Categorizing Ensembles of Real-Valued Functions

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requires:

- formalization of making statements about natural phenomena
- efficient decidability of logical entailment of such statements

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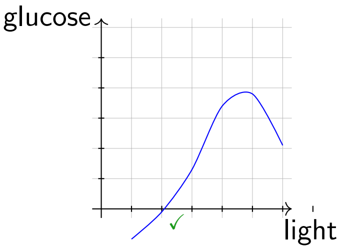

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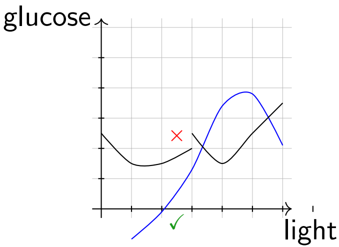

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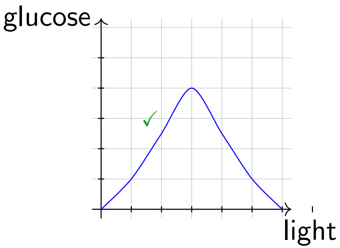

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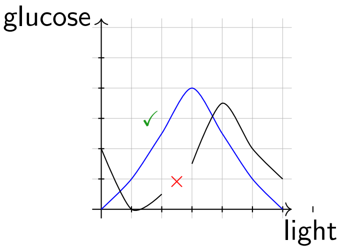

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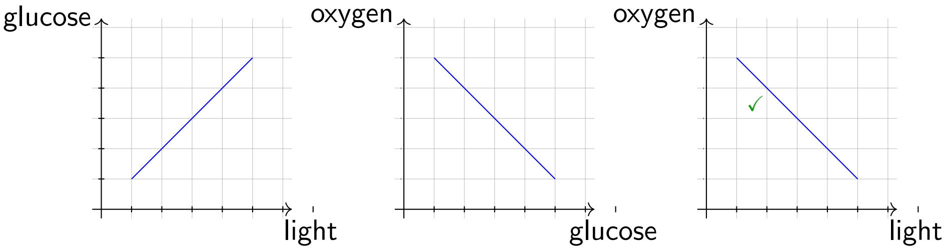

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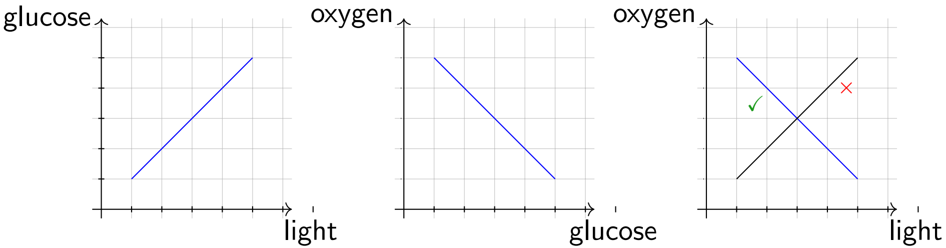

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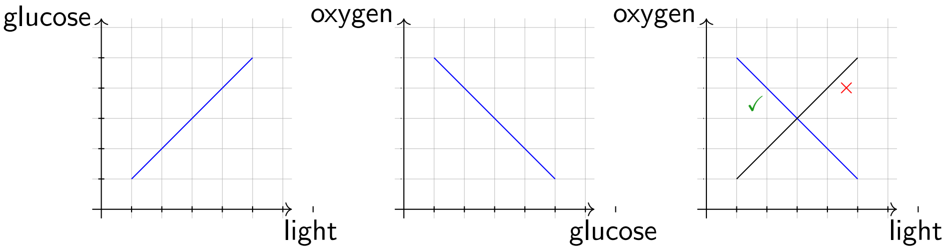
- functions defined only alongside variable order



Statements

influence statement S is $a \xhookrightarrow{I\,q\,I'} b$ where

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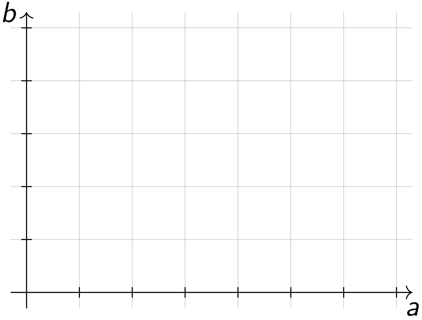

