Tight Approximation and Kernelization Bounds for Vertex-Disjoint Shortest Paths

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Univerity of Bergen

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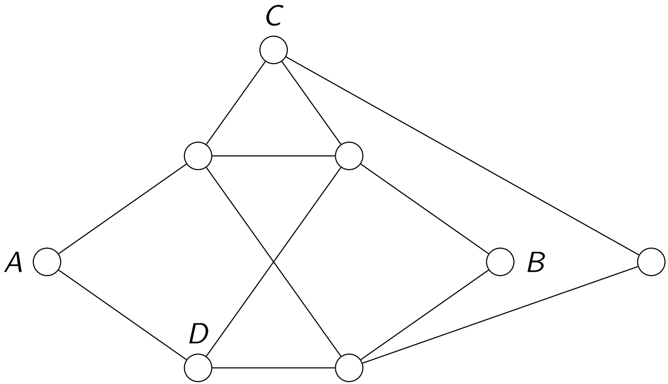
Moving



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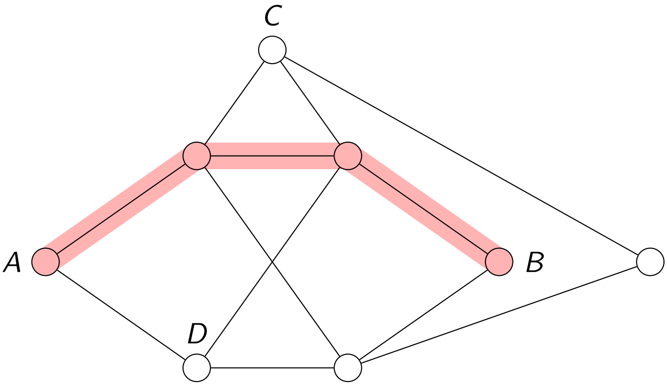

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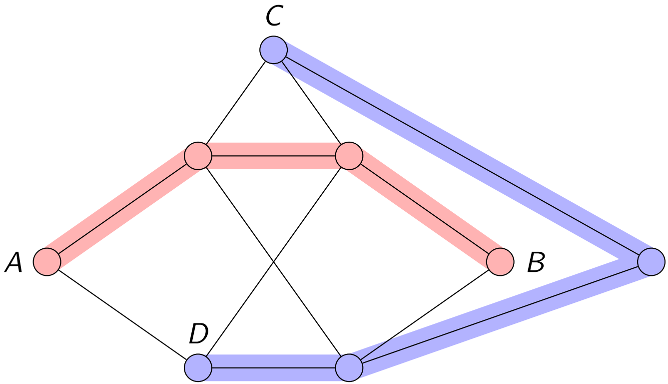

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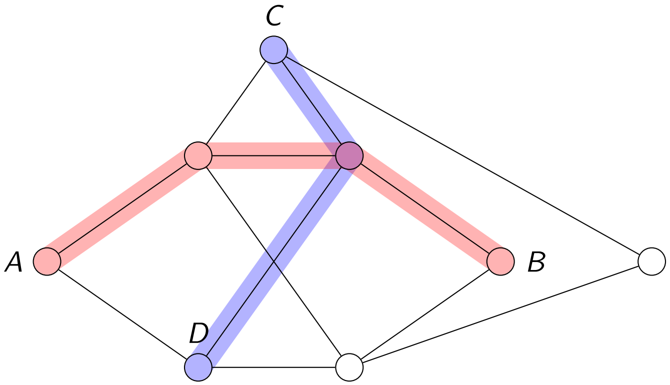

