Structure-Guided Automated Reasoning

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Structure?

- Many problems are (computationally) hard, but simpler on trees
- There is a way to capture how *"tree-like"* a structure is – the so-called treewidth, which can be defined in terms of tree decompositions (TDs)

Why tree decompositions? Efficient Solving!

$F = (\neg a \vee b \vee x) \wedge (a \vee b) \wedge (c \vee \neg x) \wedge (b \vee \neg c) \wedge (\neg b \vee \neg c \vee \neg y)$

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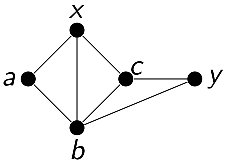

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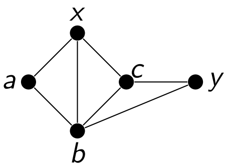

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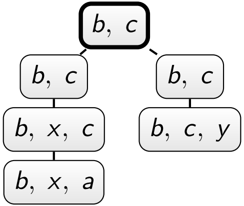

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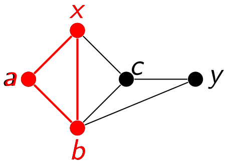

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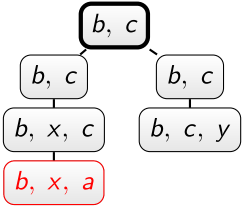
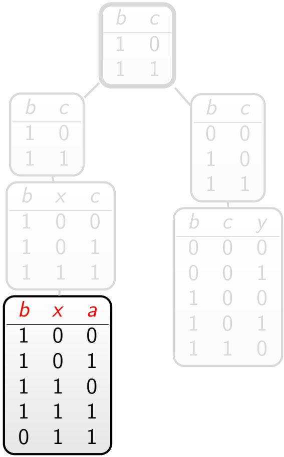
2. Solve subproblems





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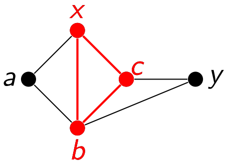

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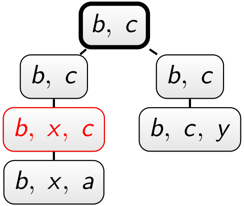

b	c
1	0
1	1

b	c
1	0
1	1

b	c
0	0
1	0
1	1

b	x	c
1	0	0
1	0	1
1	1	1

b	c	y
0	0	0
0	0	1
1	0	0
1	0	1
1	1	0

b	x	a
1	0	0
1	0	1
1	1	0
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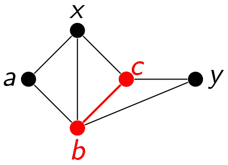

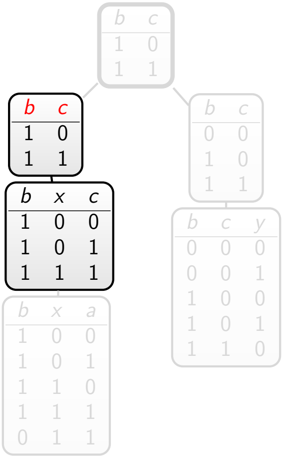

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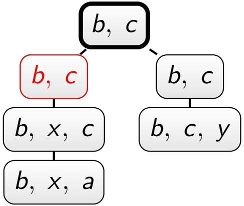

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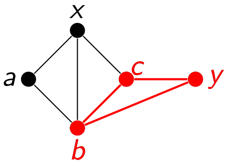

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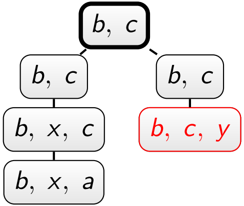

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b	x	c
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b	x	a
1	0	0
1	0	1
1	1	0
1	1	1
0	1	1

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0	0	0
0	0	1
1	0	0
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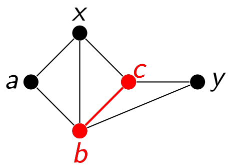

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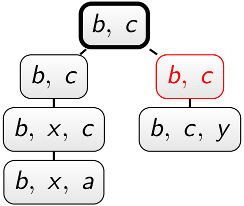

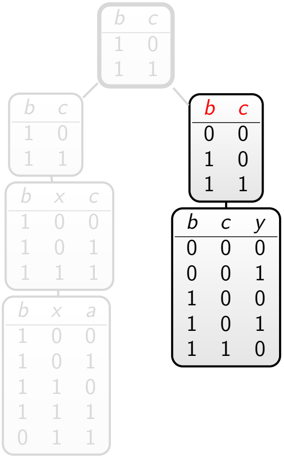

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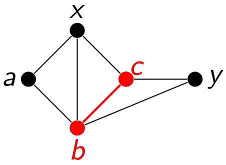

