Twin-width One

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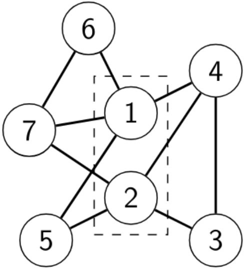
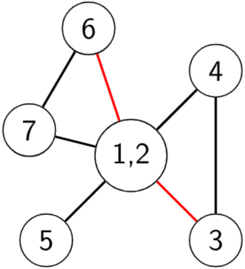
joint work with Jungho Ahn, Noleen Köhler, Christophe Paul, Amadeus Reinald, and Sebastian Wiederrecht

LIRMM

Twin-width





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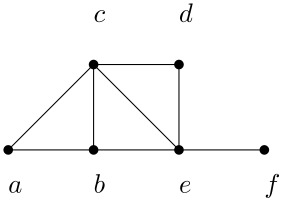


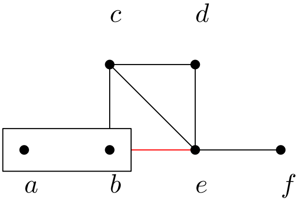
Definition

A d-contraction sequence is a sequence of trigraphs obtained by iterative contraction of pairs of vertices such that the red degree of a vertex is at most d.

Definition

The twin-width of a graph is the minimum d such that the graph admits a d-contraction sequence.

Twin-width



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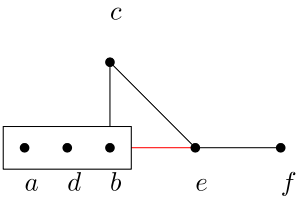

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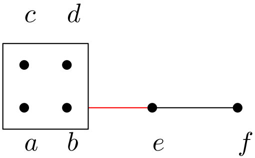

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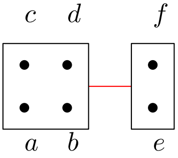

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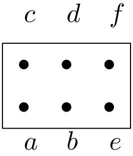

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Motivations for twin-width

Pros:

- A partial extension of sparsity theory to hereditary classes with dense graphs.
- Tractability dichotomy for FO in ordered structures.

Cons:

- Computing the twin-width exactly is NP-hard (already for $d = 4$).
- There is no known approximation algorithm

Small values of twin-width

- $d = 0$: Cographs $= P_4$-free graphs, linear time recognition.
- $d \leq 1$: Polynomial time recognition, have a linear structure.

This talk:

Theorem

Twin-width 1 *graphs can be recognized in linear time.*

Previous algorithm

Main observation:

Lemma

If a graph admits a 1-contraction sequence, it admits a 1-contraction sequence with at most one red edge per trigraph.

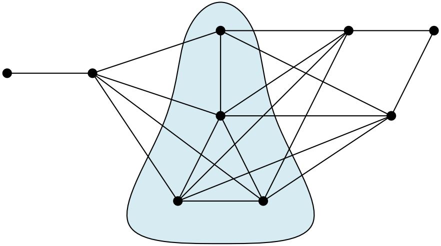
Algorithm sketch:

```
while the graph is not fully contracted:
        Guess a pair of vertices to contract.
        while there is a single red edge:
                keep contracting greedily on the red edge,
                then contract the red edge.
        if there are at least 2 red edges:
                reject.
```

Modules

Definition (Module)

A module M is a subset of vertices such that every vertex not in M is adjacent either to all vertices of M, or no vertex of M.



Modules

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A graph is prime if it has only trivial modules.

Observation

A contraction in a module creates red edges only within the module.

Definition (Splitter)

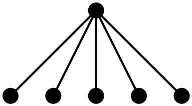
A splitter with respect to M, is a vertex v such that $vx \in E(G), vy \notin E(G)$ for $x, y \in M$. It certifies that M is not a module.

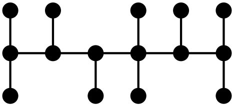
What was known

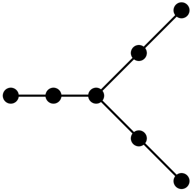
Restriction to trees

A tree has:

- twin-width 0 if and only if it is a star.
- twin-width at most 1 if and only if it is a caterpillar.
- twin-width at most 2.







Perfect graphs

Theorem (Strong perfect graph theorem)

Perfect graphs are exactly the graphs that do not contain odd holes nor odd antiholes as induced subgraphs.