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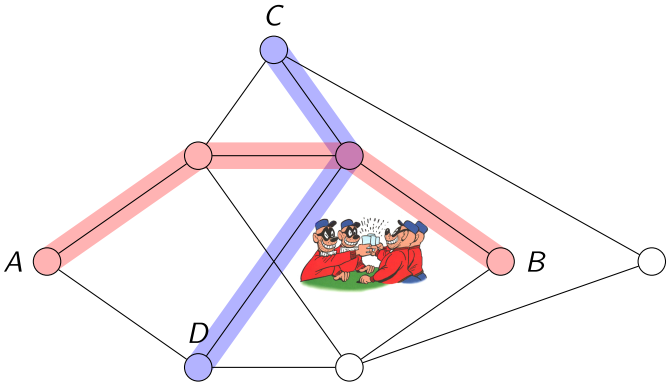

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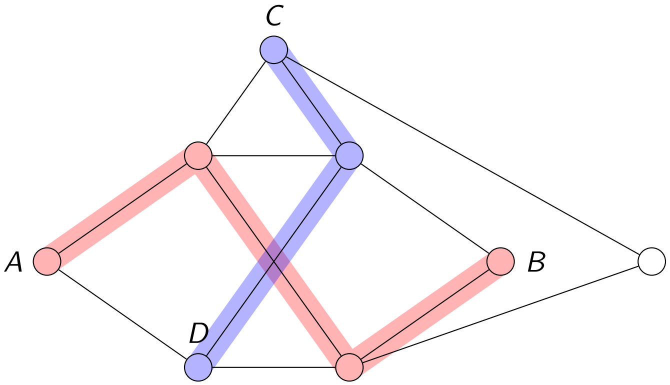

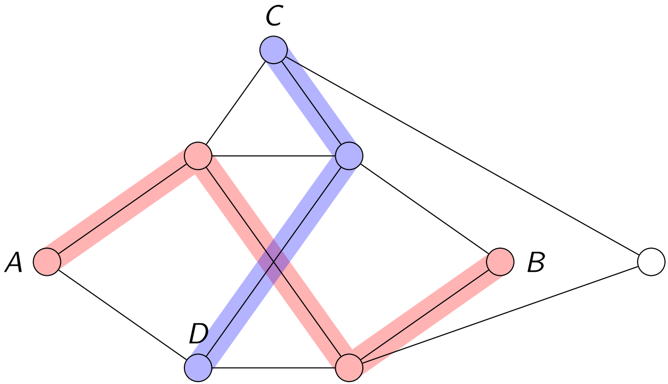

Disjoint Shortest Paths

Input: An undirected graph G and vertex pairs $(s_i, t_i)_{i \in \{1,2,\ldots,k\}}$.
Task: Connect as many terminal pairs as possible via vertex-disjoint shortest s_i-t_i-paths.

Related Work

- **2 Disjoint Shortest Paths**

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  - undirected graphs: $O(n^8)$, $O(n^6)$, $O(nm)$, $O(n+m)$ time

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  - no $(2-\varepsilon)$-approximation in $f(k)n^{o(k)}$ time assuming gap-ETH
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  - solvable in polynomial time on directed planar graphs if all terminals are incident to two faces (even min-sum) [Coli n de Verdière and Schrijver. ACM TALG 2011]

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Parameter	Exact	Approximation
	NP-complete	**$\mathbf{m^{1/2-\varepsilon}}$-approximation is NP-hard**

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Upper and Lower (FPT) Approximation Bounds

A Polynomial-time Approximation Algorithm

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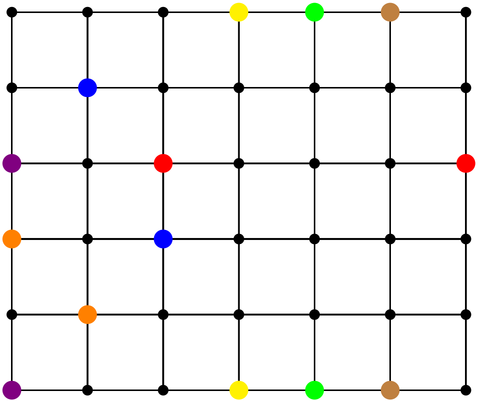

