Hereditary First-Order Logic and Extensional ESO

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joint work with Manuel Bodirsky

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TU Dresden

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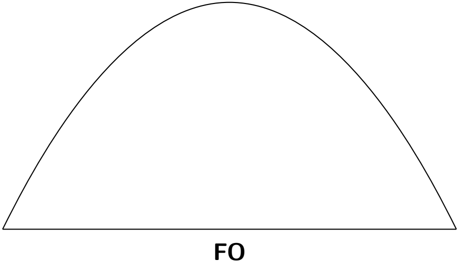
ERC Synergy Grant POCOCOP (GA 101071674)

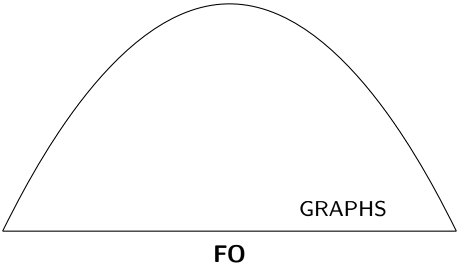
First-Order Logic



First-Order Logic

- Loopless symmetric digraphs (a.k.a. simple graphs)

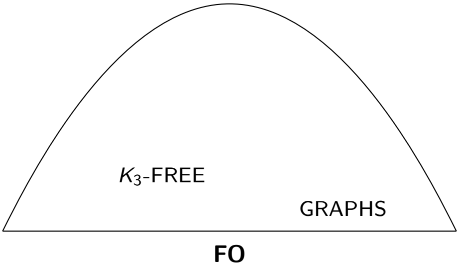
$$\forall x.\ \neg E(x,x) \wedge \forall x,y.\ E(x,y) \iff E(y,x)$$



First-Order Logic

- Triangle-free graphs

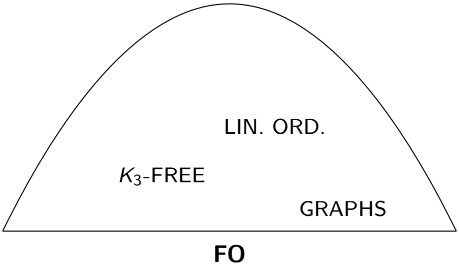
$$[\dots] \wedge \forall x, y, z.\ \neg E(x,y) \vee \neg E(y,z) \vee \neg E(z,x)$$



First-Order Logic

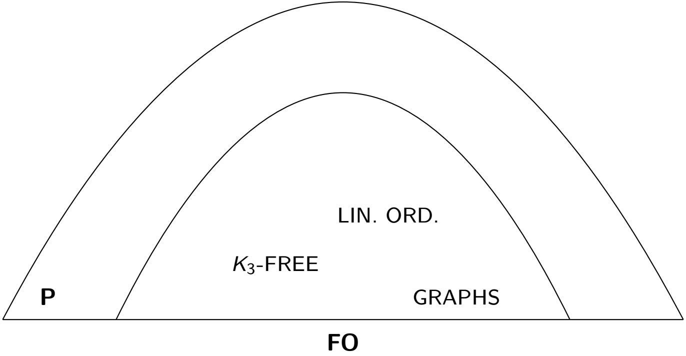
- (Strict) linear orders

$$\forall x.\, \neg E(x,x) \;\wedge\; \forall x,y.\, x = y \vee E(x,y) \vee E(y,x) \;\wedge\; E \text{ is transitive}$$