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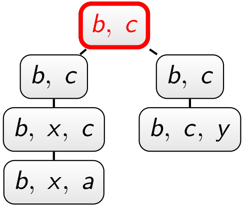

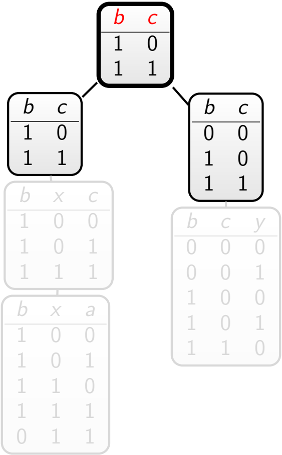

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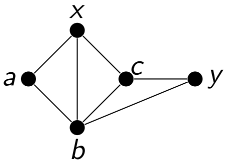

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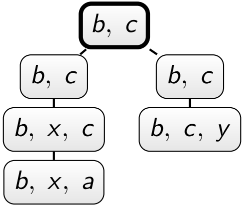

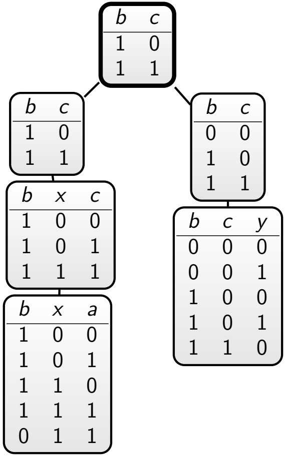

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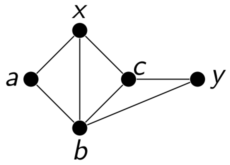

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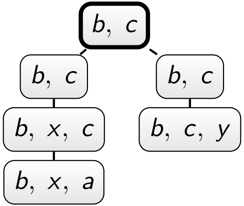

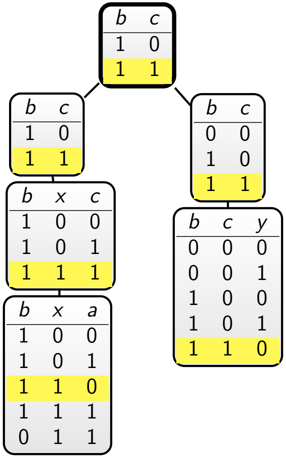

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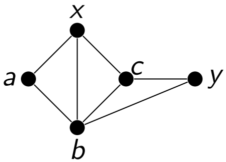

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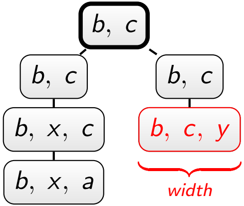

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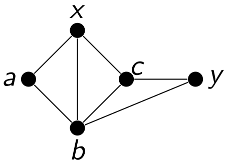

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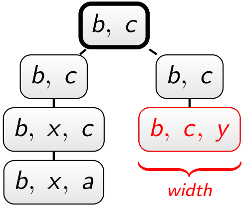
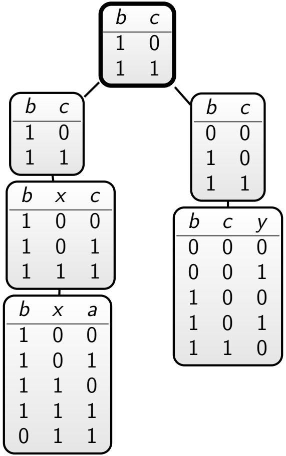


$\rightarrow 2^{width} \cdot n$



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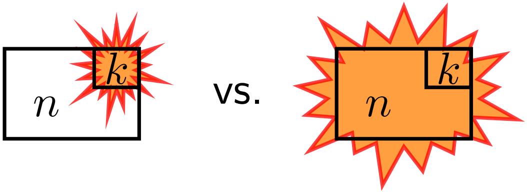
- renders large variety of NP-hard problems *fixed-parameter tractable (FPT)*



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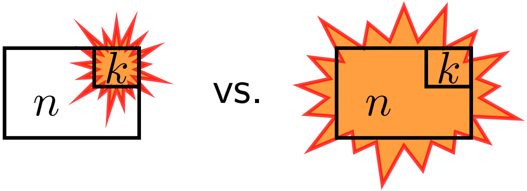