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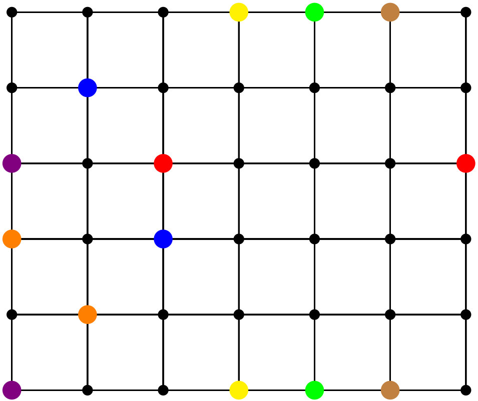

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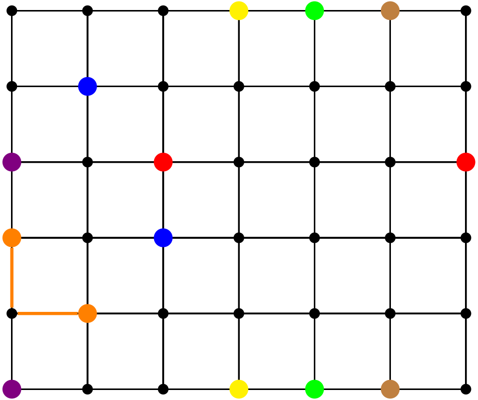

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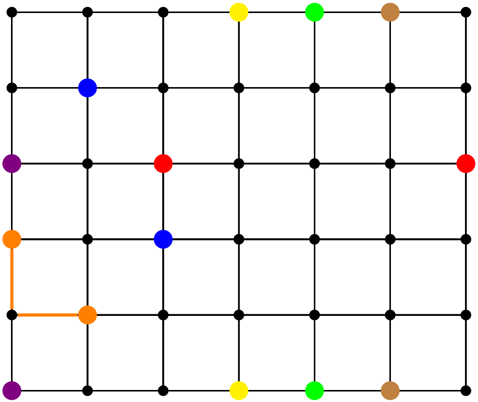

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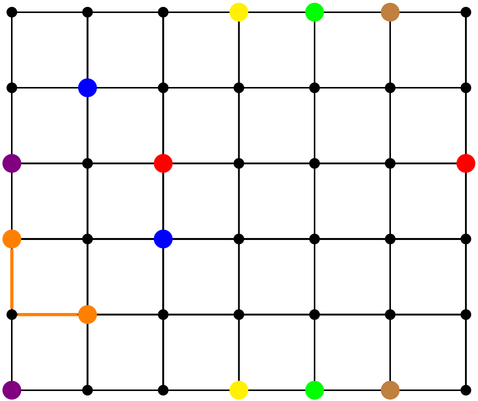

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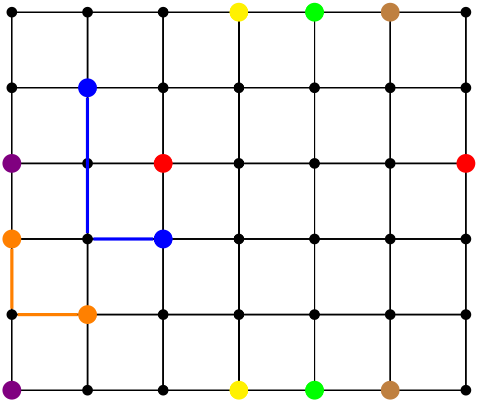

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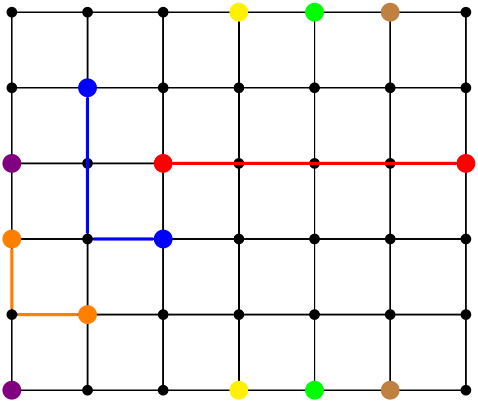

