# Tropical proof systems

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# Crash-Introduction to Propositional Proof Complexity

For those who missed Albert's tutorial, [Cook, Reckhow 1979], [Tseitin 1968], and next 1000 papers

- We are interested in the length of (shortest) proofs in various proof systems.
- We typically consider proofs of **unsatisfiability** of Boolean formulas in CNF (or proofs for some other **co-NP**-complete language):

$$\neg\, \exists x_1, x_2, \ldots, x_n \in \{0,1\} \qquad \underbrace{\Phi(x_1, x_2, \ldots, x_n)}_{\text{quantifier-free formula in CNF}}$$

- A proof system is a polynomial-time deterministic verification algorithm $V(\Phi, \pi)$:

$$\Phi \in \textbf{UNSAT} \implies \text{there is a proof } \pi \text{ such that } V(\Phi, \pi) = 1$$
$$\Phi \notin \textbf{UNSAT} \implies \text{for every candidate proof } \kappa,\ V(\Phi, \kappa) = 0$$

- A well-known proof system: `Resolution`, a proof is a derivation of the empty clause (`false`) from the input clauses using the rule

$$\frac{A \lor x \qquad B \lor \overline{x}}{A \lor B}$$

- Typical statements:
  - A specific proof system $\Pi$ has no polynomial-size proofs for some formulas $\{\Phi_n\}$.
  - System $\Pi_S$ polynomially simulates system $\Pi_W$ ($\Pi_W$-proofs can be rewritten as $\Pi_S$-proofs with at most polynomial increase in size).
  - System $\Pi_S$ has polynomial-size proofs for some specific formulas $\{\Phi_n\}$ while system $\Pi_W$ has none.

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# Good old LP systems

Axioms $x_i \geqslant 0$, $1 - x_i \geqslant 0$. Clause $\bigvee_i \ell_i$ is translated to $\sum_i \ell_i \geqslant 1$.

Cutting Planes (CP) [W.Cook, Coullard, Turán] based on Gomory-Chvátal cuts

$$\frac{f_1 \geqslant 0 \qquad f_2 \geqslant 0}{\alpha_1 f_1 + \alpha_2 f_2 \geqslant 0}\ (\alpha_1, \alpha_2 \geqslant 0), \qquad \frac{\sum_i c a_i x_i - d \geqslant 0}{\sum_i a_i x_i - \lceil d/c \rceil \geqslant 0}\ (c, a_i, d \in \mathbb{Z}).$$

Resolution over Cutting Planes (Res(CP)) [Krajíček]

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$$\frac{\quad}{f - 1 \geqslant 0 \ \vee\ -f \geqslant 0}. \qquad \text{(NEG-INT)}$$

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$$\frac{\quad}{x - 1 \geqslant 0 \ \vee\ -x \geqslant 0}\ (x \text{ is a variable}). \qquad \text{(NEG-BOOL)}$$

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# Good old LP systems

Facts:

- Exponential lower bound for CP
  [Pudlák] following [Krajíček] for CP*

  star* for unary coefficients

- Quasipolynomial simulation of treelike Res(CP*) in CP
  [Fleming et al.]

  **treelike** proofs: used twice, derived twice

- **Exponential lower bound for treelike Res(CP)**
  [Gläser, Pfetsch]

- (Daglike) Res(LP*)=Res(CP*)
  [H, Kojevnikov]

Open:

- **Exponential lower bounds for (daglike, not treelike!) Res(LP*)?**
  Res(CP) is more popular, but it may be stronger

# Tropical (min-plus) arithmetic: A more algebraic version of $\vee, \leqslant$

Min-plus arithmetic over $\mathbb{Q}_\infty$:

$$a \oplus b = \min(a, b), \qquad a \odot b = a + b$$

Tropical monomial:

$$m = x_1^{d_1} \odot x_2^{d_2} \odot \ldots \odot x_k^{d_k} \qquad (\textstyle\sum d_i x_i)$$

Tropical term:

$$t = c \odot m, \quad c \in \mathbb{Q} \qquad (m + c)$$

The empty monomial (term), $\infty$ plays the role of zero:

$$\infty$$

Tropical polynomial:

$$p = t_1 \oplus t_2 \oplus \ldots \oplus t_m \qquad (\min(t_1, \ldots, t_m))$$

Min-plus inequality:

$$p_1 \leqslant p_2$$

In the usual terms (for linear functions $L_i, L'_j$):

$$\min(L_1, \ldots, L_m) \leqslant \min(L'_1, \ldots, L'_s)$$

or

$$L_1 \leqslant L'_j \quad \vee \quad L_2 \leqslant L'_j \quad \vee \quad \ldots \quad \vee \quad L_m \leqslant L'_j \qquad \text{(for every } j\text{)}$$

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Input system of min-plus inequalities: $f_1 \leqslant g_1, \; f_2 \leqslant g_2, \; \ldots, \; f_m \leqslant g_m$.

