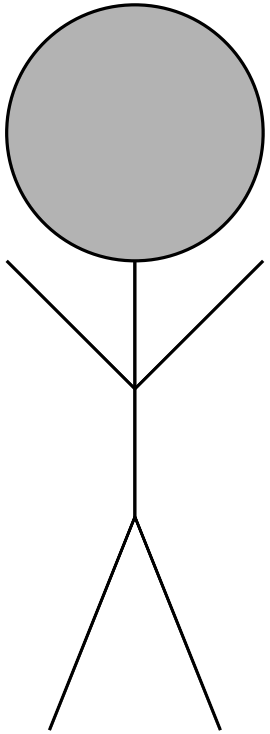
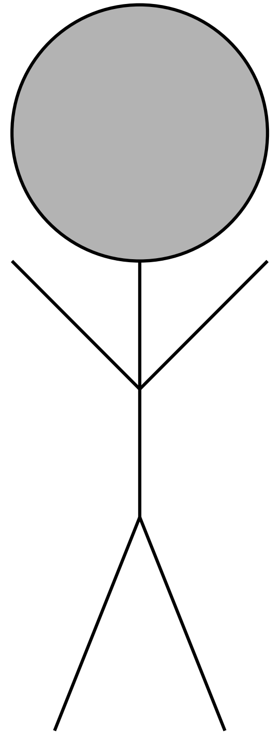
Unfairly Splitting Separable Necklaces

Patrick Schnider, Linus Stalder, and Simon Weber

The Ham Sandwich Theorem





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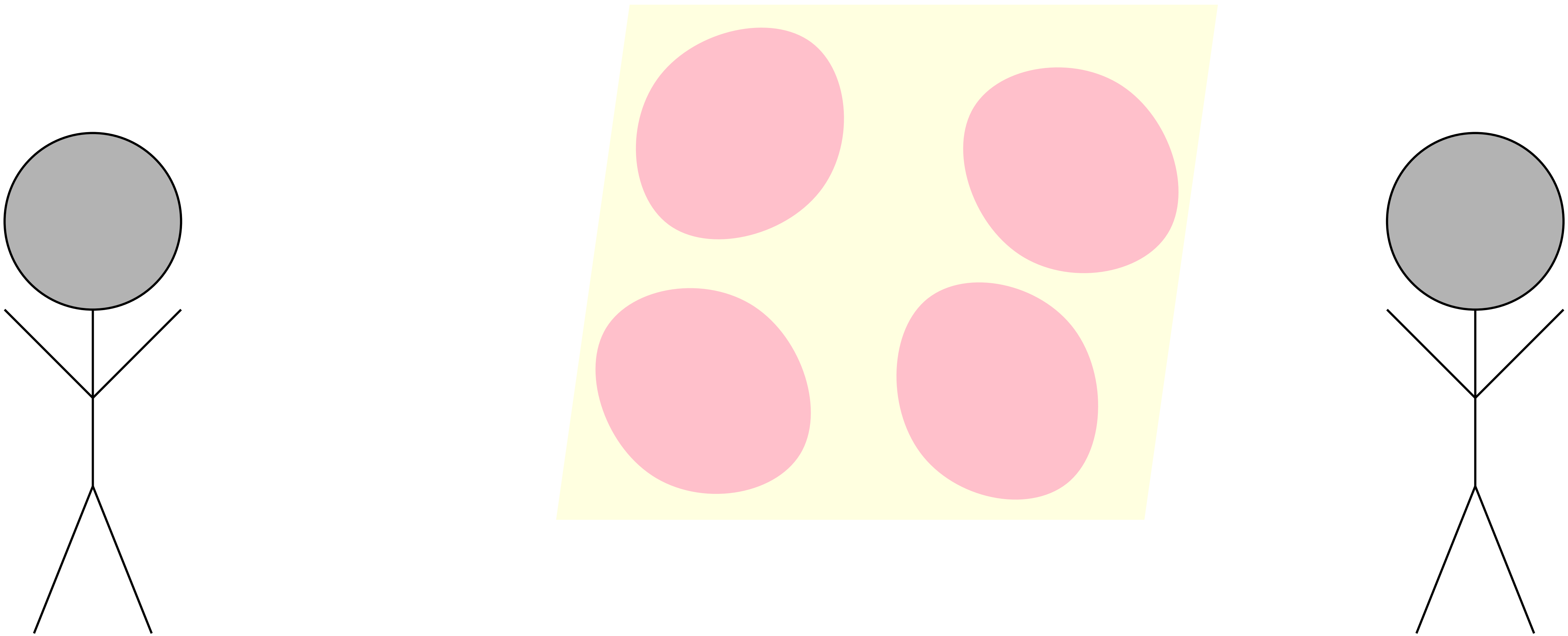

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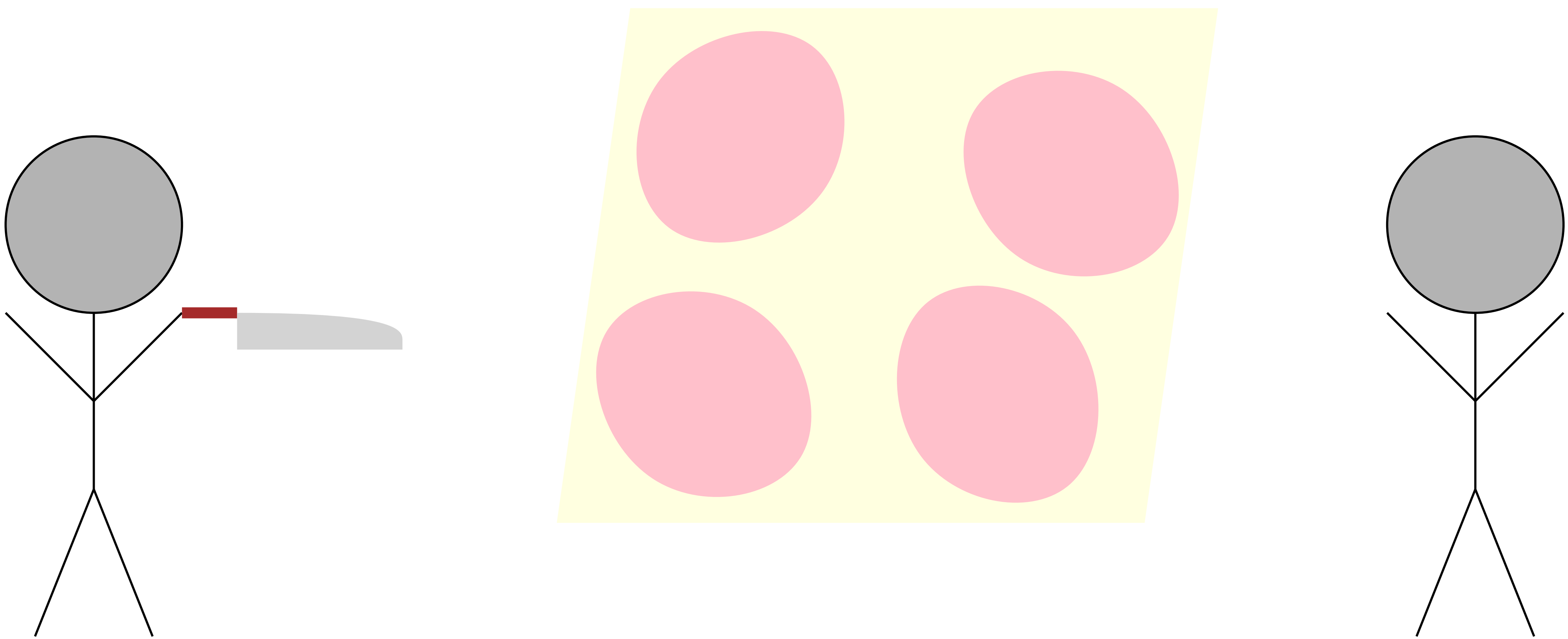

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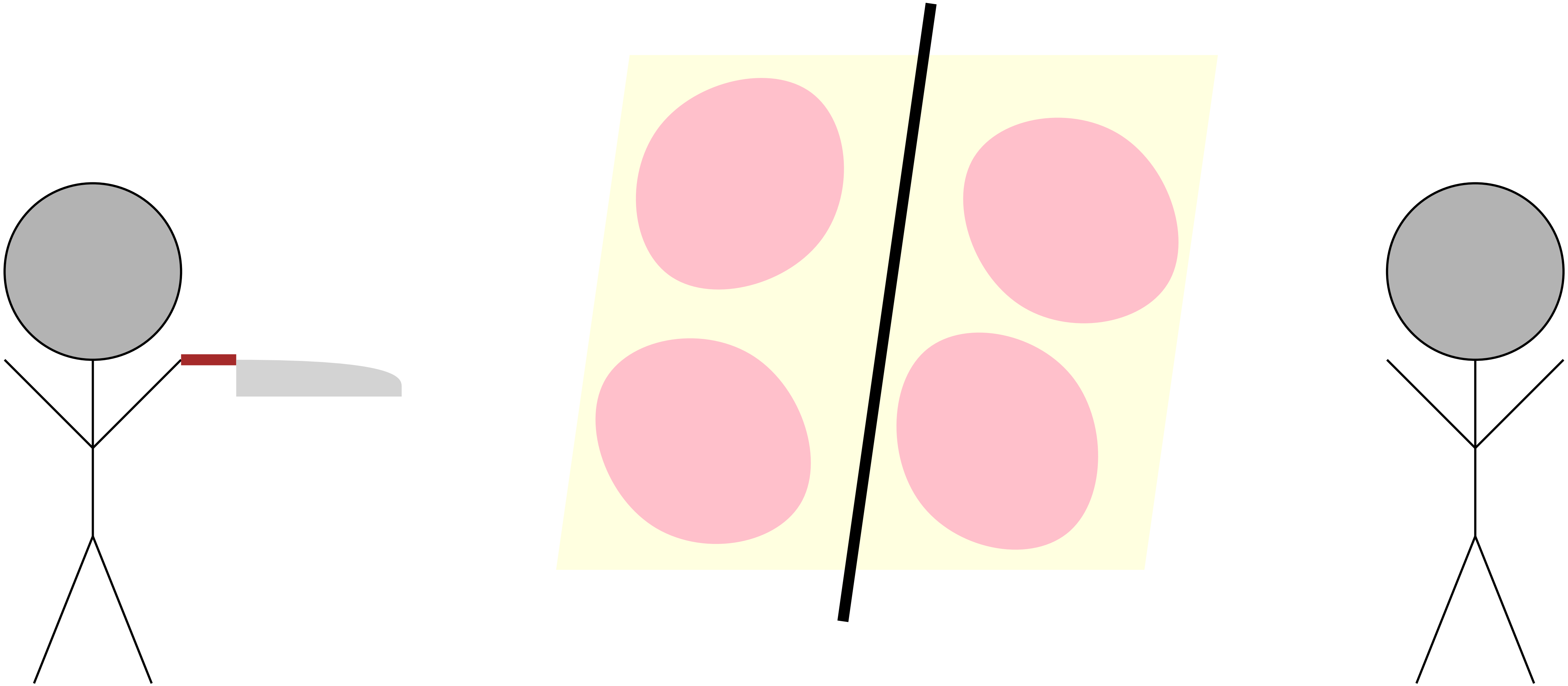