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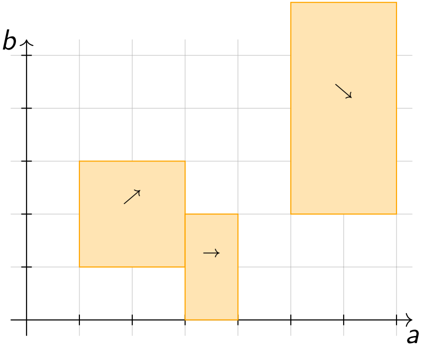

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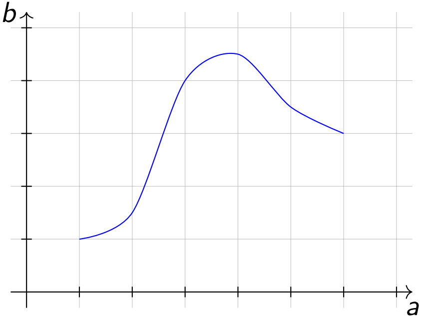

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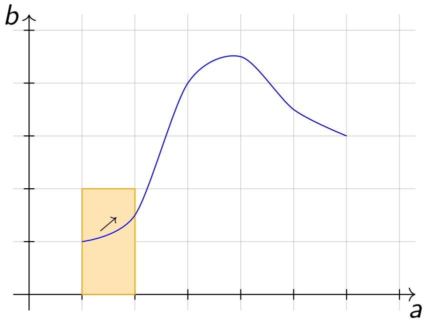

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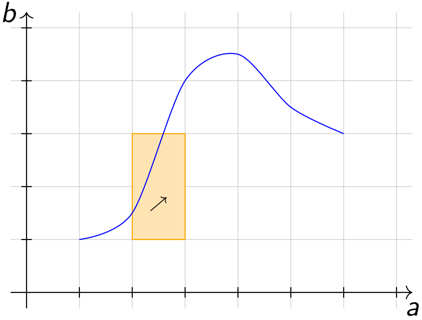

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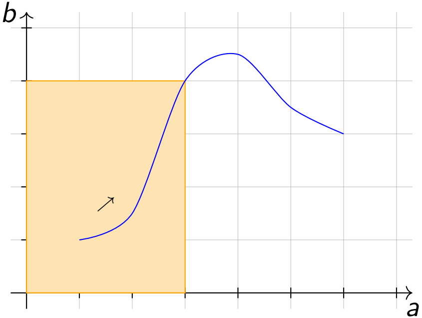

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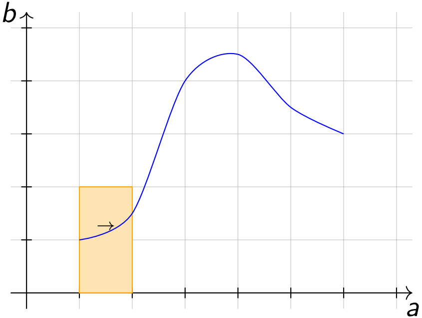

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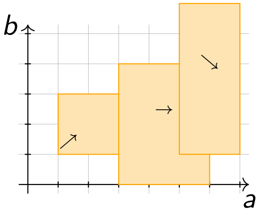

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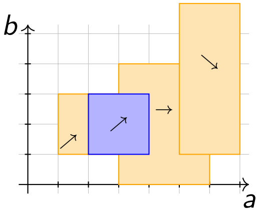

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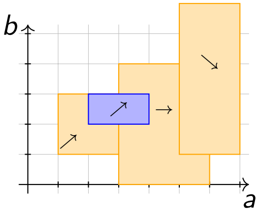

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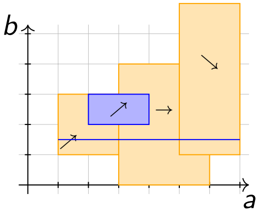

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goal: (efficient) decidability of entailment ($\models$) (cf. motivation)

first approach: Calculus of Influence (B./Lange/Möller Cade'23 + follow-up)

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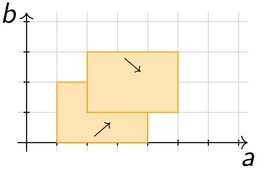

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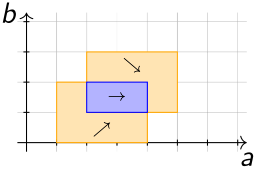