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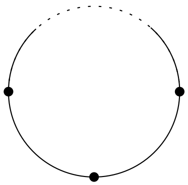

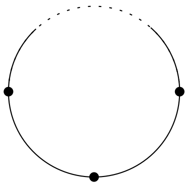

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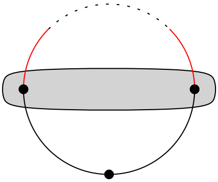

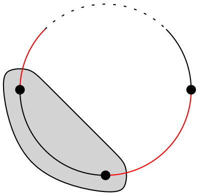

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Observation

Holes and antiholes have twin-width 2.

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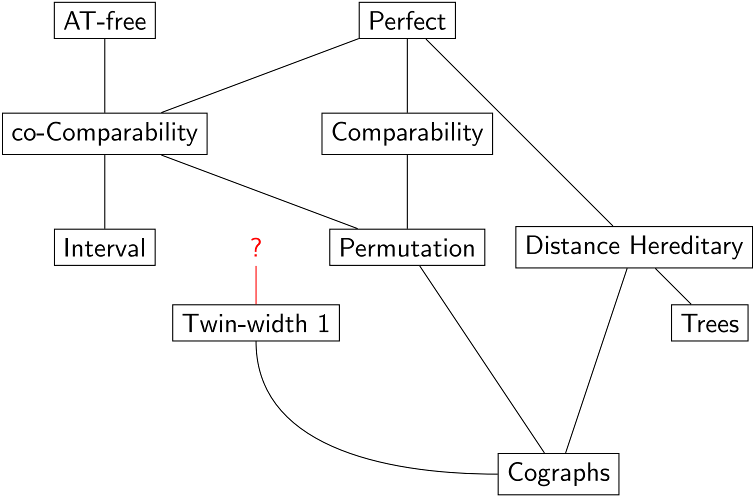
Observation

Holes and antiholes have twin-width 2.
In particular, twin-width 1 graphs are perfect.

Classes of perfect graphs



Definition

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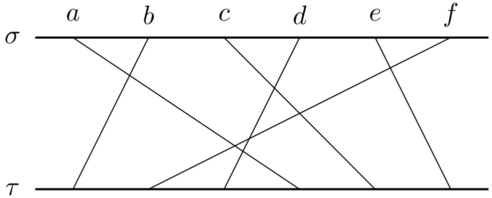
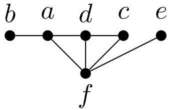
Given a permutation $\pi : [n] \to [n]$, we define the graph $G[\pi] = ([n], E_\pi)$ where E_π is the set of inversions of π (i.e. $xy \in E_\pi$ iff $x < y$ and $\pi(x) > \pi(y)$).



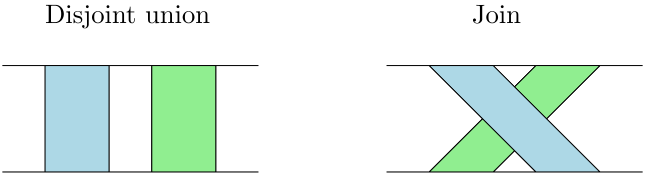


Warm up: Cographs

Lemma

Cographs are permutation graphs.

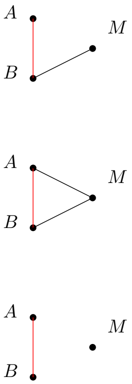
Induction on the cotree with two cases:



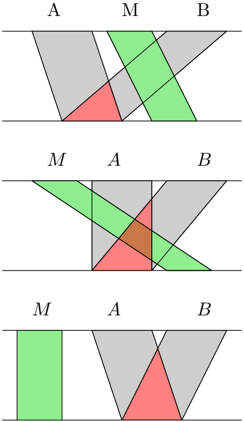
Twin-width 1 graphs are permutation graphs

Induction on the contraction sequence:

3-vertex trigraph



Permutation diagram



Twin-width 1 graphs are permutation graphs

Vertices contracted together are consecutive in the diagram.

Lemma

If G admits a 1-contraction sequence $G_n, \ldots, G_1$, then there exists a realiser (σ, τ) such that:

- *if x is a vertex of G_i, then vertices of G that are in x form an interval of σ or τ.*
- *if xy is a red edge of G_i, then vertices of G that are in xy form an interval of σ or τ*

Twin-width 1 graphs are permutation graphs

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- *if xy is a red edge of G_i, then vertices of G that are in xy form an interval of σ or τ*

Corollary

If the graph is prime, an extremal vertex of the diagram can be contracted last. There are only 4 vertices to try!