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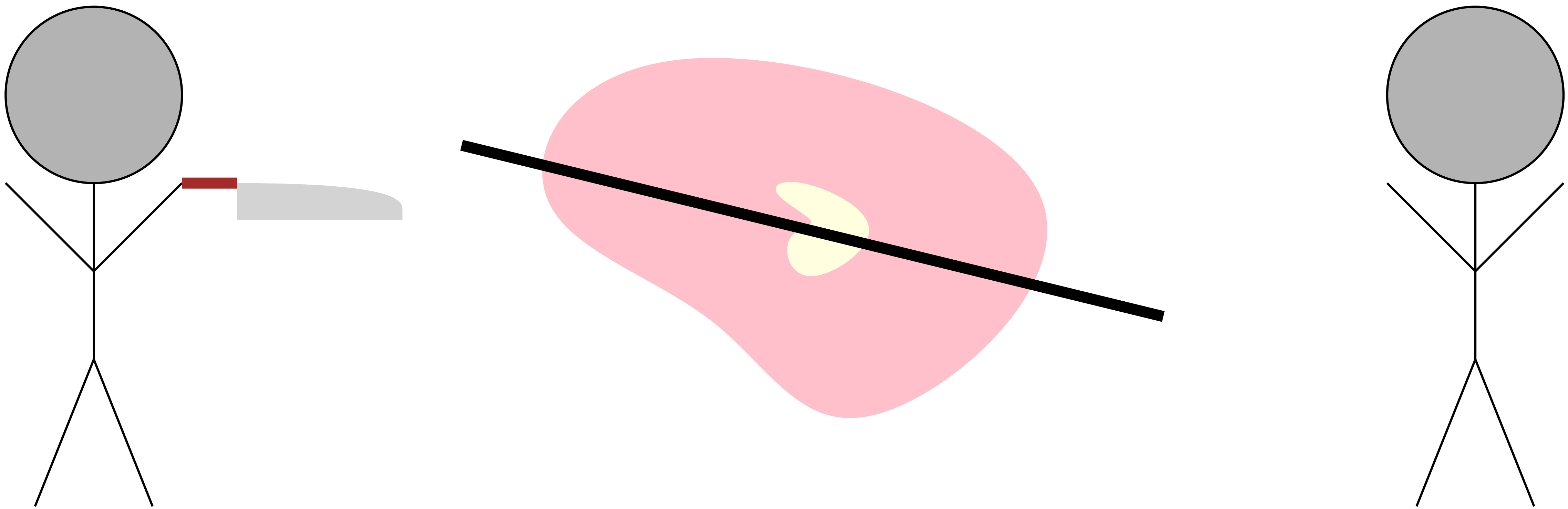
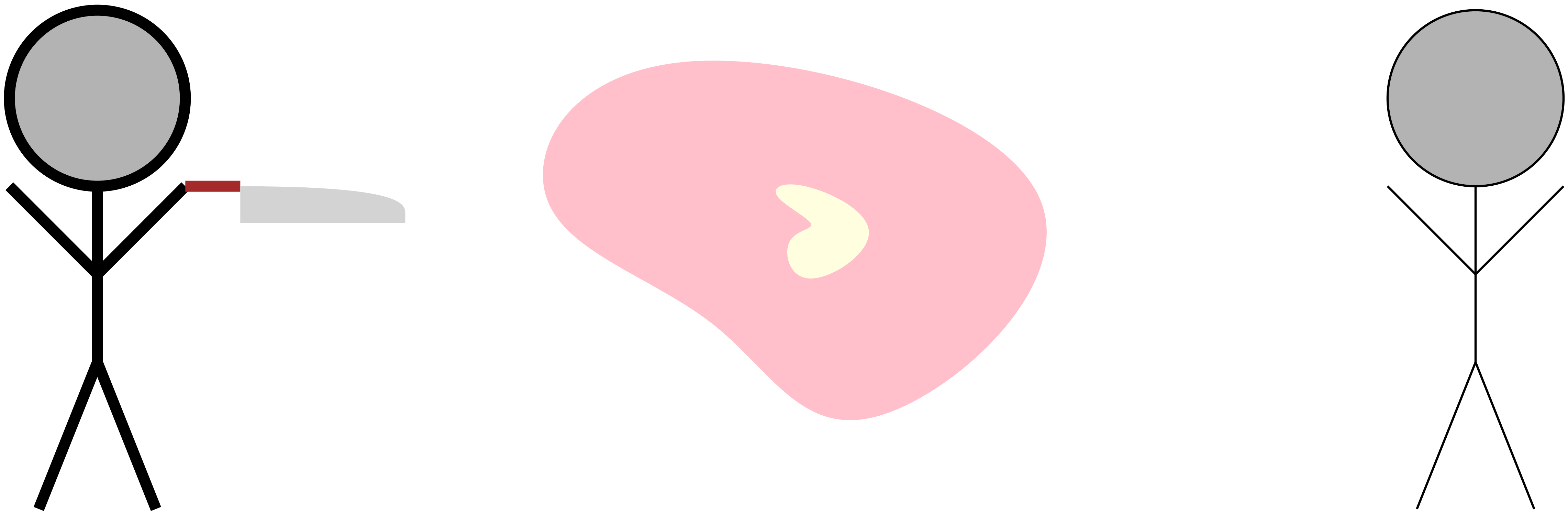


The α-Ham Sandwich Theorem



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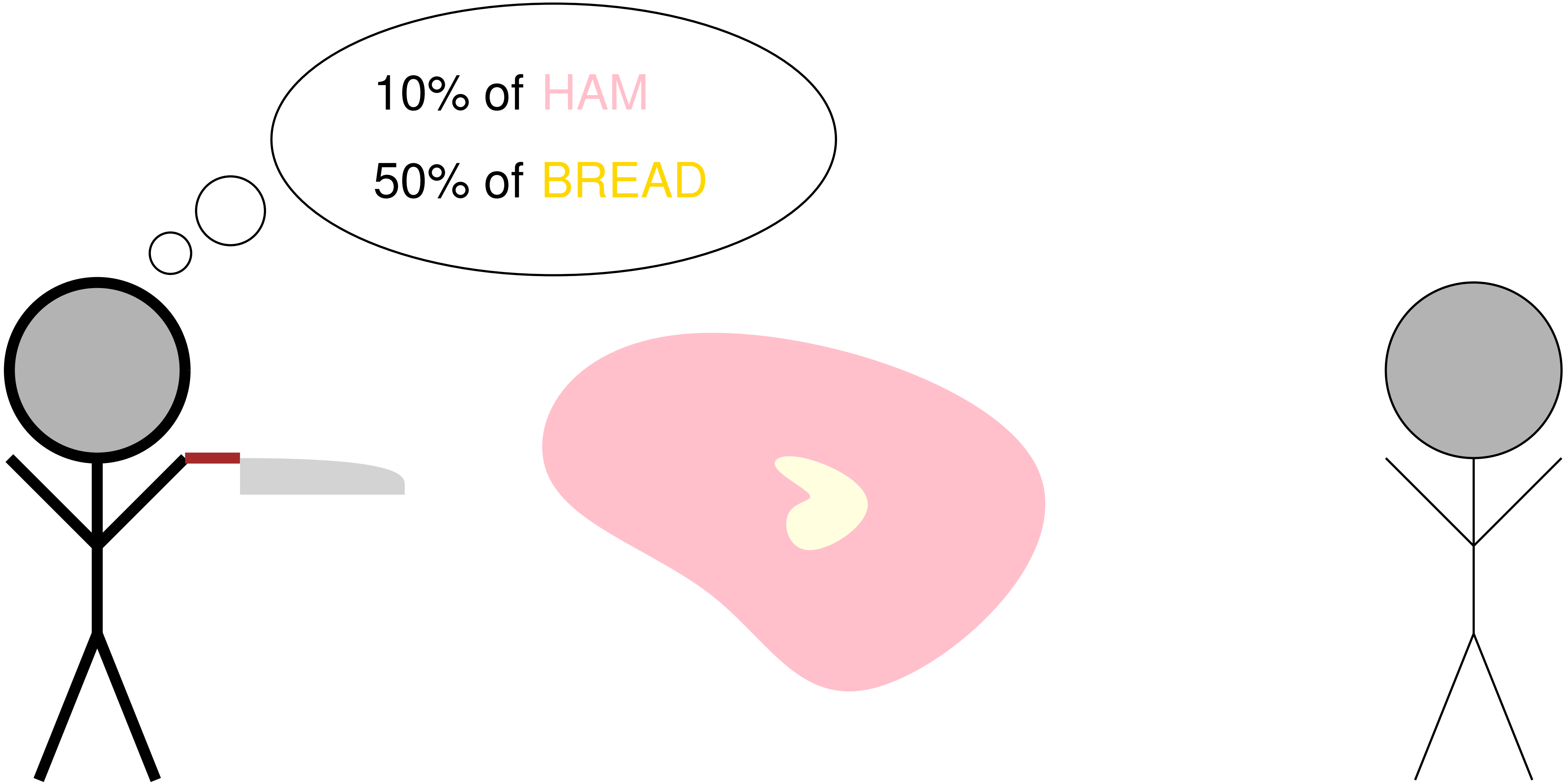