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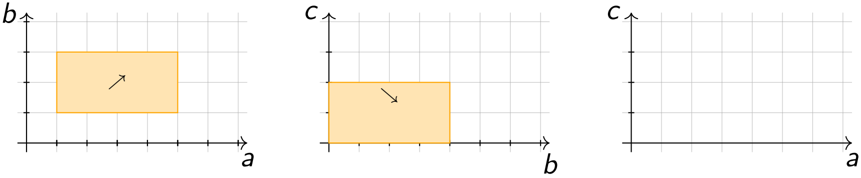

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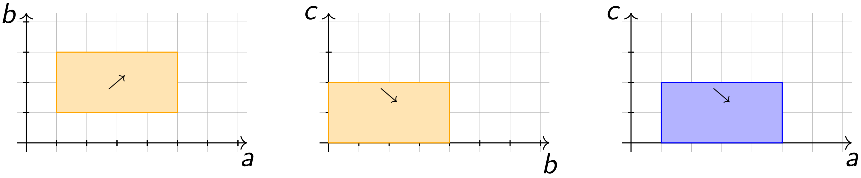

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On diamond-free, elementary schemes, the Calculus of Influence is complete, i.e., $\mathcal{C} \models H$ *iff* $\mathcal{C} \vdash H$.

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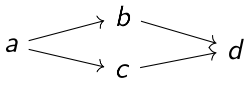

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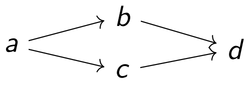
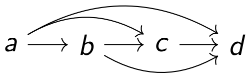
- a scheme is elementary if statements only on elementary variable pairs (i.e. not on $a \mapsto c, a \mapsto d, b \mapsto d$)





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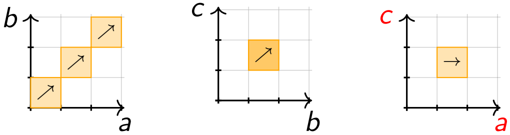

Limits of the Calculus

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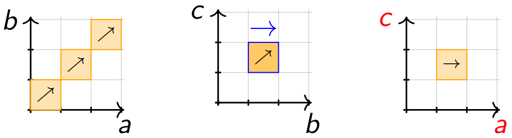

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new, direct approach: compute whether $\mathcal{C} \models H$ by exhaustive testing

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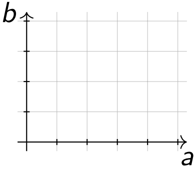

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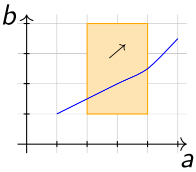

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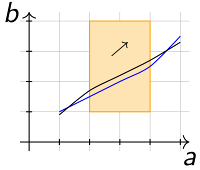

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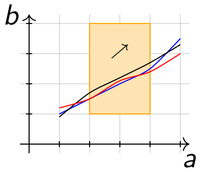

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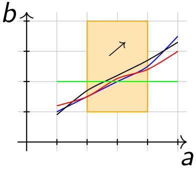

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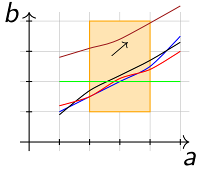

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the green one is different (it is constant), and so is the brown one (much larger b-value)

Equivalence for Influence Experiments

want: equivalence relation $\sim_{\text{good}}$ between influence experiments s.t.

- for all H (with integer bounds), all $\mathcal{F} \sim_{\text{good}} \mathcal{F}'$, have $\mathcal{F} \models H$ iff $\mathcal{F}' \models H$
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yields: whether $\mathcal{C} \models H$ reduces to testing finitely many cases

equivalence relation that satisfies above requirements is good

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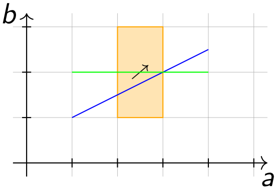

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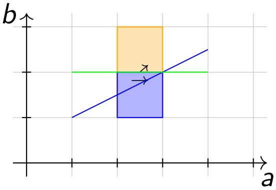

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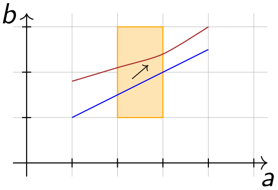

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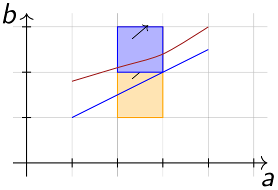

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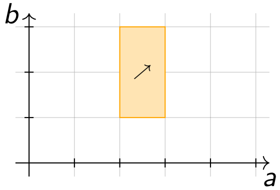

