Two-Dimensional Longest Common Extension Queries in Compact Space

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Text Indexing and Basic Queries

Index a text $T[1...n]$ over an alphabet $[\sigma]$ to support the pattern matching queries

Query: Pattern $P[1...m]$
Output: Set of occurrences of P in T

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	1	2	3	4	5	6	7	8	9	10	11	12
$T[1...n]$	G	A	**C**	**A**	**T**	C	T	**C**	**A**	**T**	T	G

Set of occurrences of P = **CAT** in the above text is {3,8}

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FFT based algorithm
Rabin Karp etc

Time is at least O(n), i.e., linear in text length

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DATA STRUCTURES (Indexes)
Suffix Trees
Suffix Arrays
Their compressed versions (FM-index, CSA, r-index, etc)

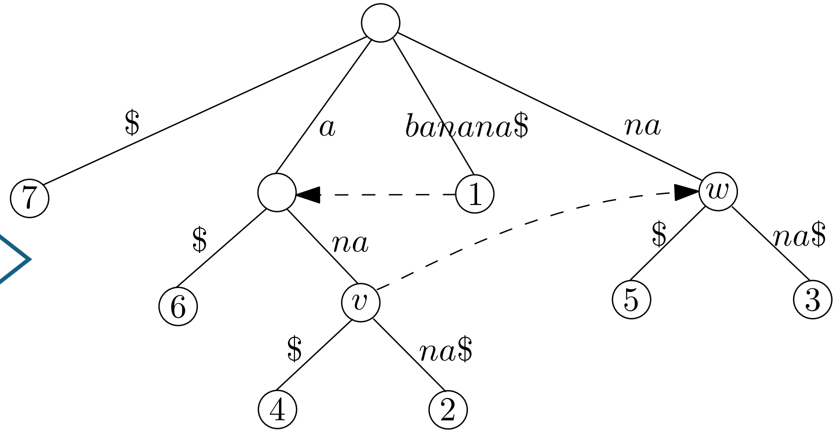
Query time is linear in "m"and output size

Suffix Trees and Suffix Arrays (text = banana$)

1 banana$
2 anana$
3 nana$
4 ana$
5 na$
6 a$
7 $

SORT

7 $
6 a$
4 ana$
2 anana$
1 banana$
5 na$
3 nana$



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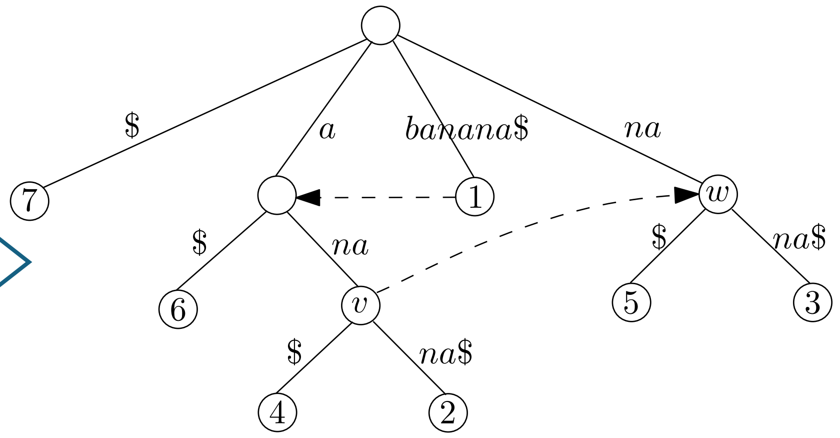

	1	2	3	4	5	6	7
SUFFIX ARRAY	7	6	4	2	1	5	3
Inverse SUFFIX ARRAY	5	4	7	3	6	2	1

$LCE(2,4) = 3$

$LCE(3,5) = 2$

$LCE(2,5) = 0$

Longest Common Extension $LCE(i, j)$ *is length of the longest common prefix of suffixes starting at i and j*

Suffix Trees and Suffix Arrays

Good news: O(1) time for all 3 operations

- **suffix array**
- **inverse suffix array**
- **LCE operation**

Bad news: $O(n \log n)$ **bits space,**

while data needs only $n \log \sigma$ **bits**

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Can we encode suffix arrays/trees in space close to text's space ~$n \log \sigma$ **bits?**

Can we encode suffix arrays/trees in space close to text's space $\sim n \log \sigma$ bits?

YES !!! ... (1D case) is well solved...