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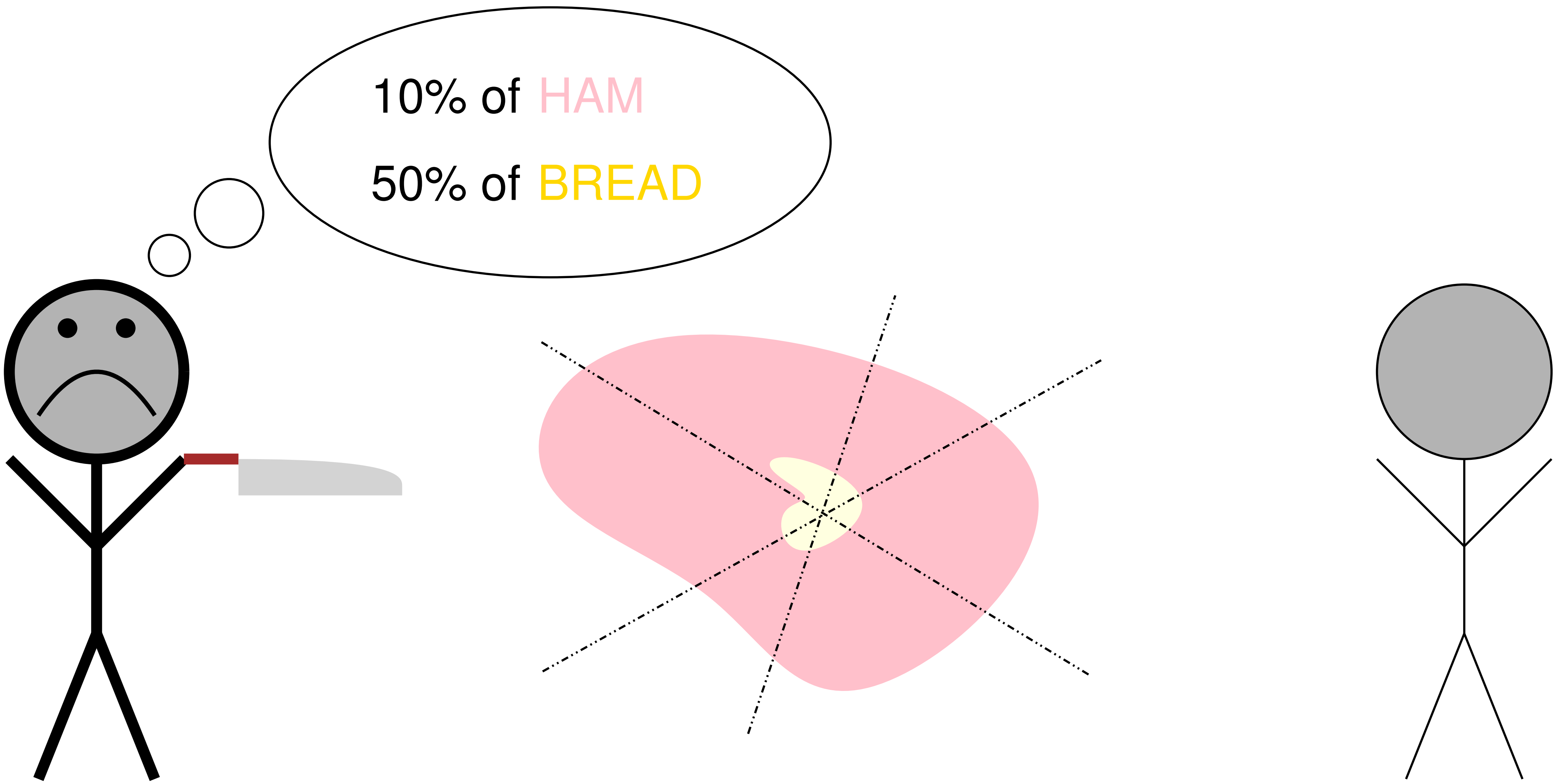

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Given d point sets $P_1, \ldots, P_d \subset \mathbb{R}^d$ that are *well-separated* and in *weak general position*, and any integers $\alpha_1, \ldots, \alpha_d$ for $0 < \alpha_i \leq |P_i|$, there exists a *unique* hyperplane h such that h goes through exactly one point of each P_i, and $|h^+ \cap P_i| = \alpha_i$.

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Well-Separation

Definition: k point sets $P_1, \dots, P_k \subset \mathbb{R}^d$ are called *well-separated* if for every non-empty index set $I \subset [k]$, the convex hulls of the two disjoint subfamilies $\bigcup_{i \in I} P_i$ and $\bigcup_{i \in [k] \setminus I} P_i$ can be separated by a hyperplane.

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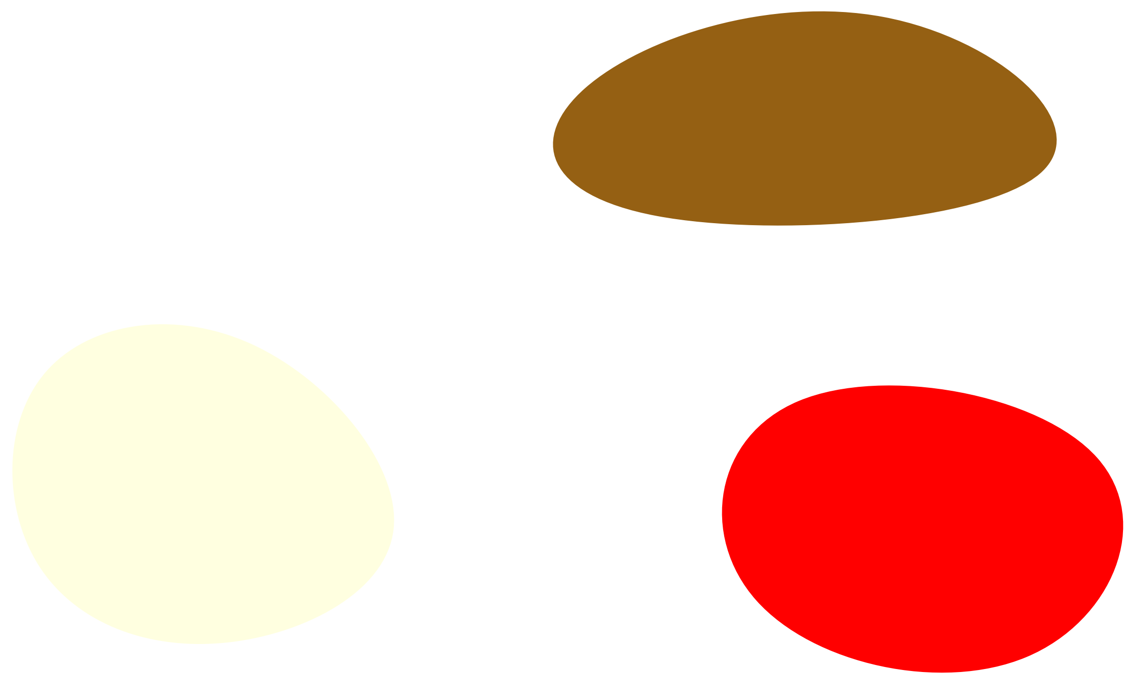

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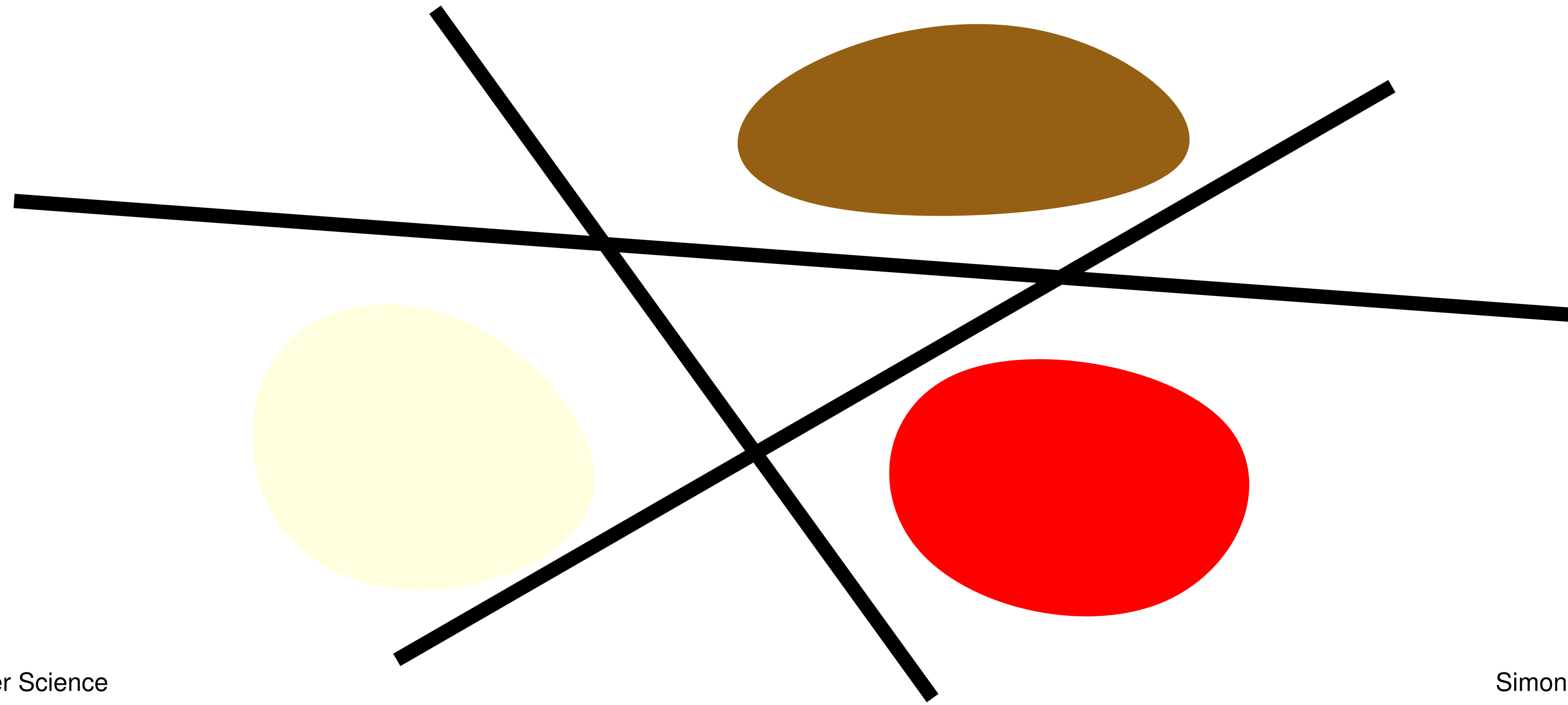

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UEOPL $\subseteq$ PPA, conjectured $\neq$!