Min-Plus Nullstellensatz (MP-NS) — complete by [Grigoriev, Podolskii].
Multiply $f_i \leqslant g_i$ by polynomials and take the sum:

$$\bigoplus_i f_i \odot p_i \leqslant \bigoplus_i g_i \odot p_i$$

so that for every monomial (including $\infty$), its coefficient on the left is $>$ than on the right.

$1 \odot x \odot y \bigoplus \frac{1}{2} \leqslant 0 \odot x \odot y \bigoplus -\frac{1}{2}$ $\qquad$ $\min(x + y + 1, \frac{1}{2}) \leqslant \min(x + y, -\frac{1}{2})$

Min-Plus Polynomial Calculus (MP-PC).

Do it step by step.

- Take the tropical sum $\oplus$.
- Multiply by a term.
- Transitivity of $\leqslant$.
- Thus compose an MP-NS-like inequality.

Tropical resolution rule:

$$\frac{t \oplus f \leqslant 0 \qquad t' \oplus f \leqslant 0}{(t \odot t') \oplus f \leqslant 0}, \text{ where } t, t' \text{ are terms.} \qquad (\odot\mathrm{RES})$$

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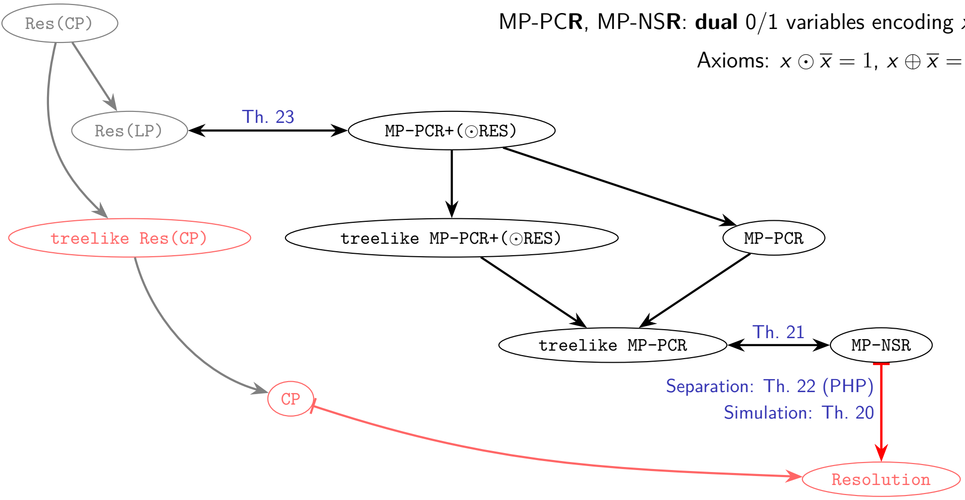
# General picture

MP-PC**R**, MP-NS**R**: **dual** $0/1$ variables encoding $x, \overline{x}$

Axioms: $x \odot \overline{x} = 1$, $x \oplus \overline{x} = 0$




Important: there is **no** equivalence of MP-PC to Res(LP) without (⊙RES), with some evidence.

# Daglike Resolution in MP-NSR

Translate $A = \ell_1 \lor \ldots \lor \ell_k$ into $[A] = 0 \odot \ell_1 \odot \ldots \odot \ell_k$.

**Input clauses**

$$1 \leqslant 0 \odot [C_i]. \tag{1}$$

At step $i = 1, 2, \ldots, s$, let $c_i = 1 - 1/(i+1)$.

**Resolution**

$$\frac{A \lor x \quad A \lor \neg x}{A},$$

becomes

$$c_i \odot x \odot [A] \bigoplus c_i \odot \overline{x} \odot [A] \leqslant c_i \odot [A]. \tag{2}$$

It's the axiom $x \oplus \overline{x} \leqslant 0$ multipled by $[A]$!

**Weakening**

$$\frac{A}{A \lor \ell},$$

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$$c_i \odot [A] \leqslant c_i \odot [A] \odot [\ell]. \tag{3}$$

Take the tropical sum of all these (1), (2), (3). They combine nicely!

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# Lower bounds

- Exponential size bound:

  MP-NSR refutation size for the binomial $x^{\odot k} = c$ is greater than $k$

- Tropical resolution rule ($\odot$RES) is not derivable in MP-PCR
- Non-integer coefficients are needed in MP-NSR

# Further research

Two sources of non-treelikeness

- Integer multiplication by a big constant: treelike in LP, daglike in tropics.
- Transitivity of min-plus inequalities: treelike in tropics, daglike in LP.

