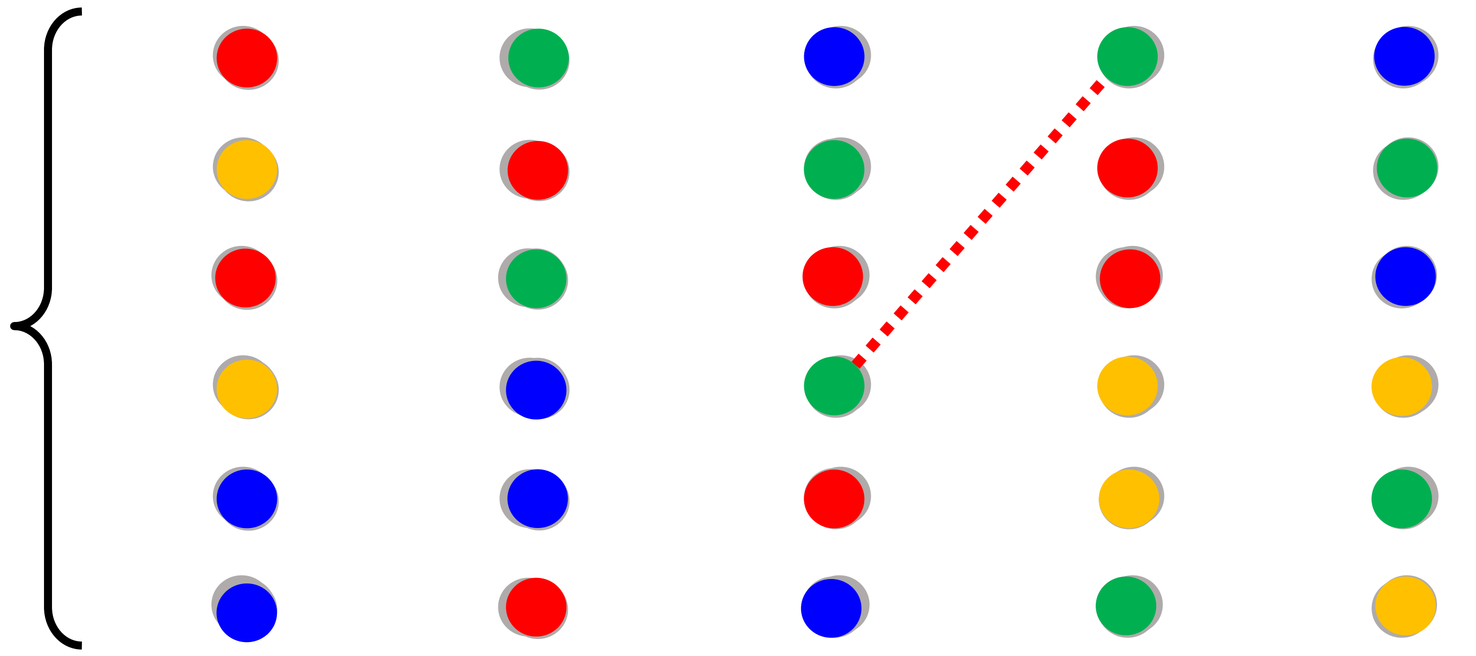


Alice: m

Bob: m



Includes all edges between vertices of $\neq$ colours

New results using approximation method

Improved on [Andreev '87, Harnik, Raz '00] $\exp(\tilde{\Omega}(n^{1/3}))$-size lower bound for f in NP

- $\exp(\tilde{\Omega}(n^{1/2}))$ lower bound for f in NP
- $n^{\Omega(k)}$-size lower bound for k-clique for any $k \leq n^{1/3-o(1)}$

[Cavalar, Kumar, Rossman '20]

- $n^{\Omega(k)}$-size lower bound for k-clique for any $k \leq n^{1/2-o(1)}$ [Błasiok, Meierhöfer '25]

- Clique lower bounds not for clique-colouring

- *Key tool:* improved sunflower lemmas [Alweiss, Lovett, Wu, Zhang '19] and further improvements [Rao '19], [Bell, Chueluecha, Warnke '20]

Very recent result for monotone circuits

> The difficulty in proving that a given boolean function has high complexity lies in the nature of our adversary: the circuit. Small circuits may work in a counterintuitive fashion, using deep, devious, and fiendishly clever ideas. How can one prove that there is no clever way to quickly compute the function? [Jukna '12]

- How deep must we go? Are circuits of depth $> n$ stronger?

- $\exists f$ computable by monotone circuits of size $s = n^{O(1)}$ [dR, Fleming, Janett, Nordström, Pang '25]
  - any monotone circuit of depth-n^2 requires size $s^{1.4}$

- $\exists f$ computable by size-$n^{O(\log n)}$ monotone circuits [Göös, Maystre, Risse, Sokolov '25]
  - any monotone circuit of depth-$n^{O(1)}$ requires size $\exp(\Omega(n^{\epsilon}))$

Some open problems

- Truly exponential size lower bound for f in NP (and in P)
  - Best known $\sim \exp(n^{1/2})$ and $\exp(n^{1/3})$, respectively
- Super-poly lower bound for f in AC^0 [Grigni, Sipser '92] (or even in NC^1)
  - Best known for f in NC^2 [GKRS '19] and for f in $\mathrm{AC}^0[\oplus]$ [Cavalar, Oliveira '23]
- Exhibit function f that has poly-size monotone circuits s.t. any monotone circuit computing f in depth $\leq n$ requires super-poly size