Compressed Suffix Arrays (CSA) [Grossi and Vitter, 2000]
FM-index [Ferragina and Manzini, 2000]
Compressed Suffix Trees [Sadakane, SODA 2004]
.....
....
....
r-index [Gagie, Prezza, Navarro, SODA 2018]
Suffix array in delta-compressed space [Kempa Kociumaka, FOCS 2023]

2D texts & 2D suffix Arrays & Trees

Index a 2D text for matching SQUARE patterns

T	0	1	2	3	4
0	a	a	b	b	\$
1	a	b	b	c	\$
2	b	b	a	a	\$
3	b	c	a	b	\$
4	\$	\$	\$	\$	\$

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3	b	c	a	b	\$
4	\$	\$	\$	\$	\$

Define L-suffixes L(2,1) = b aac ab$$$...

If an mxm square pattern occurs at a positions,
the its L-suffix is a prefix of the (corresponding) L-suffix of the text

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Suffix Tree: Compacted Trie over L-suffixes

$O(N \log N)$ bits space ($N = n \times n$; matrix size)

O(1) for all 3 operations (SA/ISA/LCE)

Efficient (square) pattern matching

LCE((i,j), (i',j')) = m if mxm is the largest square matrix matching ...

2D texts & 2D suffix Arrays & Trees

Space Efficient Encoding for 2D strings?

$N \log \sigma$ **or even any** $o(N \log N)$ **bits?**

Space Efficient Index for 2D strings

T	0	1	2	3	4
0	a	a	b	b	$\$$
1	a	b	b	c	$\$$
2	b	b	a	a	$\$$
3	b	c	a	b	$\$$
4	$\$$	$\$$	$\$$	$\$$	$\$$

Theorem: $O(N \log \sigma)$-bit index for LCE in ~ $O(\log^{2/3} n)$ time

LCE((0,2)&(2,0)) = 2

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LCE((1,2)&(2,0)) =1

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Corollaries

$O(N \log \sigma + N \log \log N)$

bits index for

ISA queries via O(log n)* LCE queries
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2D LCE

Give two positions in the matrix,
find the largest common square sub-matrix LxL
whose top-left-corner align to those positions

- Linearize the 2D text into 1D text(s)
- Reduce the 2D-LCE query into logarithmic number of 1D-LCE's

- 1D LCE can be answered in O(1) time using an $O(n \log \sigma)$ bit structure
- 1D LCE can be answered in O(t) time using O(n/t) words
  [& text in read only], t is any parameter.
- **Difference Cover ......**

2D LCE - A simple O(L) time solution

T	0	1	2	3	4
0	a	a	b	b	\$
1	a	b	b	c	\$
2	b	b	a	a	\$
3	b	c	a	b	\$
4	\$	\$	\$	\$	\$

Concatenate all ROWS into a single 1D text,
make a compact space (1D) LCE structure over it

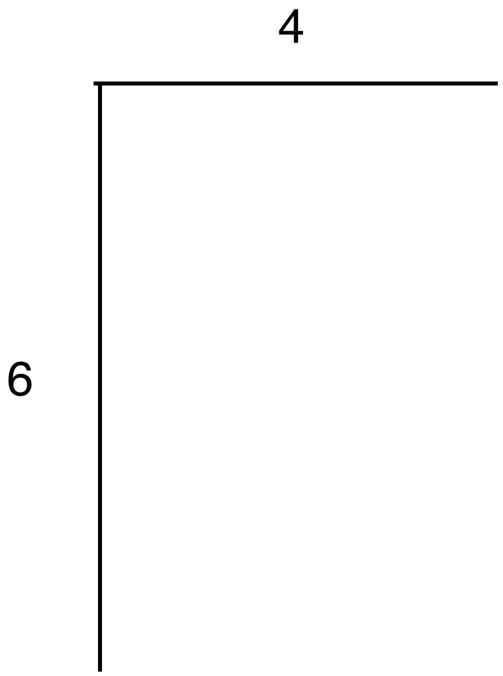
Similarly, concatenate all COLUMNS into a single 1D text,
make a compact space (1D) LCE structure over it

Maintain 1D LCE structure with space $O(n \log \sigma)$ bits and time O(1)

2D LCE - A simple O(L) time solution

Given two positions (a,b) and (c,d), initialize i = a, j = b, k = c, l = d

- *Compute LCE of R_i[j...] & R_k[l...] and LCE of C_j[i..]&C_l[k..]*
- *increment all 4 values are repeat (keep guessing L until we get it)*