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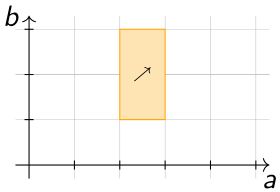

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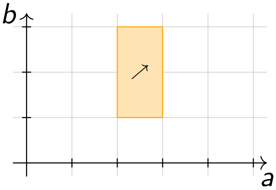

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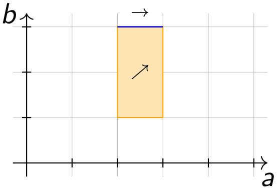

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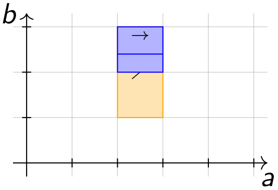

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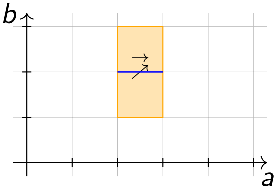

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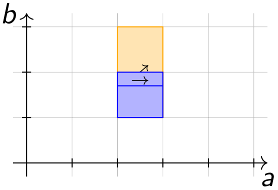

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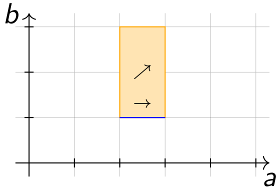

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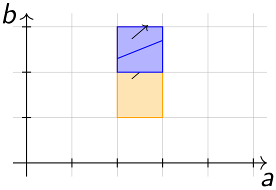

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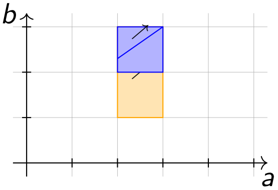

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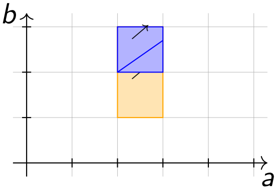

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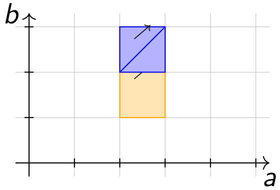

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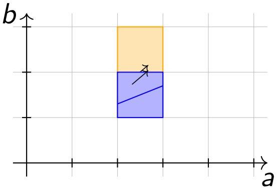

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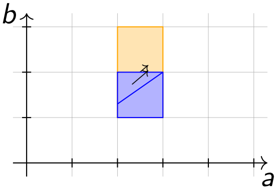

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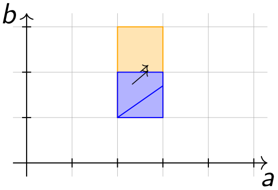

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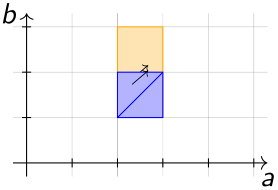

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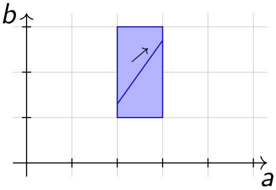

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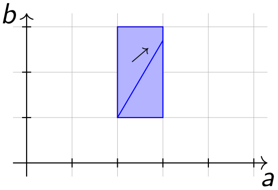

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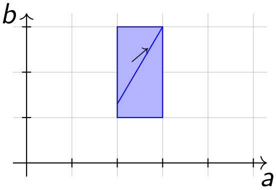

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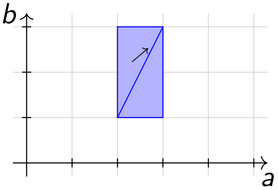

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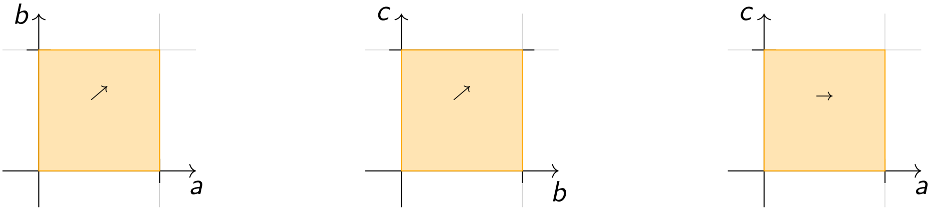