Some more open problems

- Prove $\Omega(n^3)$ lower bound for st-connectivity [Jukna '12]
- Prove $n^{O(\log n)}$ upper bound for matching or prove better lower bound
  - Or explain why it's hard to prove exponential lower bounds
- Prove $n^{\Omega(k)}$ lower bound for k-clique for $k > \sqrt{n}$

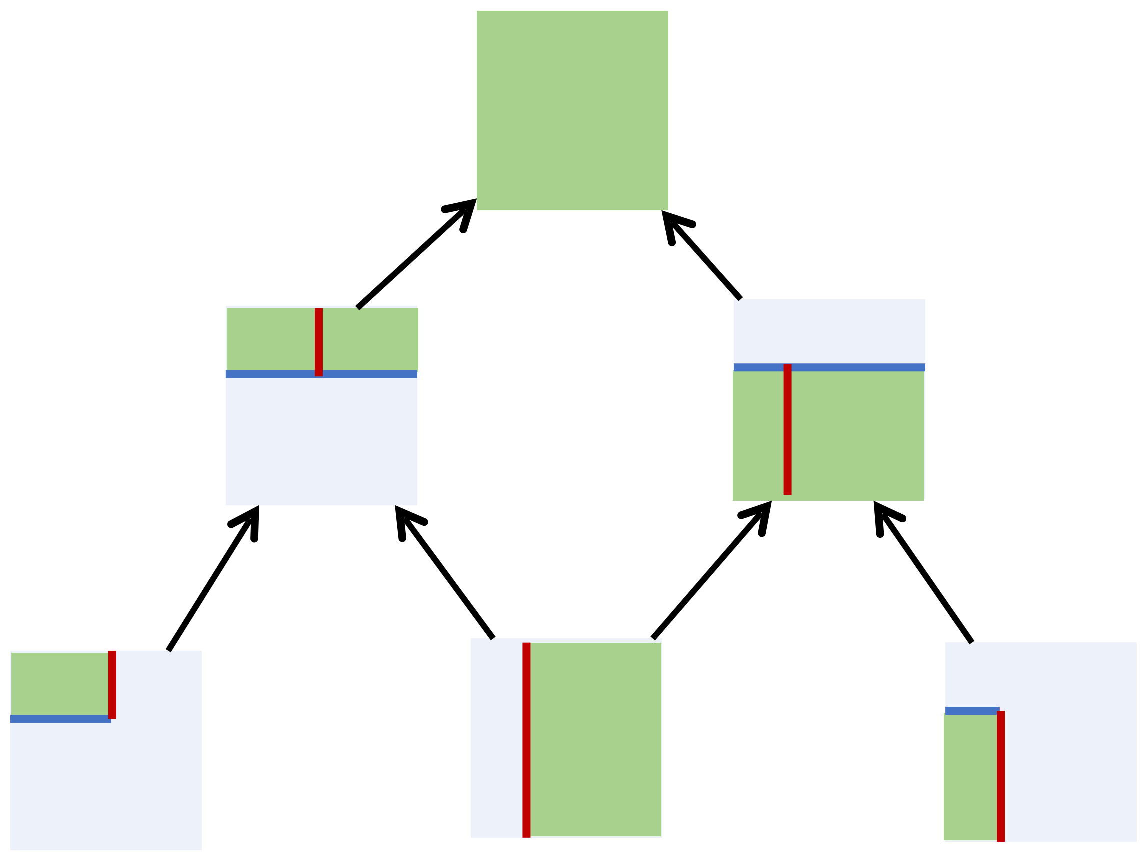
Thank you!

Thoughts on methods

- Methods to obtain monotone circuit lower bounds
  - Approximation method
  - Bottle-neck counting
  - DAG-like lifting

  } Specialised per problem (Approximation method, Bottle-neck counting)

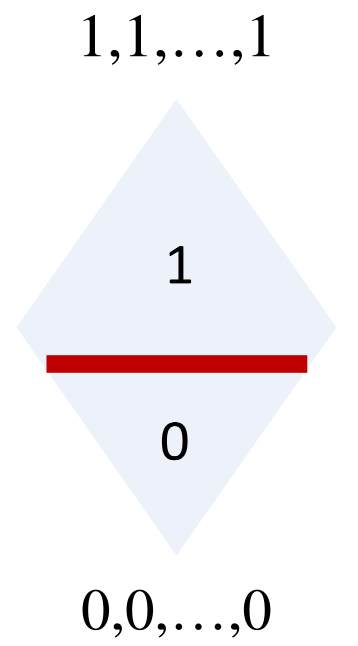
  Generalised method (DAG-like lifting)



- Generalised method: black-box, main argument done once
- Specialised method: can perhaps get better bounds
- Common flavour: is lifting a special case of approximation method?
- Also common: use of sunflowers

Thoughts: challenges for slice functions

- All(?) size lower bounds for monotone circuits hold for monotone ***real*** circuits
  - Linear size monotone real formulas can compute slice functions
  - Current lower bound methods don't work for slice functions



- Monotone real circuits introduced to study cutting planes in proof complexity
  - Corresponding "communication" model: triangle-DAGs
  - Only way we know of proving lower bounds for cutting planes
  - Gives important insight on the methods

Some more open problems

- Prove lower bounds with limited number of negations $\gg \log \log n$ [Jukna '12]
  - Methods work only to $\leq \log \log n$ [Amanoand, Maruoka '05]
  - $\lceil \log(n+1) \rceil$ negation gates are enough [Markov '57]
- Can all (monotone) circuit lower/upper bounds be seen naturally as communication lower/upper bounds for $(\mathrm{m})\mathrm{KW}$? (Majority?) [Karchmer '89]
- Understand power and limitations of lifting? [Cavalar, Oliveira '23]
- General TFNP framework: other lifting theorems & communication l.b.?