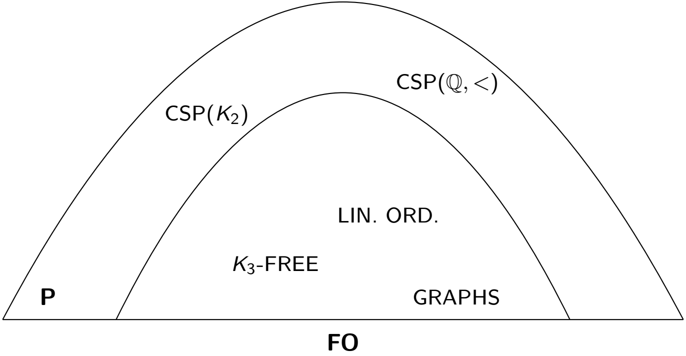
First-Order Logic

- If $\mathcal{C}$ is expressible in FO, then $\mathcal{C}$ is in P



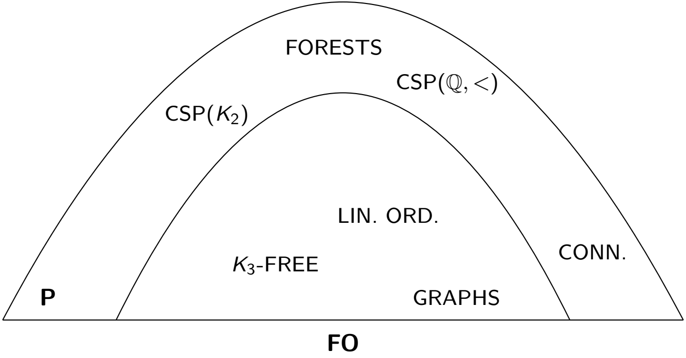
First-Order Logic

- CSP(H) := class of finite digraphs (structures) G such that $G \to H$



First-Order Logic

- Many classes are not expressible in FO



Existential Second-Order Logic

Existential Second-Order Logic

- Bipartite graphs

$$\exists W, B.\, \forall x, y.\, (W(x) \Leftrightarrow \neg B(x)) \wedge (E(x,y) \Rightarrow (W(x) \Leftrightarrow B(y)))$$

- Finite domain CSPs

$$\exists U_1, \ldots, U_k \forall x_1, \ldots, x_n.\, (\ldots)$$

- Acyclic digraphs = CSP($\mathbb{Q}, <$)

$$\exists T \forall x, y.(E(x,y) \Rightarrow T(x,y) \wedge T \text{ is a strict linear order})$$

- Strict NP (SNP)

$$\exists R_1, \ldots, R_k \forall x_1, \ldots, x_n (\ldots)$$

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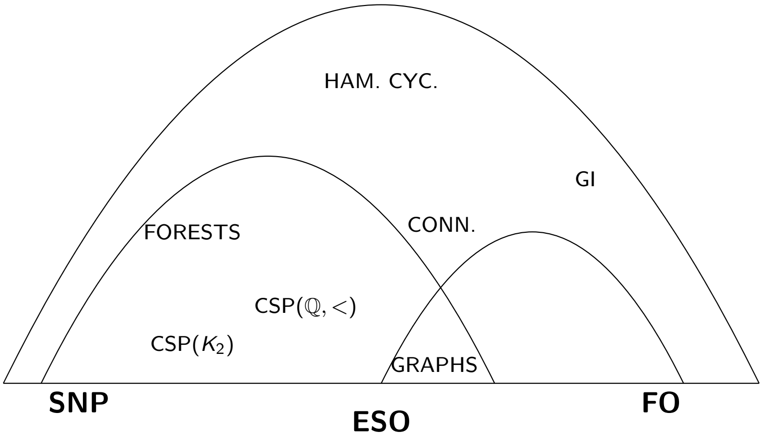

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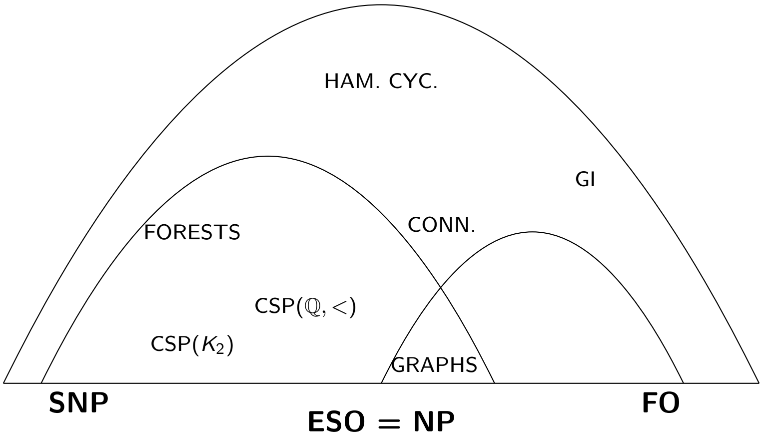

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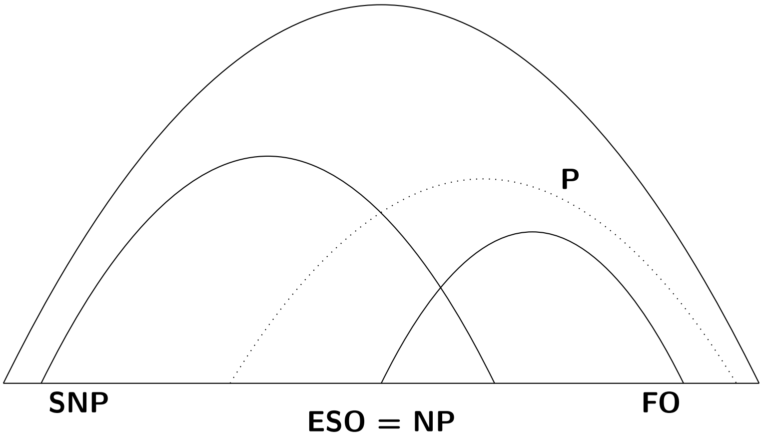