Is α-Ham Sandwich also UEOPL-hard?

PPA-hardness of Ham Sandwich: Necklace Splitting

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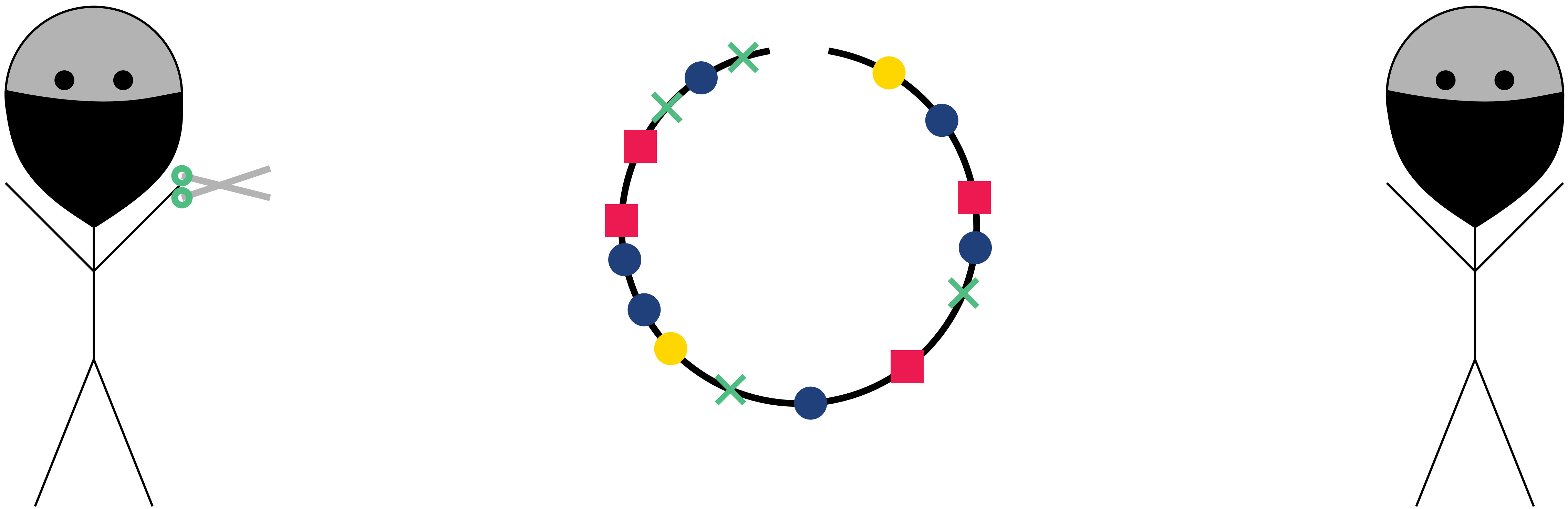

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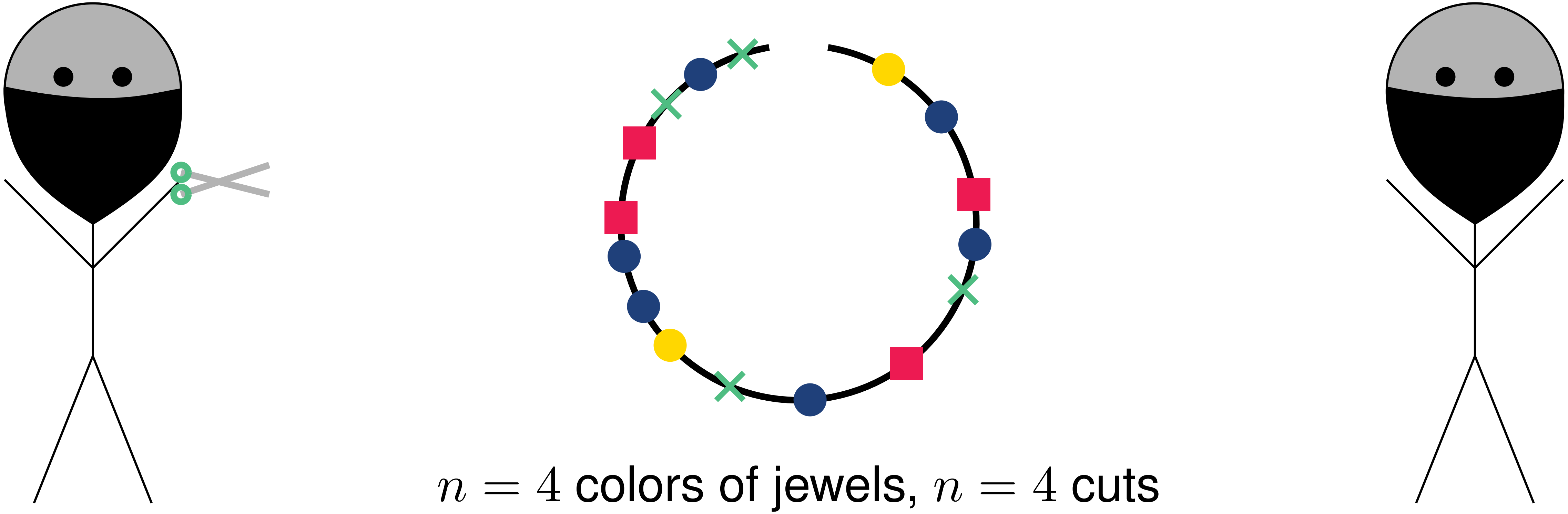

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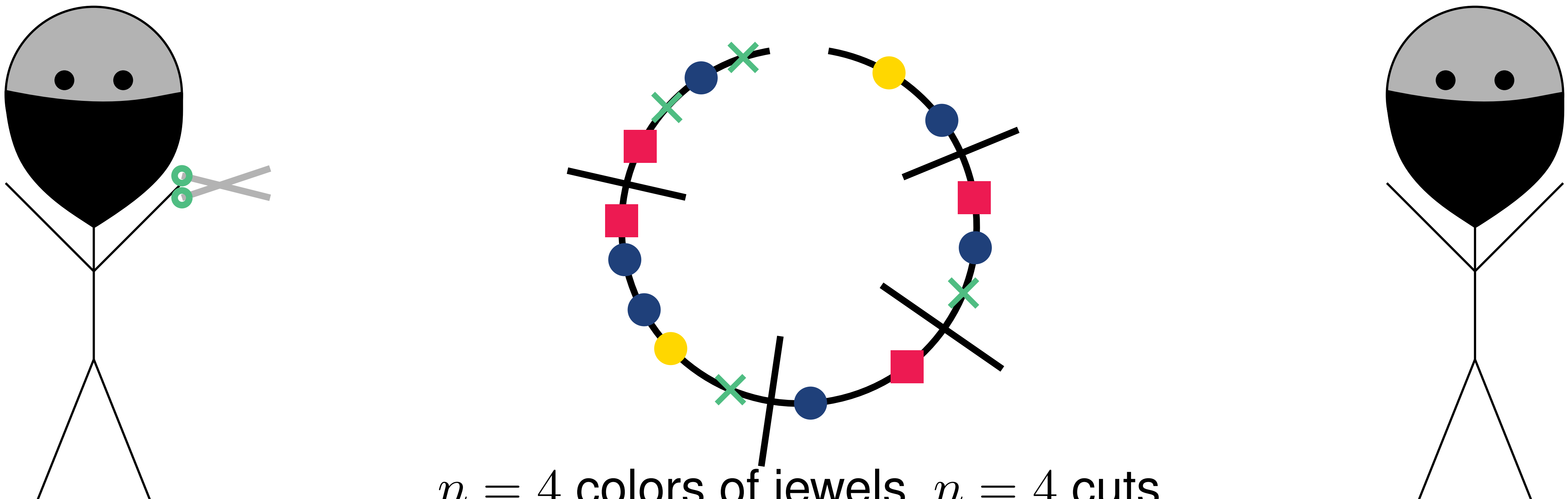