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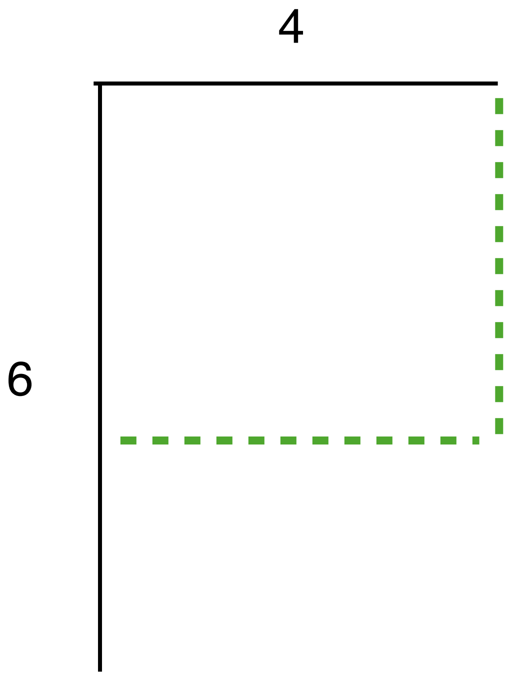
$L \in [1,4]$



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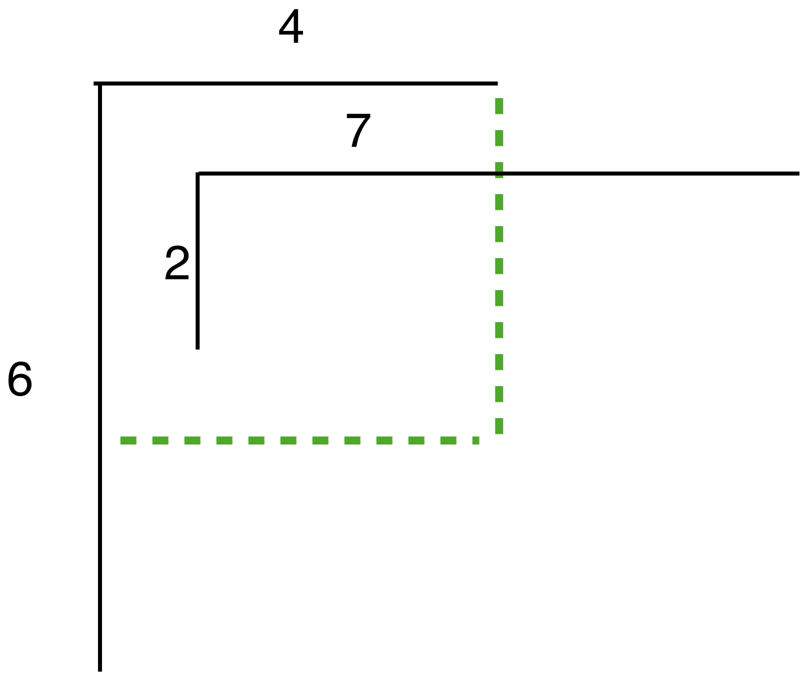

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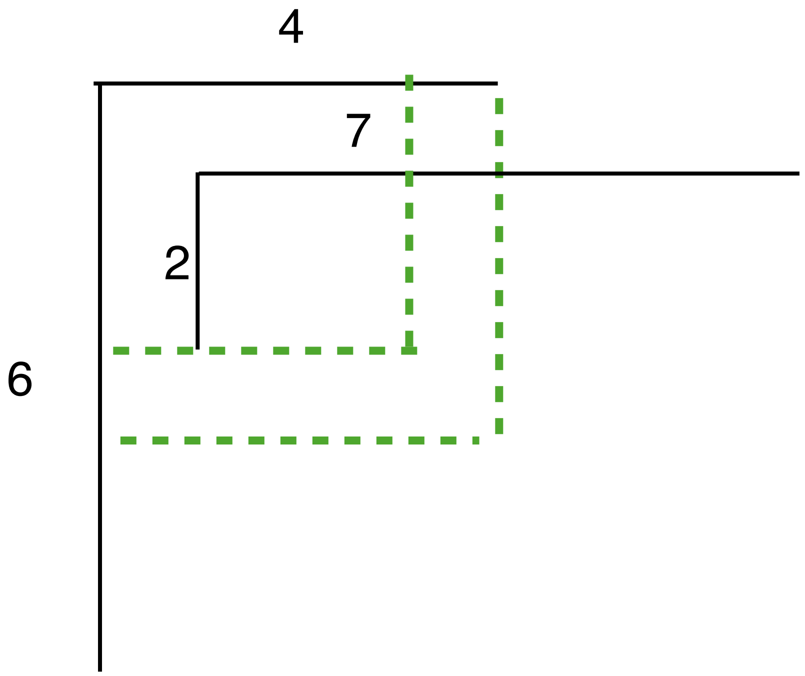