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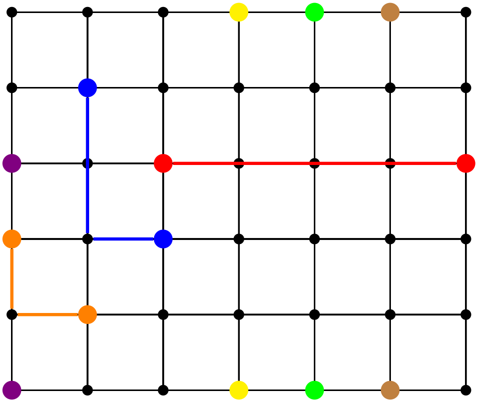

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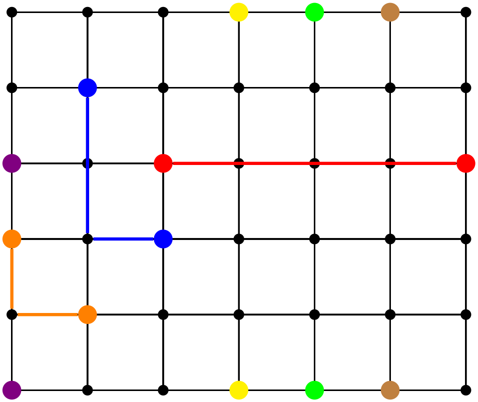

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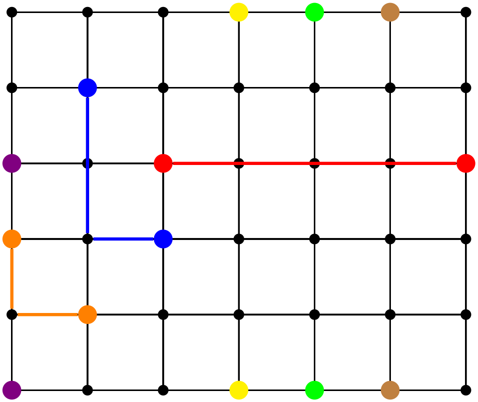

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$\rightsquigarrow$ $\sqrt{\ell}$-approximation in polynomial time

The General Idea

for excluding $m^{1/2-\varepsilon}$-approximations

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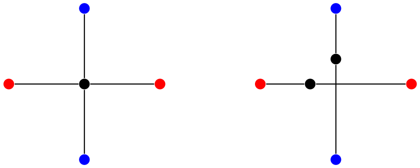

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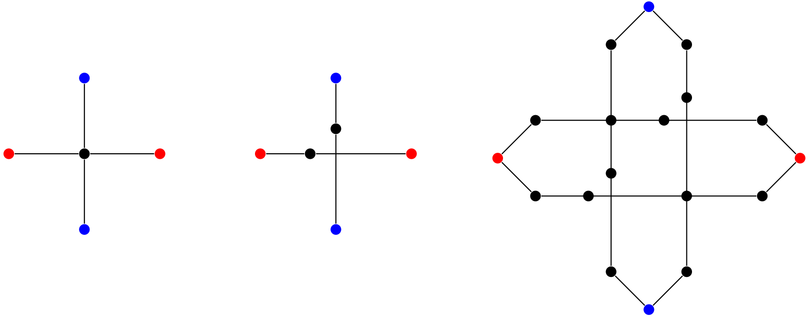

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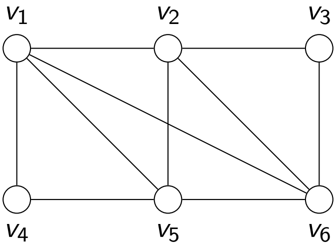

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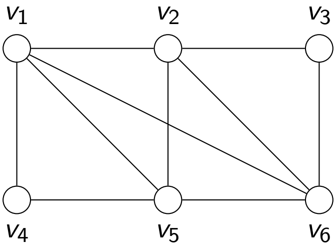