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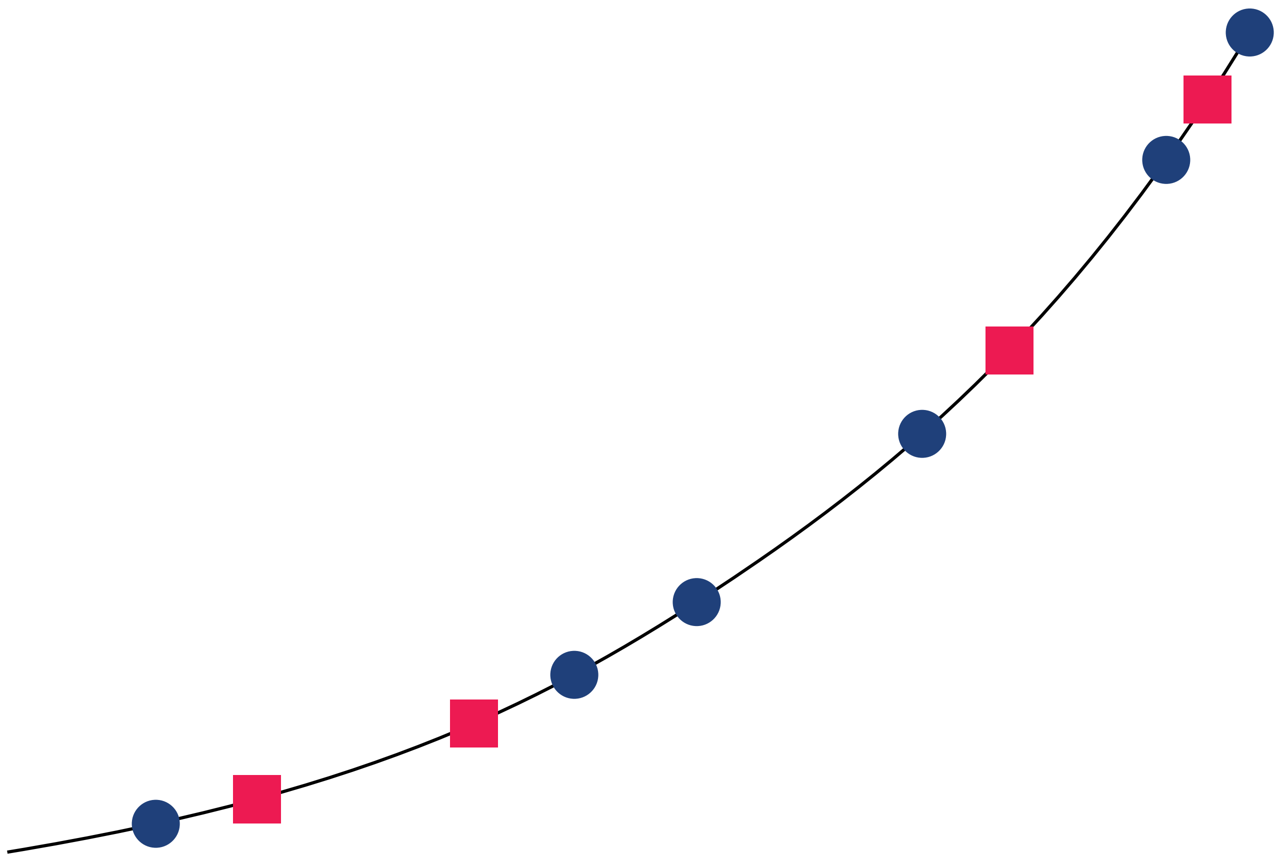

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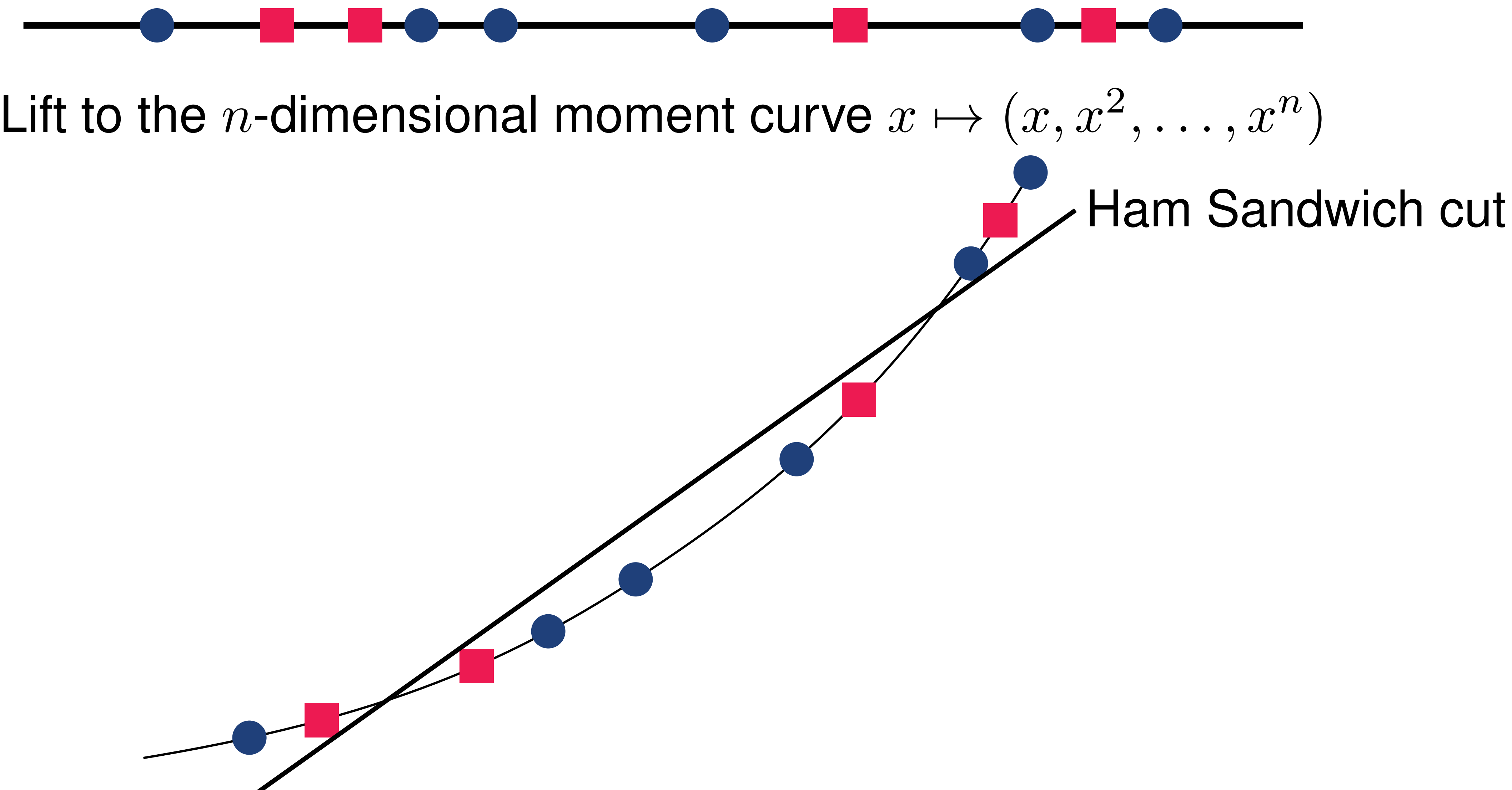

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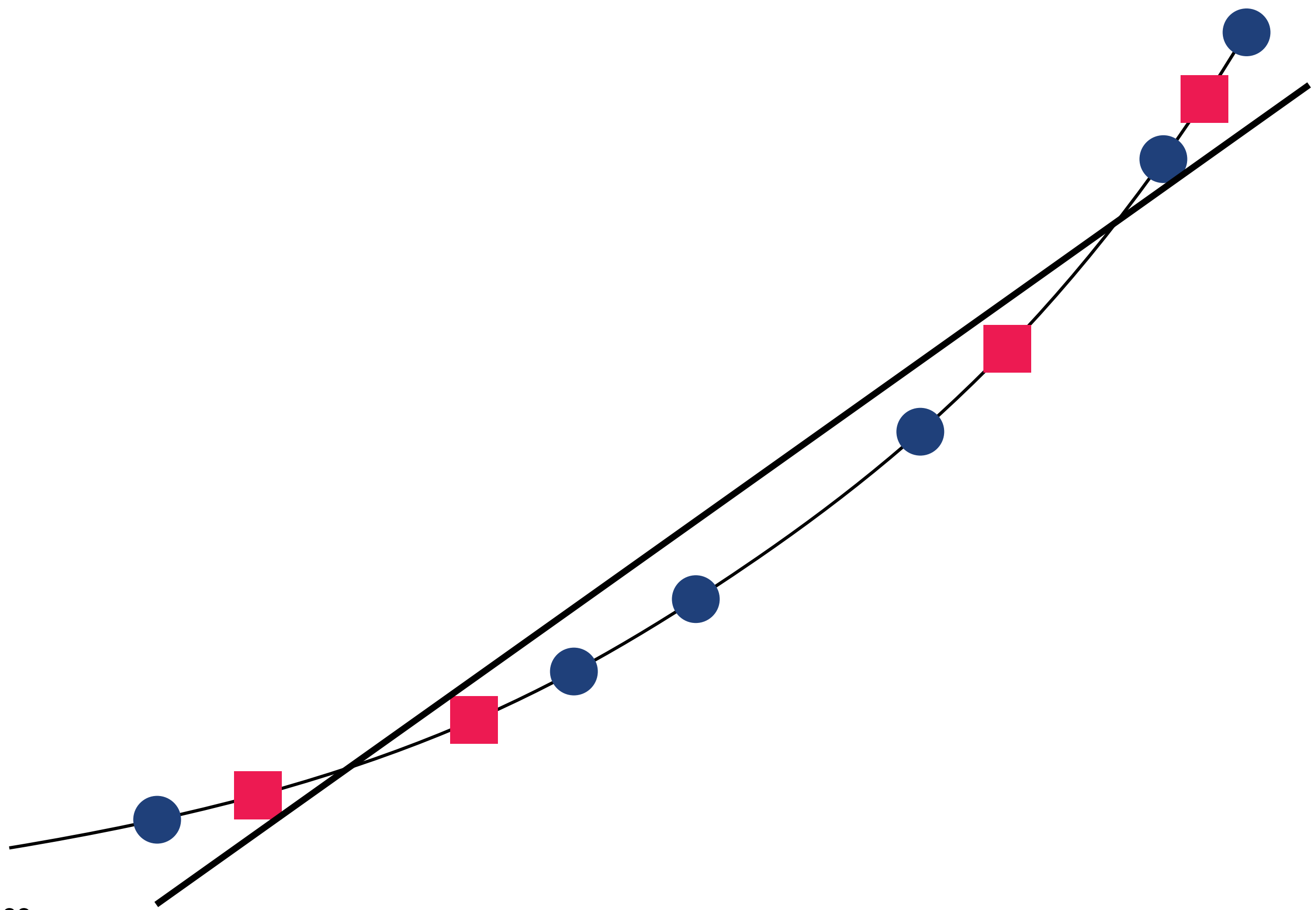

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Is Necklace Splitting on n-separable necklaces UEOPL-hard?