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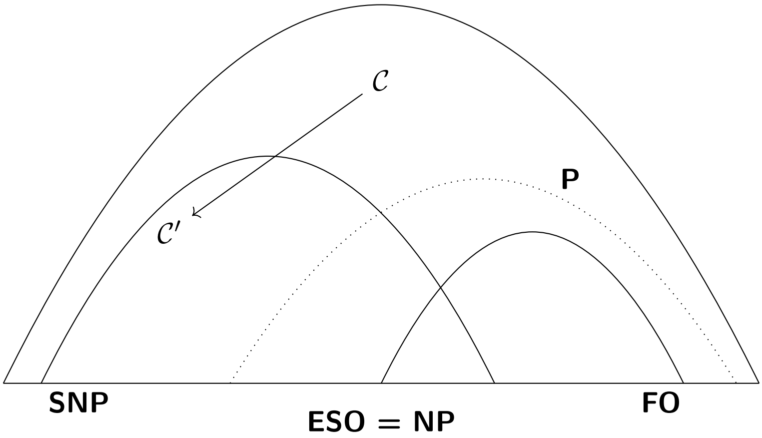

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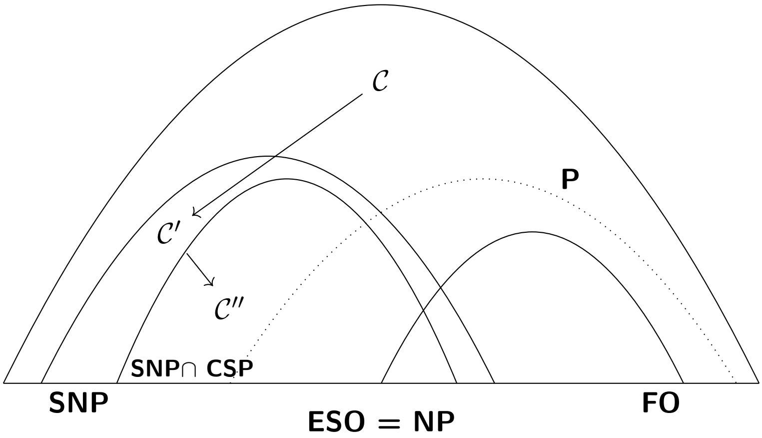

Hereditary First-Order Logic

A structure $\mathbb{A}$ *hereditarily satisfies* ϕ if every (induced) substructure $\mathbb{A}'$ of $\mathbb{A}$ satisfies ϕ.

Ex. 1 Acyclic digraphs: $\phi :=$ exists a source.

Ex. 2 Forests: $\phi :=$ exists a vertex of degree 1.

Ex. 3 Chordal graphs: $\phi :=$ exists a simplicial vertex (Rose, 1970).

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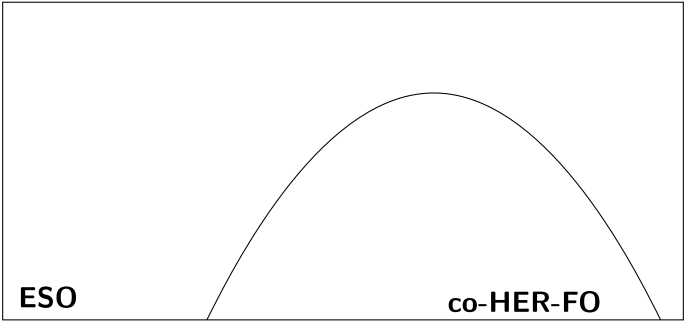
Two more. CSP($\vec{P}_3$); and every directed cycle induces a symmetric edge.

Hereditary First-Order Logic

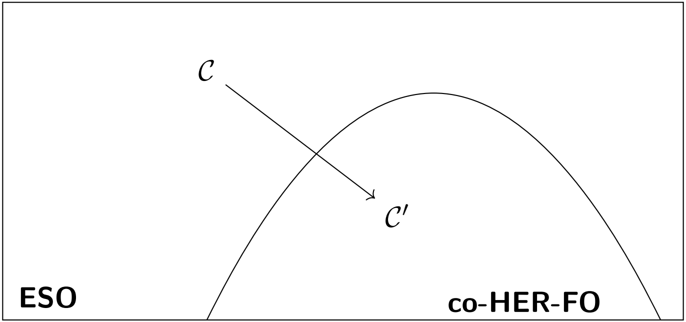
Obs. If $\mathbb{A}$ does not hereditarily satisfy ϕ, then there is a set of vertices S such that $\mathbb{A}[S]$ satisfies $\neg\phi$. Equivalently,