$f(k) \cdot n$ vs. $g(n)$

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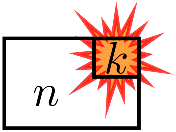

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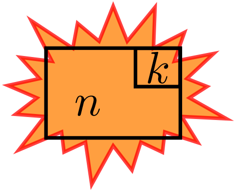

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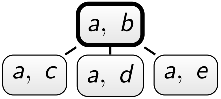

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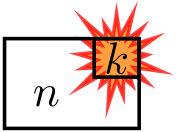

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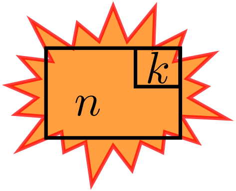

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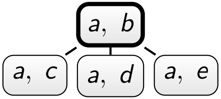

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⇝ How do these $f(k)$ look like? Can we do better?

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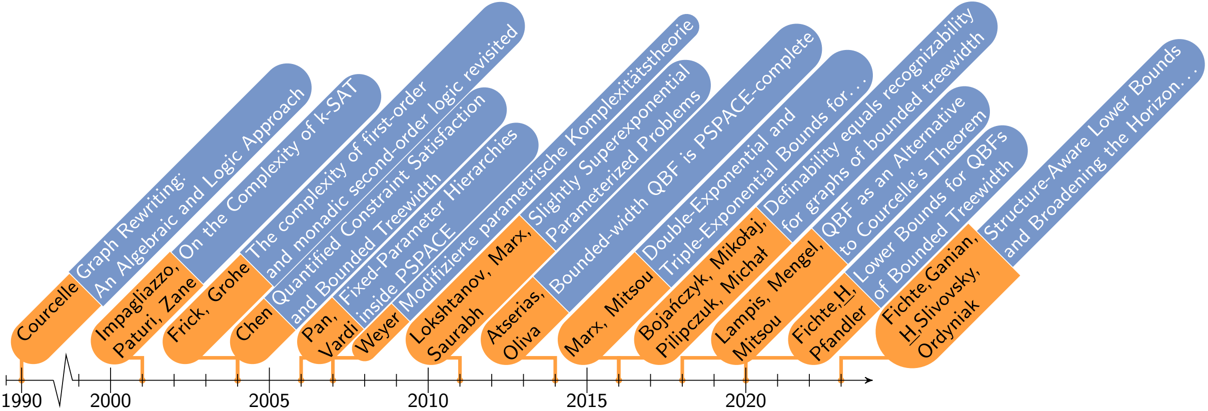
QSAT_ℓ	$\mathrm{tower}(\ell, \mathcal{O}(k))$	$\mathrm{tower}(\ell, o(k))$	Σ_ℓ^{P} / Π_ℓ^{P}

Biased Excerpt of Related Work



Thanks to Arne Meier (Uni Hannover) for the timeline latex package.

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We have  $\models \varphi_{\mathrm{3col}}$ and  $\not\models \varphi_{\mathrm{3col}}$

Can We Efficiently Translate To SAT?

MSO To SAT?

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propositional variables

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What about the treewidth?

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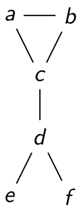

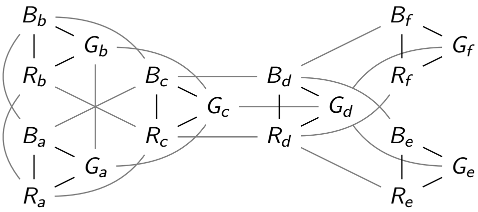

MSO To Sat?

- Can we derive Sat formula ψ with $tw(\psi) \leqslant g\big(tw(\mathcal{S})\big)$?
  ($\mathcal{S} \models \varphi$ iff $\psi \in$ Sat)

$\rightsquigarrow$ Naive Approach: $\forall$ translates to $\bigwedge$, $\exists$ translates to $\bigvee$

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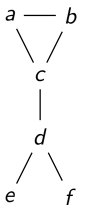

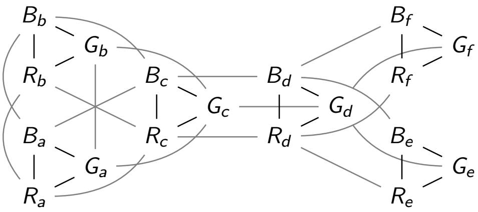

Factor 3 overhead: $\rightsquigarrow$ 3-Coloring with $f(k) = 2^{3k}$

MSO To Sat?