Previous Results

Theorem [BSW, '23]: Fair Necklace Splitting can be solved in time $2^{O(\ell \log \ell)} + O(m^2)$ on every $(n - 1 + \ell)$-separable necklace with n colors of jewels and m total jewels.

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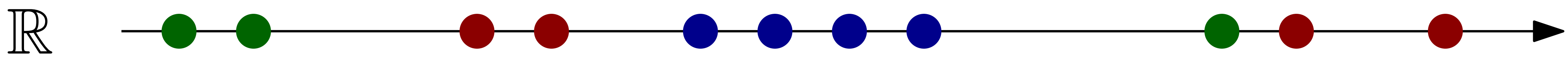
What about *unfair* splitting?

The Unfair Splitting Problem

$$\alpha = \begin{pmatrix} 4 & 3 & 1 \end{pmatrix}$$



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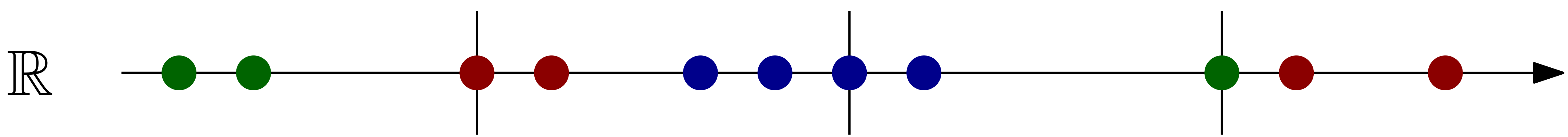

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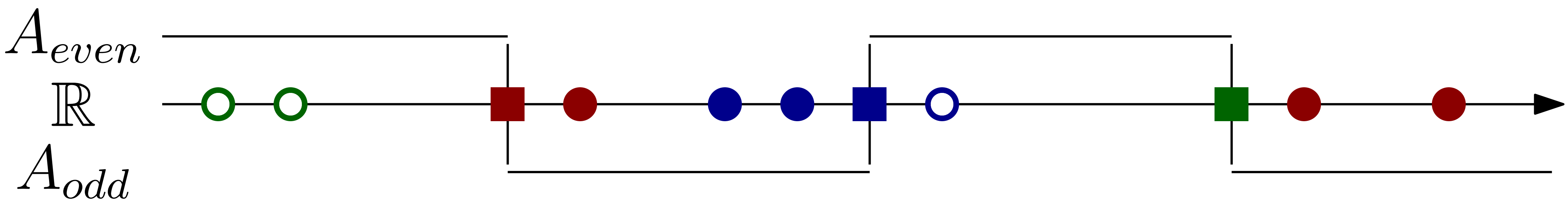

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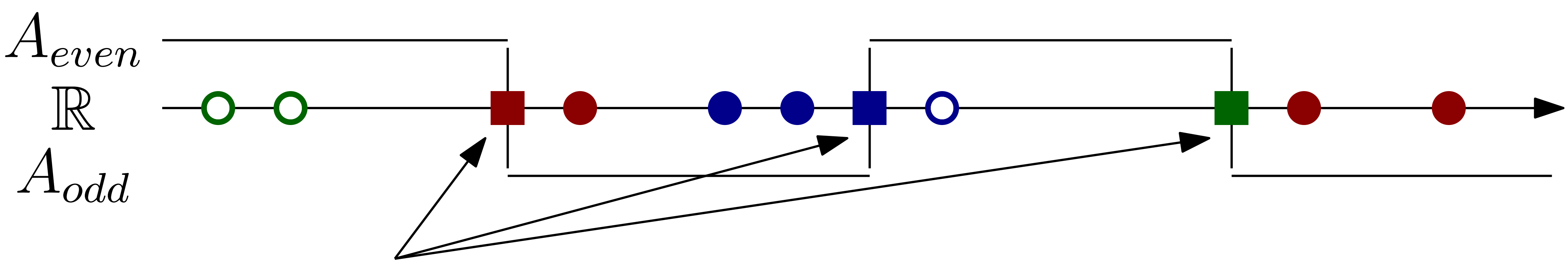

Permutation of cut point colors decides parity

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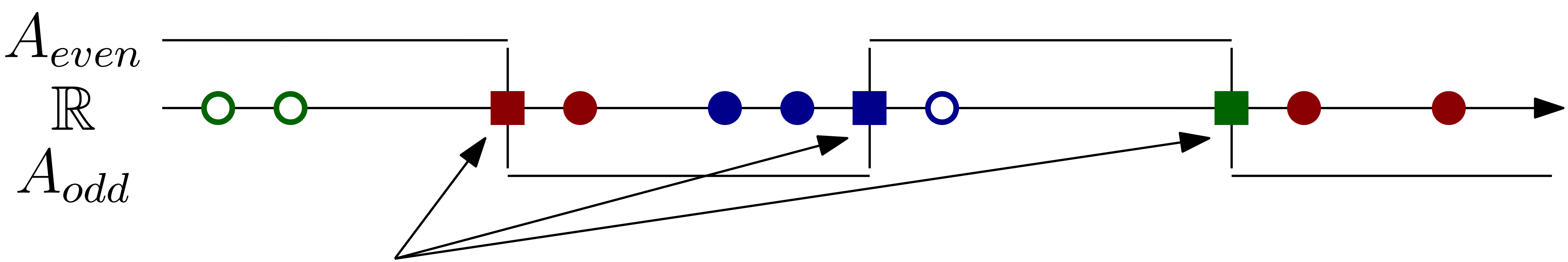

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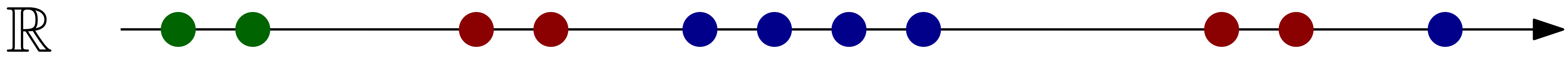
Theorem: Unfair Necklace Splitting can be solved in time $O(n \cdot m)$ on every n-separable necklace with n colors of jewels and m total jewels.

Solving Unfair Splitting



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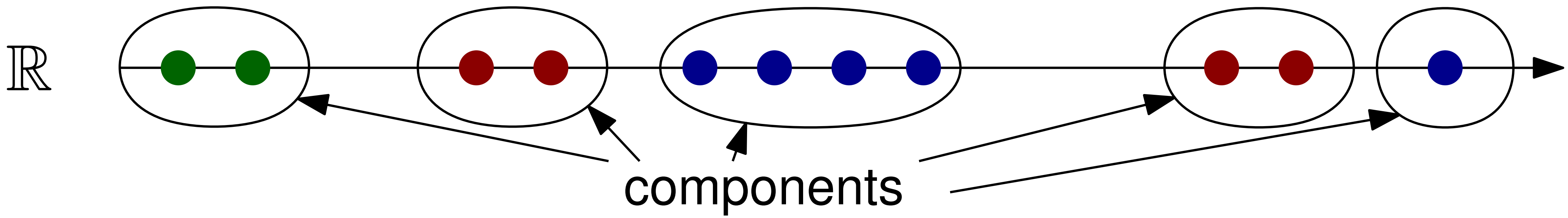

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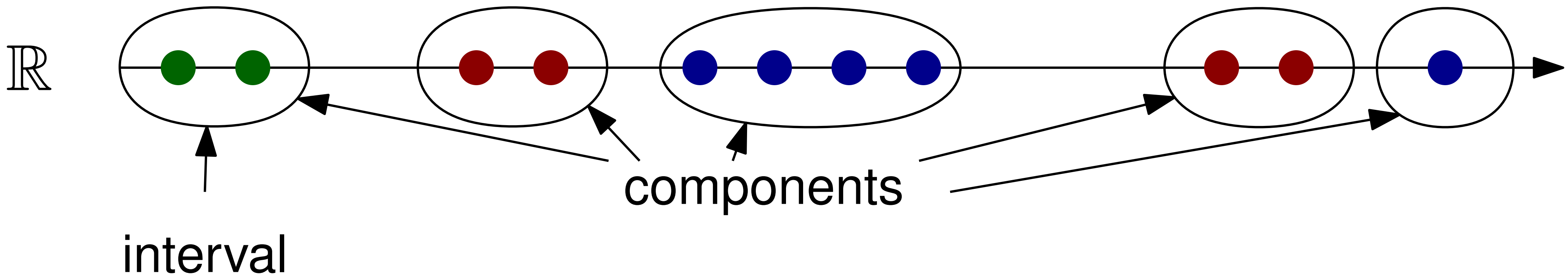

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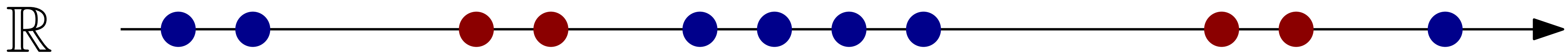

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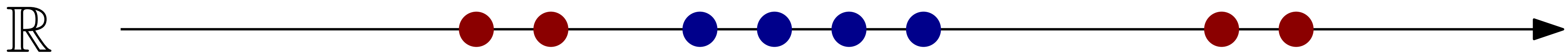

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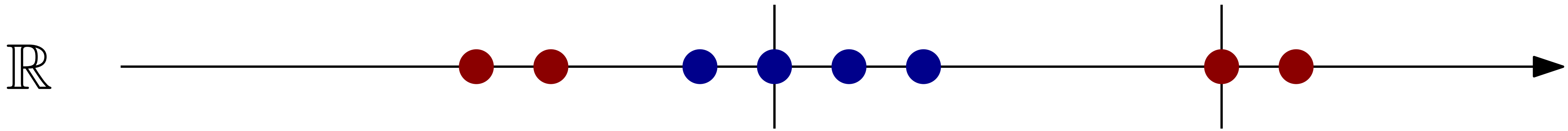