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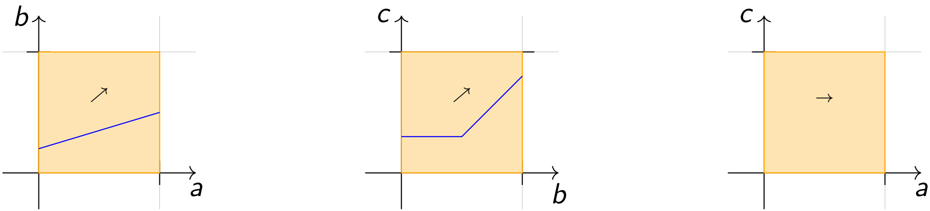

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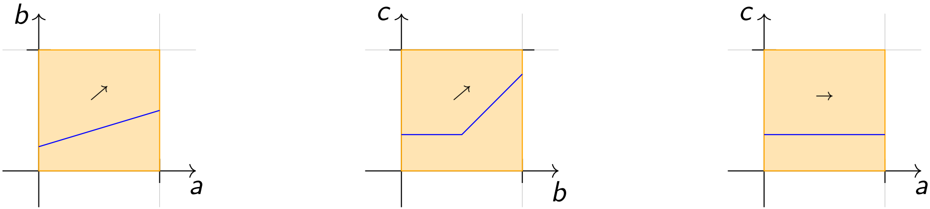

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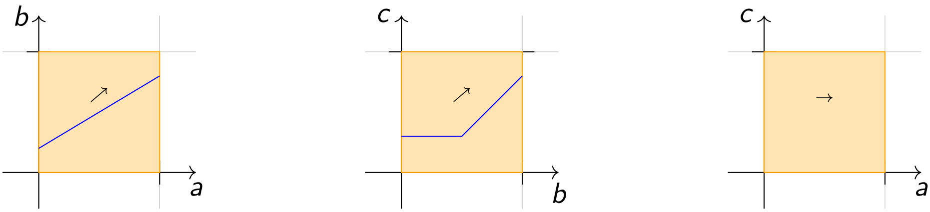

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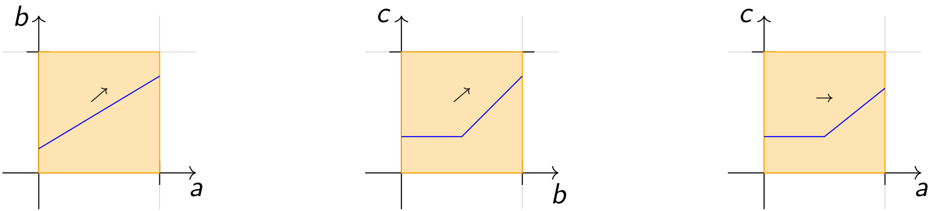
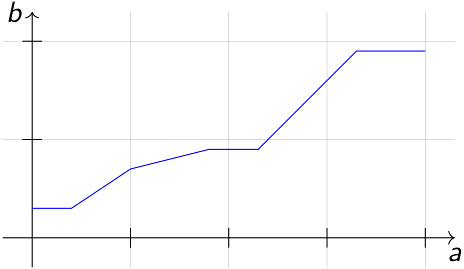


Points of Interest

goal: characterize points related directly to integer-value coordinates



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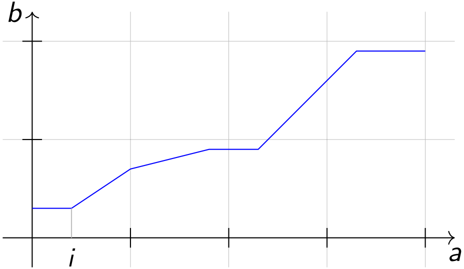

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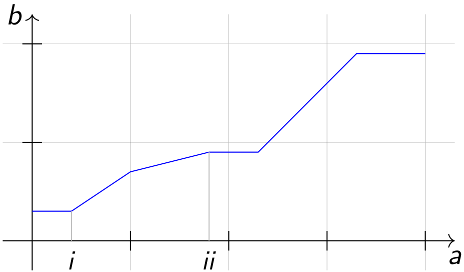

- (i) rightmost a value with same b-value as $a = 0$
- (ii) leftmost a with same b-value as $a = 2$

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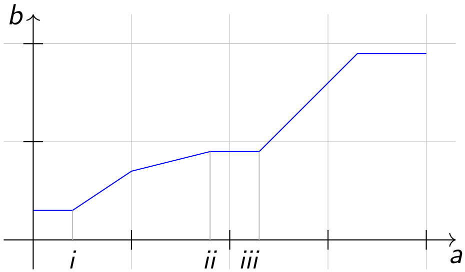