- Does this always work?

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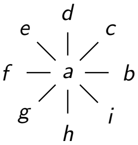
Example: Dominating Set

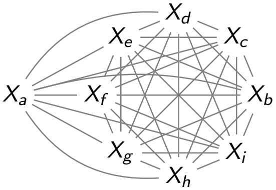
$$\varphi_{\mathrm{ds}}(X) = \forall x \exists y \, . \, Xx \vee (Exy \wedge Xy)$$

- Does this always work? NO!

Example: Dominating Set

$$\varphi_{\mathrm{ds}}(X) = \forall x \exists y \,.\, Xx \lor (Exy \land Xy)$$





Contributions

1. Faster Structure-Guided Reasoning . . .
   - . . . for $\textsc{QSat}_\ell$: $f(k) = \mathsf{tower}(\ell, k + 3.92)$
   - . . . for $\#\exists\textsc{Sat}$: $f(k) = 2^{2^{k+3.59}}$

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   - … for $\#\exists\textsc{Sat}$: $f(k) = 2^{2^{k+3.59}}$
2. A $\textsc{Sat}$ version of Courcelle's Theorem
   - $\text{MC}(\text{MSO}_\ell)$ to $\textsc{Sat}$ of size: $\text{tower}(\ell - 1, (k + 9)|\varphi| + 3.92)$
   - $\text{FD}(\text{MSO}_\ell)$ to $\textsc{MaxSat}$ of size: $\text{tower}(\ell, (k + 9)|\varphi| + 3.92)$
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$\rightsquigarrow$ Implies translations to ILP

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     $\rightsquigarrow$ compress treewidth k to $\frac{k}{c}$; costing block size increase to $c \cdot \text{bs}(\varphi)$

How Do We Translate To SAT?

Warmup: Eliminate QBF Quantifiers

Standard Translations	Structure-Aware Encoding
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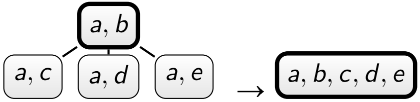

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$(\textsc{QSat}, k)$	$(\textsc{QSat}_\ell, k)$
$\downarrow$	$\downarrow_{SAW}$
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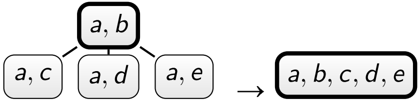

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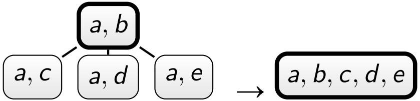

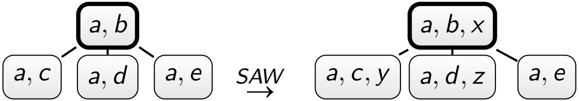

Structure-Aware Quantifier Block Elimination

Given: $\exists V_1 \; \forall V_2 \; \ldots \forall V_\ell \,.\, \psi$ (DNF), TD $\mathcal{T}$ of P_ψ,

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$$\bigwedge_{\alpha \in 2^{\chi(\mathrm{root}(\mathcal{T})) \cap V_\ell}} \mathrm{sat}_{\leqslant \mathrm{root}(\mathcal{T})}^\alpha$$ (root assignments are satisfied)

Example: $\text{QS}\textsc{at}_2$

$$\psi = \exists a, b\, \forall c, d \,.\, (\neg a \wedge \neg b) \vee (a \wedge b \wedge \neg c) \vee (c \wedge \neg d)$$

Structure-Aware Quantifier Block Elimination

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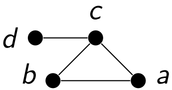

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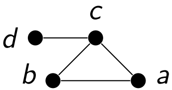
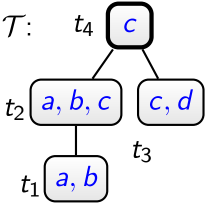


$\mathcal{T}$:



$\overrightarrow{SAW}$

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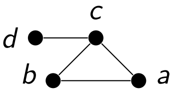