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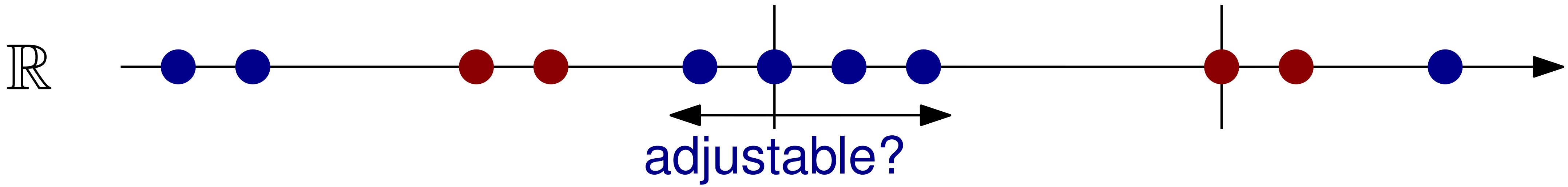

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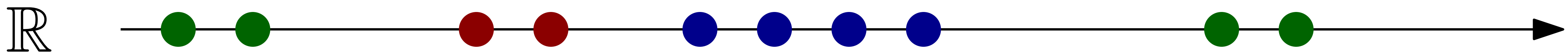

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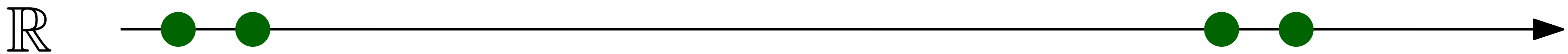

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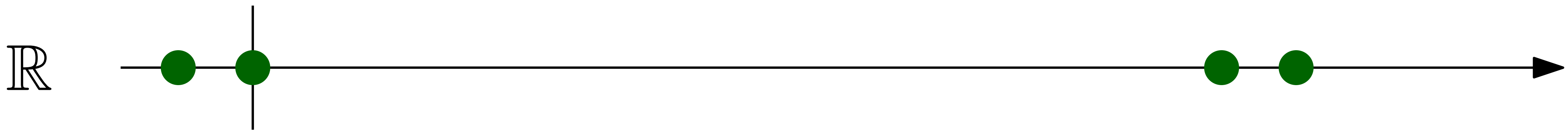

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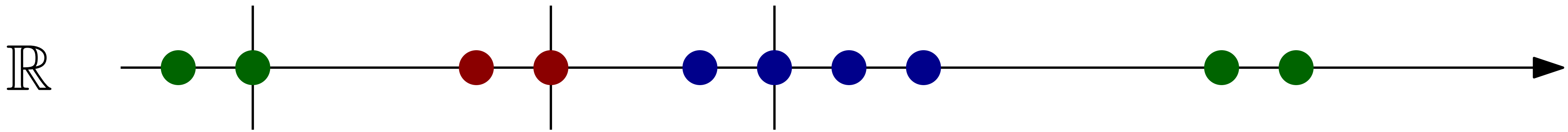

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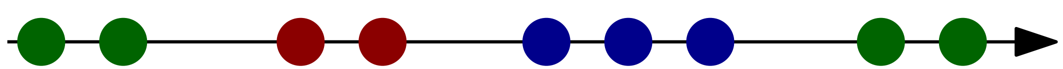
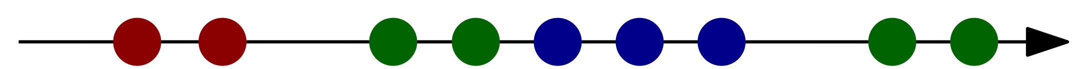
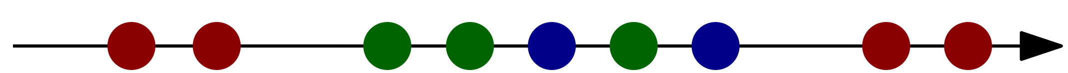


neighboring intervals



interval at end



ends are colored same

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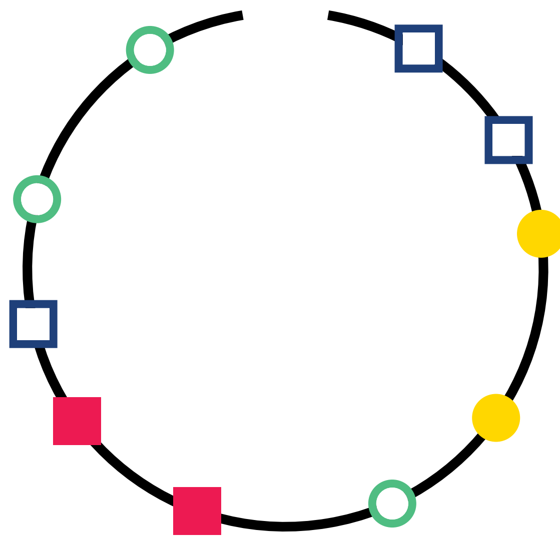
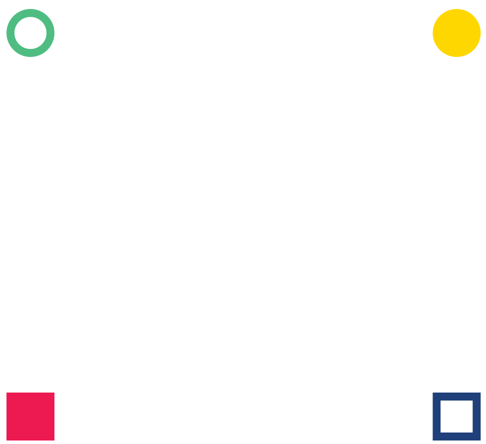
Step 3: Solve *irreducible* necklace using ILP

Irreducible Necklaces

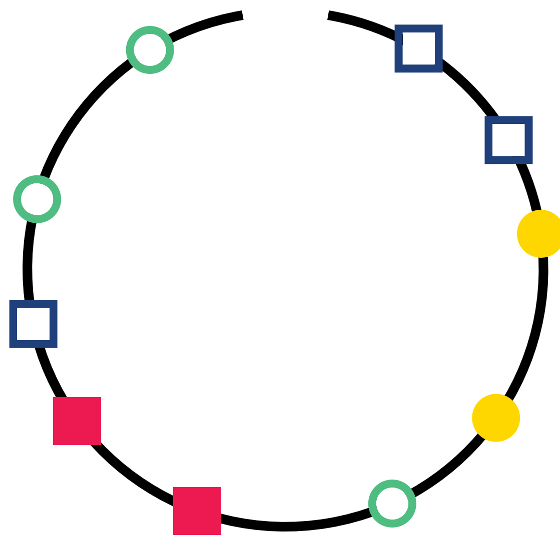


Necklace



Walk Graph

Irreducible Necklaces



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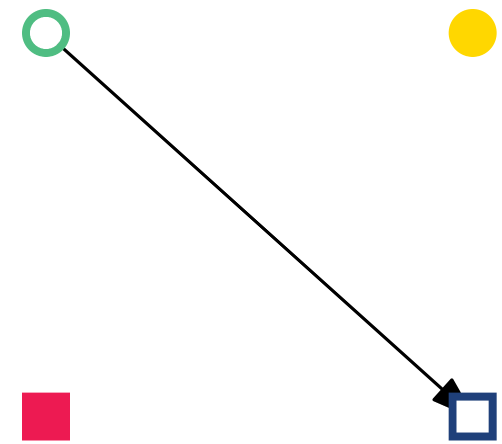

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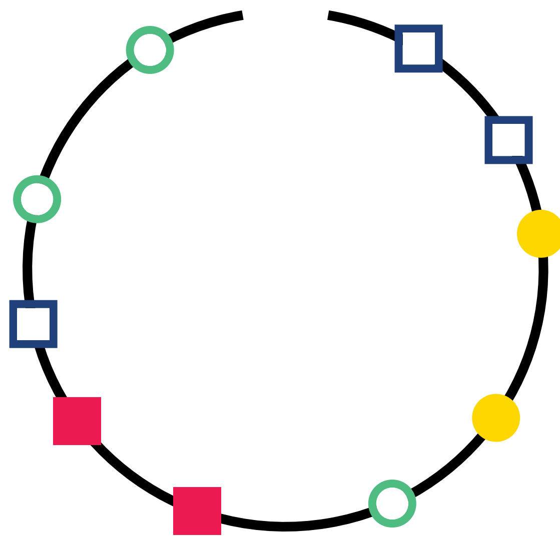

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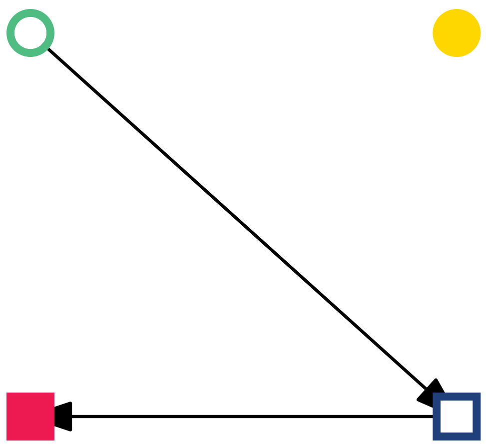

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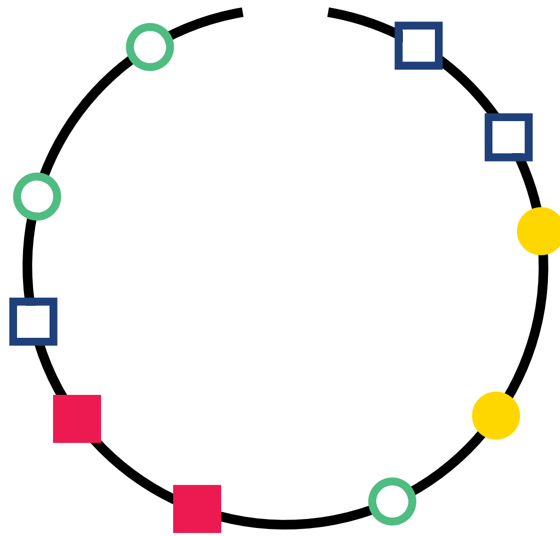