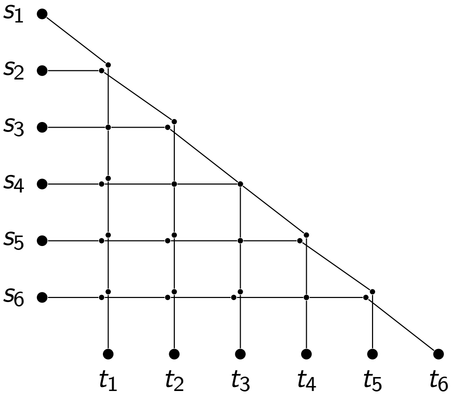

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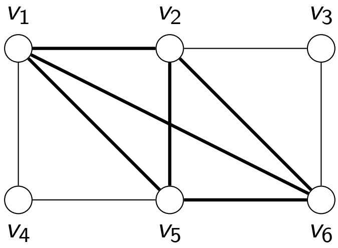
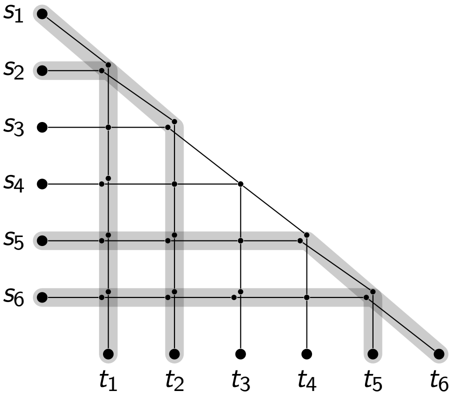
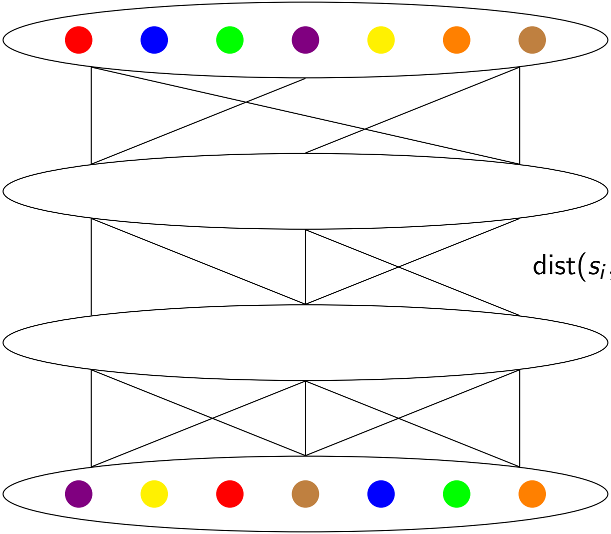




Excluding Polynomial Kernels

A Special Case



dist(s_i, t_i) = # layers - 1

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input: t instances $I_1, I_2, \ldots, I_t$ of special case all with same number of layers and terminal pairs

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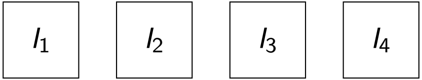

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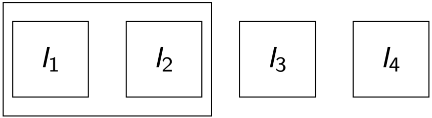

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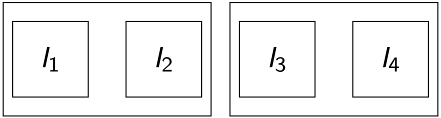

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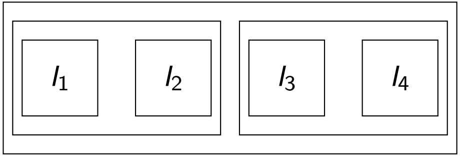