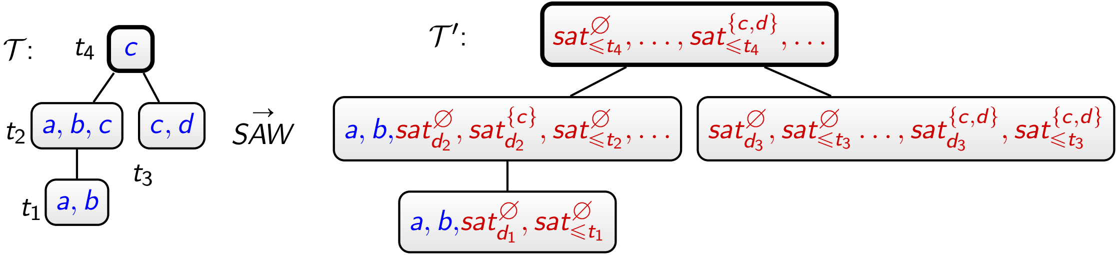

Structure-Aware Quantifier Block Elimination

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  - perfectly in line with known lower bounds (ETH) [Fichte, H & Pfandler 20]

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  - Replace SO variables X by X_u for every $u \in U(\mathcal{S})$
  - Replace FO variables x by x_u for every $u \in U(\mathcal{S})$

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    1. FO variables require unique assignment

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  - Remaining issues
    1. FO variables require unique assignment
    2. We need to evaluate the actual MSO expressions!

② How do we evaluate MSO Expressions?

- Different types of MSO atoms
  - $x = y$ for FO variables x, y
  - X_x for SO variable X, FO variable x
  - $R_{x_1 \ldots x_a}$ for relation R in $\mathcal{S}$, FO variables $x_1, \ldots, x_a$

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- How do we process atoms in the SAT way?

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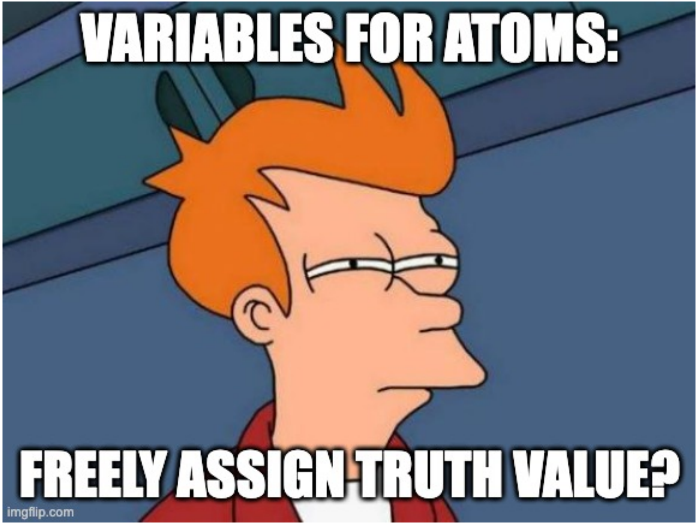
  ⇝ Not an issue; we assume prenex form


VARIABLES FOR ATOMS:
FREELY ASSIGN TRUTH VALUE?
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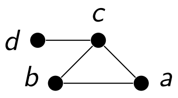

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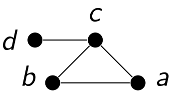

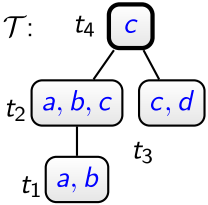

$\overrightarrow{SAW}$

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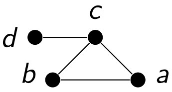
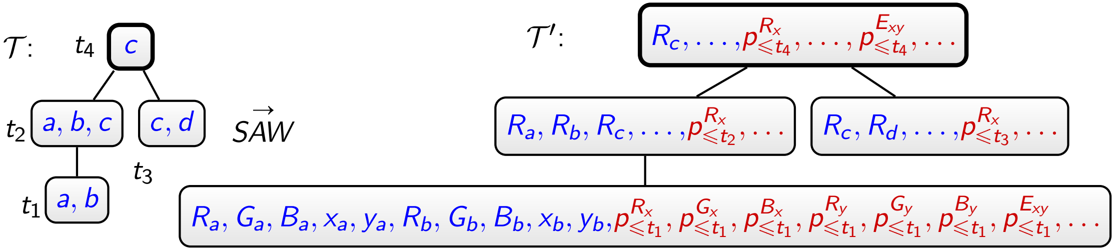
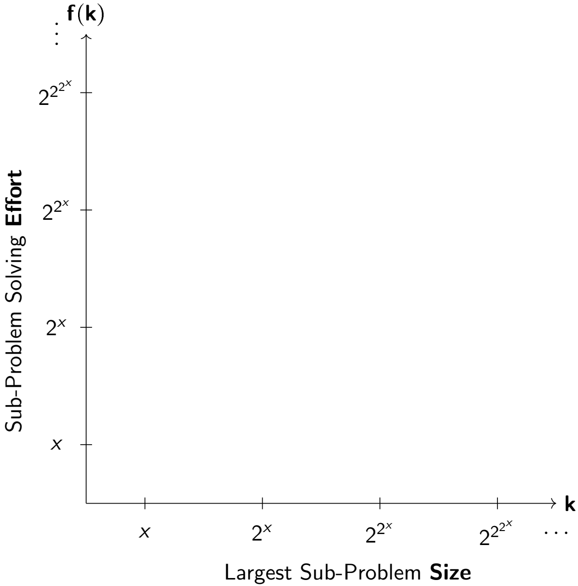