Exploiting arbitrary realisers

- For prime graphs, the realiser is unique (up to symmetry) so all realisers have the consecutivity property.
- For cographs, we can always find a pair of consecutive twins.

Corollary

For any realiser (σ, τ) of a twin-width 1 graph, there exists a 1-contraction sequence that contracts only vertices consecutive in σ or τ.

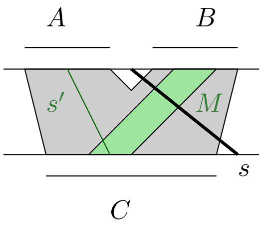
We do not need to find a specific permutation diagram!

High level description

Algorithm sketch:

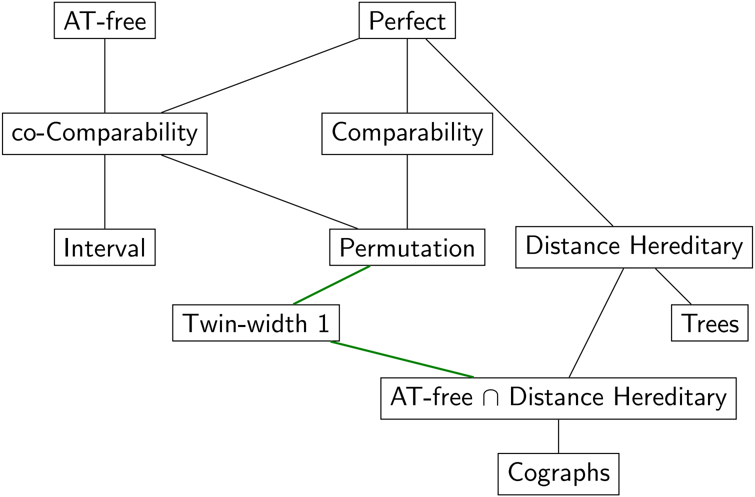
```
Compute permutation diagram, reject when not permutation graph
Compute modular decomposition
for each prime node of the decomposition:
        Guess last vertex s to be contracted
        Try to decompose recursively using s as splitter
```

Prime case description




```
while an endpoint of C is also an endpoint of A or B:
        Remove it and move indices correspondingly
if |A| > 1 and |B| > 1: Reject
else if |A| > 1 or |B| > 1:
        Deduce (M,s') from (A,B)
        Recurse on (M,s') with s' as splitter
else: Accept
```

Our results


AT-free
Perfect
co-Comparability
Comparability
Interval
Permutation
Distance Hereditary
Twin-width 1
Trees
AT-free ∩ Distance Hereditary
Cographs

Going further

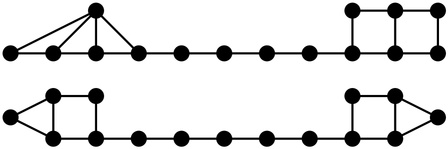
Our results:

- Linear time recognition algorithm for twin-width 1
- Permutation diagram and consecutivity properties

Open questions:

- Can we recognize twin-width 2 graphs in polynomial time?
- Is recognition of twin-width 3 NP-hard?
- Can we approximate twin-width in XP time?
- Can we compute the twin-width of permutation graphs in polynomial time?

Some obstructions to twin-width 1



These obstructions correspond to an infinite family of forbidden patterns for permutations of twin-width 1.

Cographs and modular decompositions

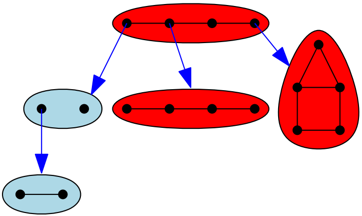
Cographs can be characterised in two ways:

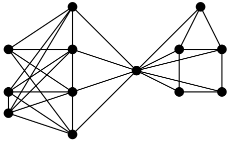
- A cograph is a single vertex graph or a graph G obtained from two cographs H_1 and H_2 by either the disjoint union of H_1 and H_2 or adding all edges between H_1 and H_2 (join).
- A cograph is a graph that admits a twin-elimination ordering.

Both characterisations follow from a top-down or bottom-up description of the cotree of the cograph

Modular decomposition

For every graph, its modular decomposition is a canonical object that recursively decomposes it into prime graphs, disjoint unions, and joins.





Modular decomposition and contraction sequence

Lemma

The twin-width of a graph is the maximum twin-width over the prime nodes of its modular decomposition.

Construct the contraction sequence by following the modular decomposition bottom-up. We iteratively contract twins, and contract prime nodes using their optimal sequence.