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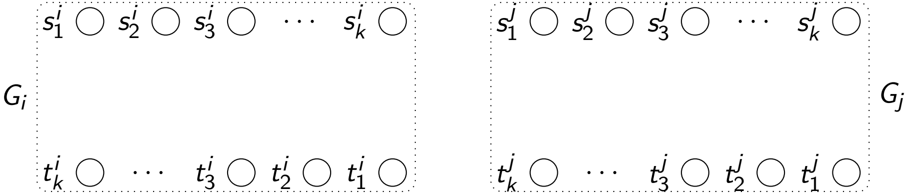

G_i
s_1^i s_2^i s_3^i $\cdots$ s_k^i
t_k^i $\cdots$ t_3^i t_2^i t_1^i
G_j
s_1^j s_2^j s_3^j $\cdots$ s_k^j
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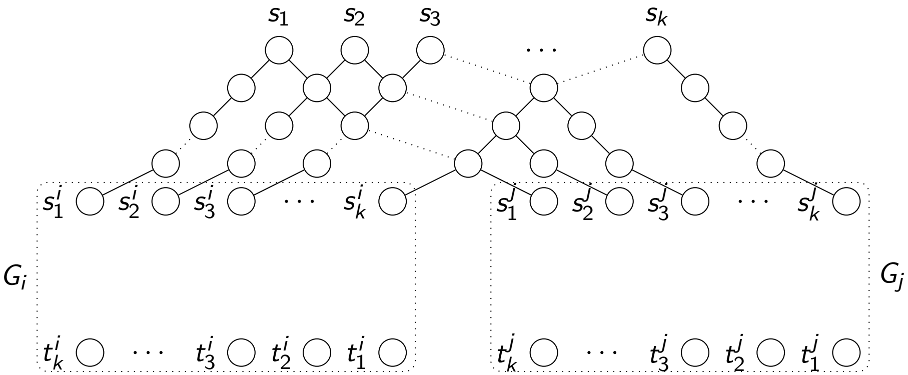

s_1 s_2 s_3 … s_k
s_1^i s_2^i s_3^i … s_k^i
s_1^j s_2^j s_3^j … s_k^j
G_i
G_j
t_k^i … t_3^i t_2^i t_1^i
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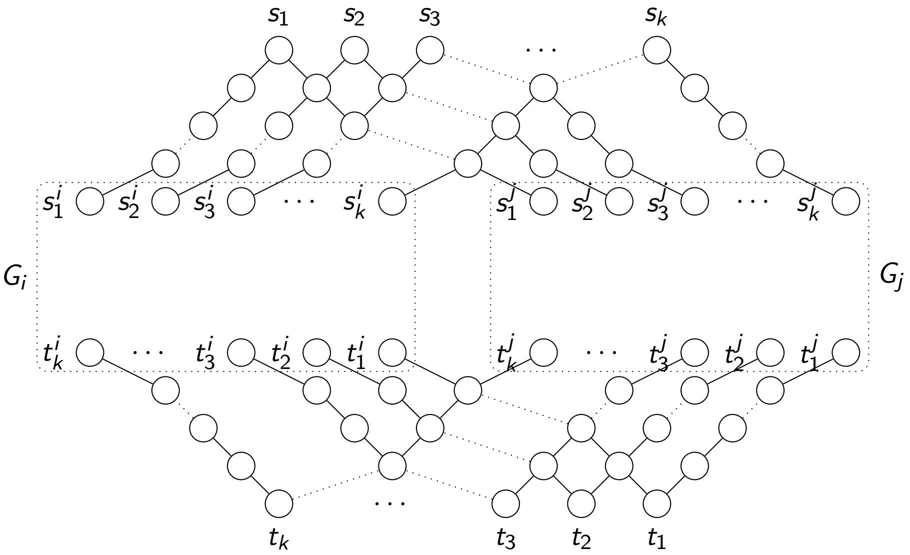
s_1
s_2
s_3
$\cdots$
s_k
s_1^i
s_2^i
s_3^i
$\cdots$
s_k^i
s_1^j
s_2^j
s_3^j
$\cdots$
s_k^j
G_i
G_j
t_k^i
$\cdots$
t_3^i
t_2^i
t_1^i
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t_2
t_1

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Thank you.