Outlook and Conclusion

Time vs. Space, you say?



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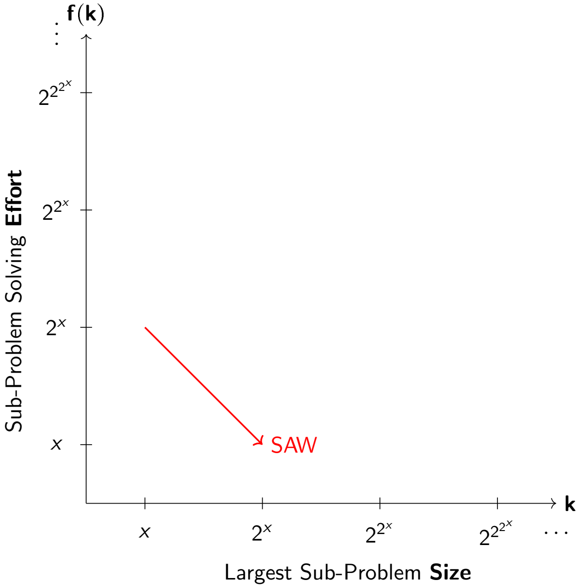

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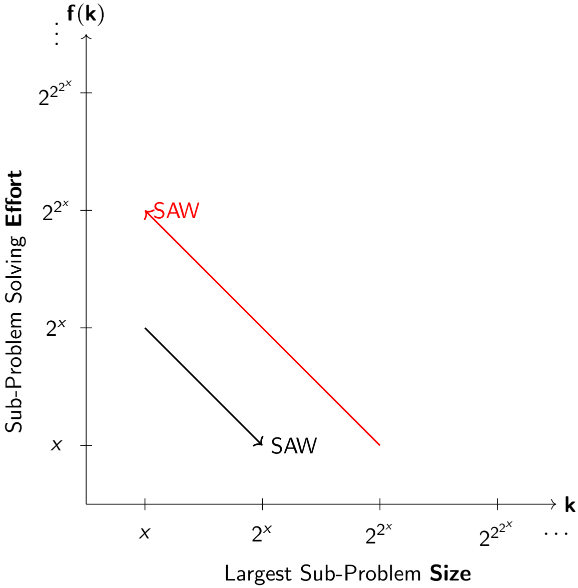

Conclusion & Future Work

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- What is the precise role of FO variables?
  - ⇝ Do we get tight bounds via "guarded" formulas?

Unique Assignment for FO Variables!

Given: $Q_1 V_1 Q_{q-1} V_{q-1} \ldots Q_q v_q \ldots Q_\ell v_\ell . \psi$, structure $\mathcal{S}$, TD $\mathcal{T}$ of $P_\mathcal{S}$

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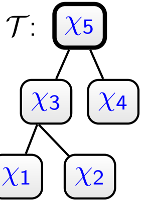
Key: Structure-Aware Encodings



$\overrightarrow{SAW}$

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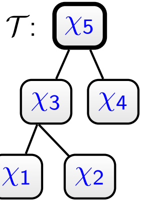

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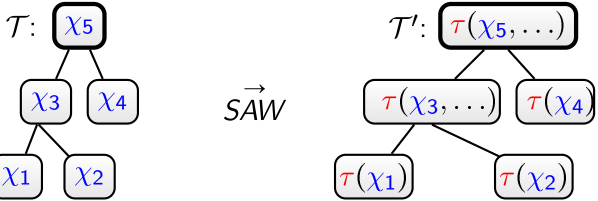

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