Tropical resolution.

- Separate or simulate `Res(LP)` in `MP-PCR` without ($\odot$RES).
- What about treelike versions, what would replace tropical resolution for `MP-NSR`?

Unary coefficients?

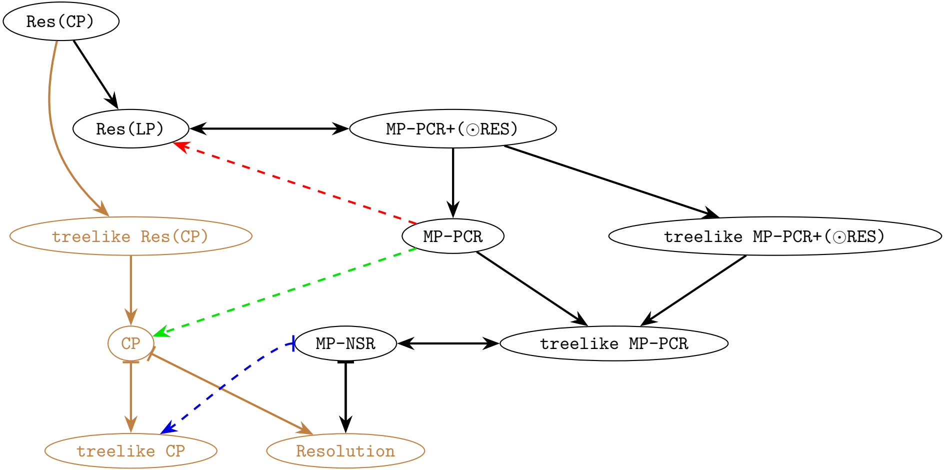
`MP-NSR` vs `CP`.

- Relations between `MP-NSR` and `CP` are unclear, both for unary and binary coefficients, and even for `treelike CP`.
- Does adding the integer negation (as in `Res(CP)`) to `MP-NSR` allow it to simulate at least `treelike CP`?

Lower bounds for CNFs. Show exponential lower bounds for `MP-NSR` for CNFs (or small degrees).

(There are more open questions...)

# General picture




Pictures are inspired by Proof Complexity Zoo by Marc Vinyals

# PHP in treelike Res(LP) and MP-NSR

Treelike Res(LP) proof (motivated by the known treelike CP proof, but avoids rounding).

## Lemma

$S_n := x_1 + \ldots + x_n$.
*There is a treelike Res(LP) derivation of $S_n \leqslant 1$ from $x_i + x_j \leqslant 1$.*

## Proof.

- $S_{n-1} \leqslant 1$

  $\vdash S_{n-1} + x_n \leqslant 1 + x_n$.
- $x_i + x_n \leqslant 1$ (for $1 \leqslant i \leqslant n-1$)

  $\vdash S_{n-1} + (n-1)x_n \leqslant n-1$

  $\vdash S_{n-1} + x_n \leqslant 1 + (n-2)\overline{x}_n$.
- Add both inequalities to $x_n \leqslant 0 \vee (n-2)\overline{x}_n \leqslant 0$.

□

It does not use full (+RES), one premise does not use the disjunction ⇒ Switch to MP-PCR.

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# PHP in `treelike Res(LP)` and `MP-NSR`

Switching to tropical Nullstellensatz (or `treelike MP-PCR`).

## Lemma

*$S_n := x_1 \odot \ldots \odot x_n$.*
*There is a `treelike MP-PCR` derivation of $S_n \leqslant 1$ from $x_i \odot x_j \leqslant 1$.*

## Proof.

- $S_{n-1} \leqslant 1$

  $\vdash S_{n-1} \odot x_n \leqslant 1 \odot x_n$.

- $x_i \odot x_n \leqslant 1$ (for $1 \leqslant i \leqslant n-1$)

  $\vdash S_{n-1} \odot x_n^{n-1} \leqslant n-1$

  $\vdash S_{n-1} \odot x_n \leqslant 1 \odot \overline{x}_n^{n-2}$.

- Take the tropical sum $S_{n-1} \odot x_n \leqslant (1 \odot x_n) \bigoplus (1 \odot \overline{x}_n^{n-2})$
  and substitute $(1 \odot x_n) \bigoplus (1 \odot \overline{x}_n^{n-2}) \leqslant 1$ □

Now it's a treelike `MP-PCR` refutation.
Some technicalities are needed for $x \oplus \overline{x}^{n-2} \leqslant 0$ and to make the proof static.