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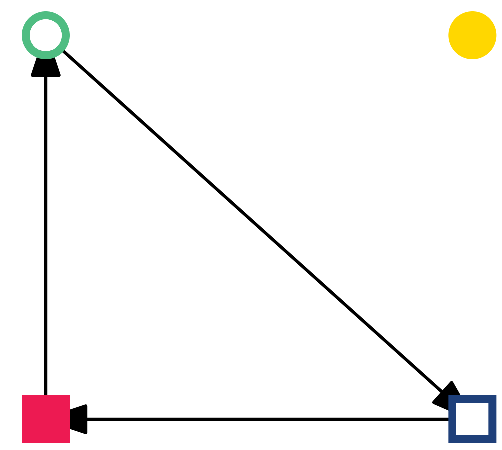

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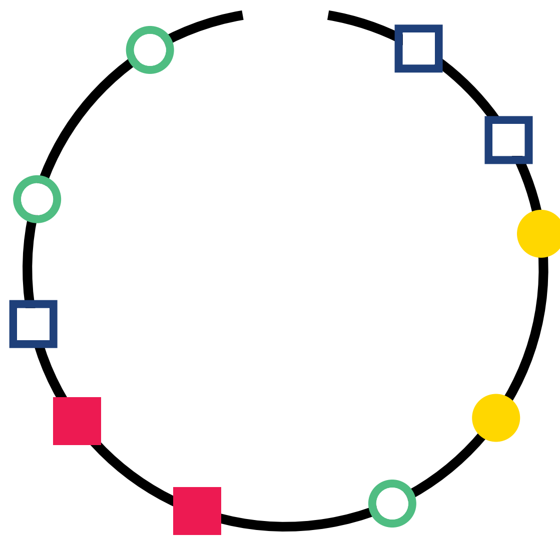

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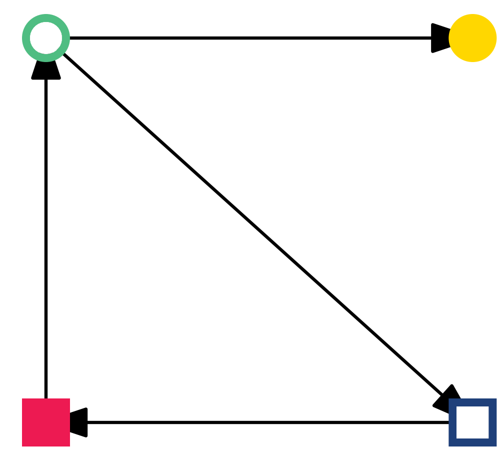

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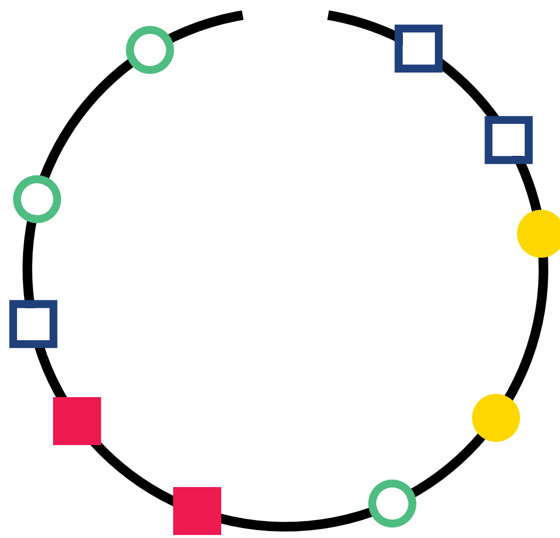

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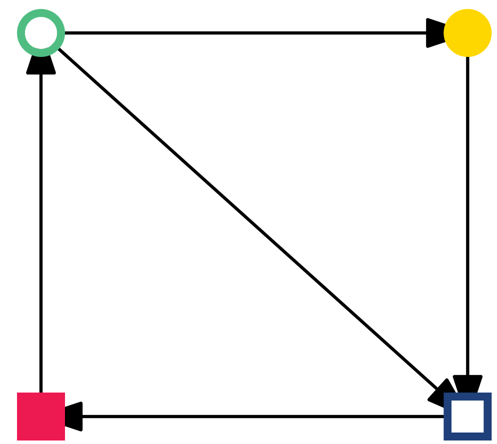

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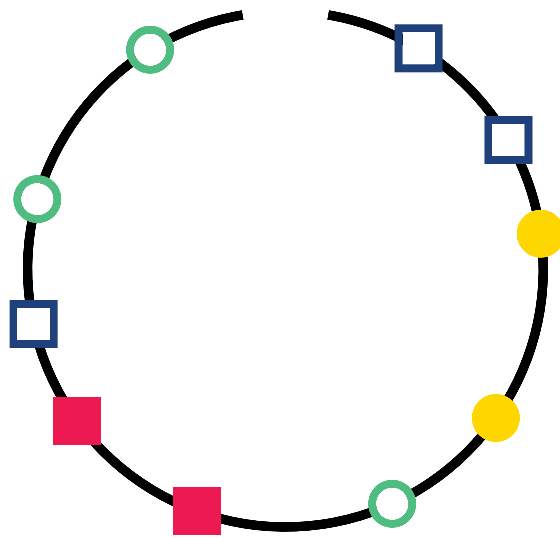

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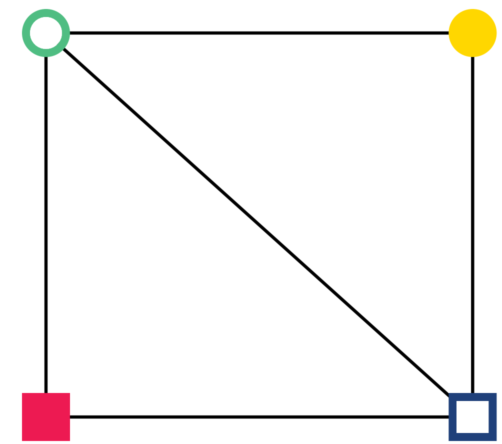

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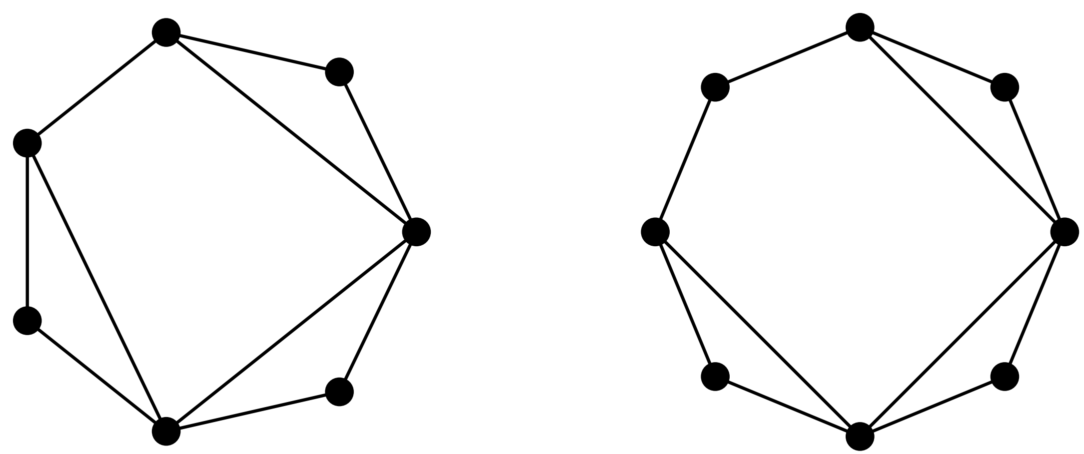
Irreducible Necklaces

Theorem: The walk graph of any irreducible necklace with n colors is isomorphic to the graph N_n.



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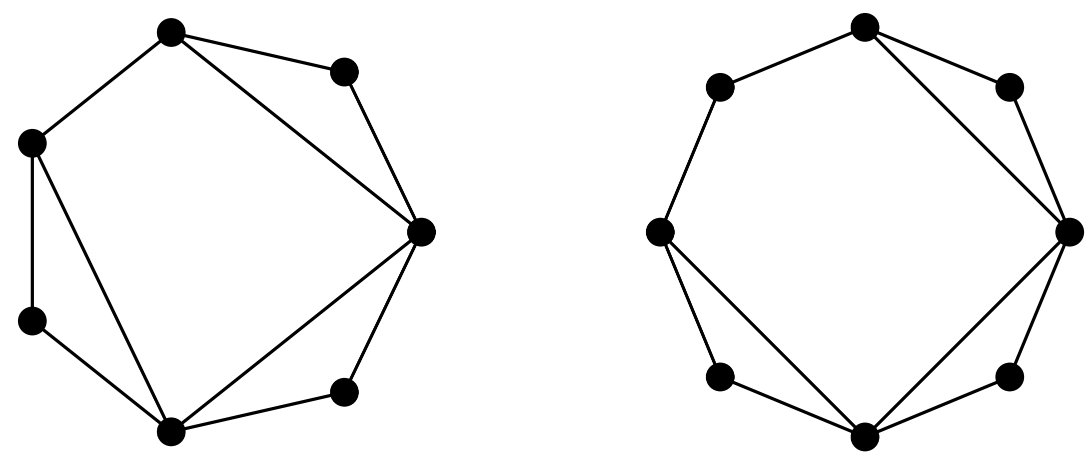
Irreducible Necklaces

Theorem: The walk graph of any irreducible necklace with n colors is isomorphic to the graph N_n.



N_7 N_8

Theorem [Jansen, Kratsch, '15]: A binary ILP is fixed-parameter tractable in the *treewidth* of its *primal graph*.

Contrast: NP-Hardness of Decision

Theorem: It is NP-complete to decide whether any given necklace on n colors has an α-cut for any given vector $\alpha = (\alpha_1, \ldots, \alpha_n)$.

Conclusion

n-separable necklaces are polynomial-time *recognizable*, and splittable both *fairly* as well as *unfairly*.

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Thank you for your attention!