- (i) rightmost a value with same b-value as $a = 0$
- (ii) leftmost a with same b-value as $a = 2$
- (iii) rightmost a with same b-value as $a = 2$

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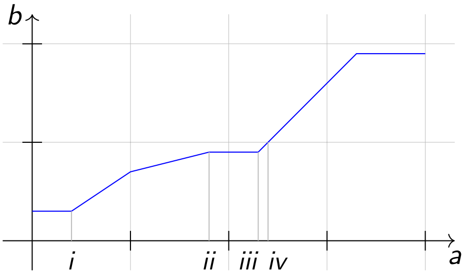

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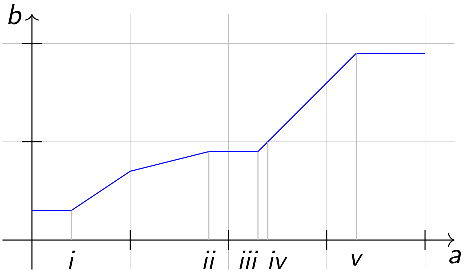

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- (v) leftmost a-value with same b-value as $a = 4$

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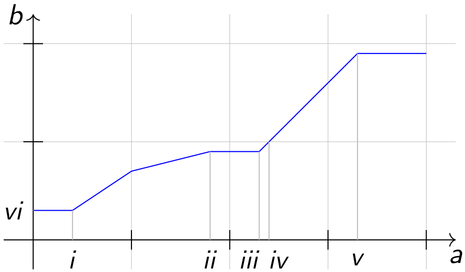

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- iv a-value where b-value is 2
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- vi b-value of $a = 0$

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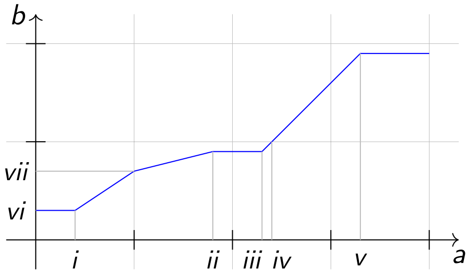

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- (vii) b-value of $a = 1$

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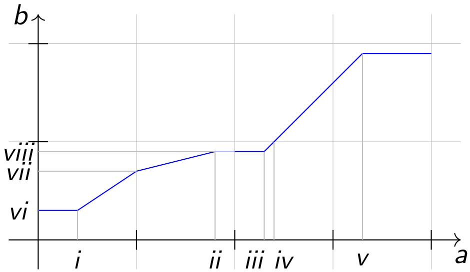

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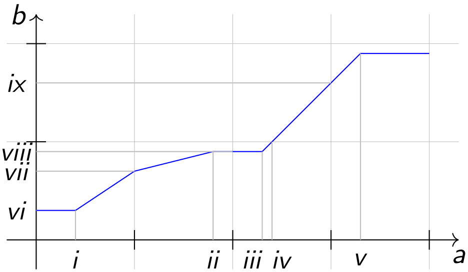

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- (ix) b-value of $a = 3$

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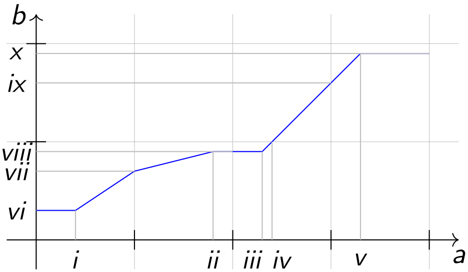

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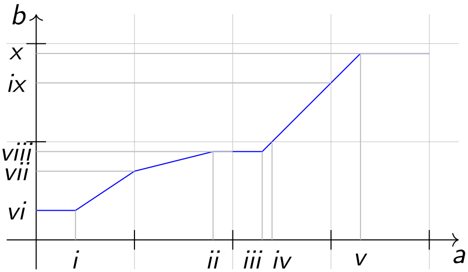

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exact value of points not important, just their order and relation to integer values

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  important: set remains finite in spite of this

generate relation $\sim_{\text{PoI}}$ by making influences equivalent if their PoI have the same order and same relation to integer values

Conjecture 3

$\sim_{\text{PoI}}$ is good, even on non-elementary schemes with diamonds.

Conclusion

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questions?