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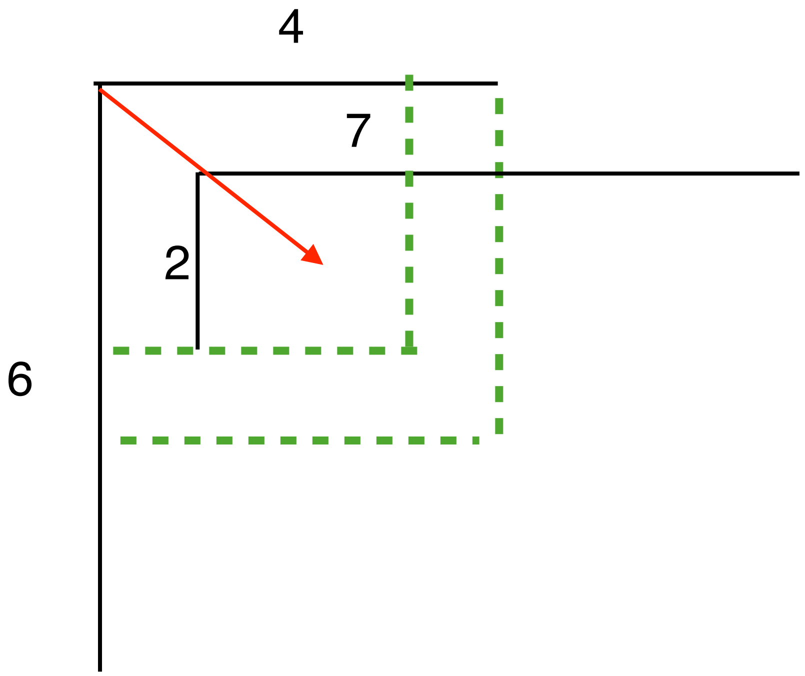

Continue .... O(L) steps and O(L) time

2D LCE - $O(\log^2 n)$ time solution

2D LCE - $O(\log_\sigma^2 n)$ time solution

Difference Cover (DC)

There exists an $S \subseteq \{1,2,3,...,n\}$ of size $O(n/\sqrt{d})$ such that $\forall(i,j), \exists h \in [0,d)$ such that $i+h, j+h \in s$

Solution: Fix $d = \log_\sigma^2 n$, Make a sparse (2D) suffix tree of sampled L-suffixes @DC across diagonals

Space = $O((n/\sqrt{d})\log n) = O(n \log \sigma)$ bits

Given two positions (i,j) and (i',j') for LCE, we know that there exists an $h < d$, such that
LCE of (i+h,j+h) & (i'+h,j'+h) are sampled positions are their LCE can be obtained in O(1) time

This means, we can run 1D-LCE queries previous algorithm just O(d) time and then jump to sparse ST

Time complexity: $O(d) = O(\log_\sigma^2 n)$

2D LCE in $\sim O(\log_\sigma n)$ time solution

Key Ideas:

Instead of taking one row/column at a time, we make slabs of sizes 1,2,4,8,... log n

However, we cannot maintain the corresponding 1D texts explicitly. So, we maintain the implicitly, and the following 1D LCE structure explicitly (we adjust "t" to keep the space low):

LCE can be answered in O(t) time using O(n/t) words [& text as some compressed structures]

The number of queries will be $O(\log \log n)$, but each query is now costly $O(\log_\sigma n)$

2D LCE in ~ $O(\log_\sigma^{2/3} n)$ time solution

Combining both Ideas:

We will maintain O(1) time 1D LCE structures for all rows and columns

Then LCE structures for selected slabs (optimized to keep the space/time small)

The final answer is obtained 3 queries

~ $O(\log_\sigma^{2/3} n)$ number of O(1) 1D-LCE queries on rows/columns
~ $O(\log\log_\sigma n)$ number of 1D-LCE queries on slabs (each costing $O(\log_\sigma^{2/3} n)$ time)
~ a single 2D LCE query on the sparse suffix tree

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We also have a REPETITION AWARE structure for 2D LCE queries

Thanks for your Listening!!!