$$\mathbb{A} \models \exists S.\ S \neq \varnothing \wedge (\neg\phi)_S$$



Hereditary First-Order Logic

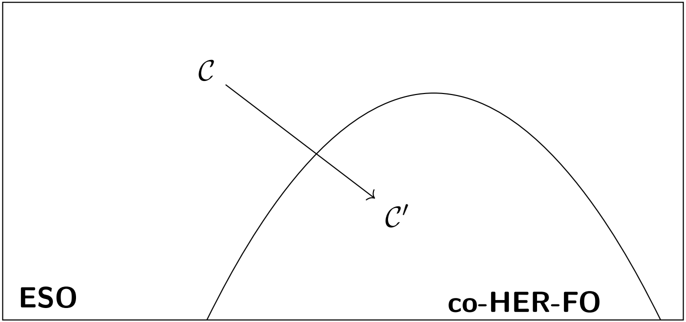
Qst 1. Are the complements of HER-FO NP-rich?



Hereditary First-Order Logic

Qst 1. Are the complements of HER-FO NP-rich?

Qst 2. Are there NP-intermediate problems in (co-)HER-FO?



Extensional ESO

Extensional ESO

Example. Acyclic digraphs = CSP($\mathbb{Q}, <$)

$$\exists T \forall x, y. (E(x,y) \Rightarrow T(x,y) \wedge T \text{ is a strict linear order})$$

Non-example. 2-COL

$$\exists W, B.\ \forall x, y.\ (W(x) \Leftrightarrow \neg B(x)) \wedge (E(x,y) \Rightarrow (W(x) \Leftrightarrow B(y)))$$

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Example.

$$\exists W, B.\ \forall x, y.\ W'(x) \Rightarrow W(x) \ \wedge \ B'(x) \Rightarrow B(x) \ \wedge \ (\dots)$$

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Example. Pre-coloured 2-COL

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Example. Pre-coloured CSP(H)

$$\exists U_1, \ldots, U_k \left(\forall x \bigwedge_{i=1}^{k} U_i'(x) \Rightarrow U_k \right) \wedge (\forall x_1, \ldots, x_n(\ldots))$$

Extensional ESO

Dear colleagues,

We are pleased to invite you to the online event:

Event: *30 Years of Graph Sandwich Problems: A Celebration*
Date: March 27, 2025
Time: 2:00 PM (GMT -3), São Paulo, Brazil

Link: https://meet.google.com/sur-pmun-evy

In 1995, the publication of the seminal paper:

"M.C. Golumbic, H. Kaplan, R. Shamir, "Graph Sandwich Problems," Journal of Algorithms 19 (1995) 449-473"

opened a rich and extensive research area that continues to inspire publications worldwide.

Extensional ESO

Graphs Sandwich Problems (GSP(Π))

Input: A vertex set V and sets of edges $E_1 \subseteq E_2$.
Question: Is there a graph (V, E) that satisfies property Π and $E_1 \subseteq E \subseteq E_2$.

Extensional ESO

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Ex 1. P_4**-free graphs:** GSP(Π) is in P (Golumbic, Kaplan, Shamir 1995).

Ex 2. C_4**-free graphs:** GSP(Π) is NP-complete (Dantas, de Figueiredo, da Silva, Teixeira 2011).

Ex 3. $(K_4 - e)$**-free graphs:** GSP(Π) is in P (Dantas, de Figueiredo, da Silva, Teixeira 2011).

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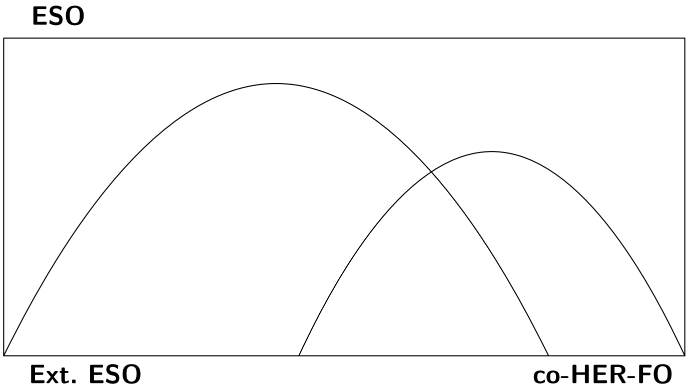
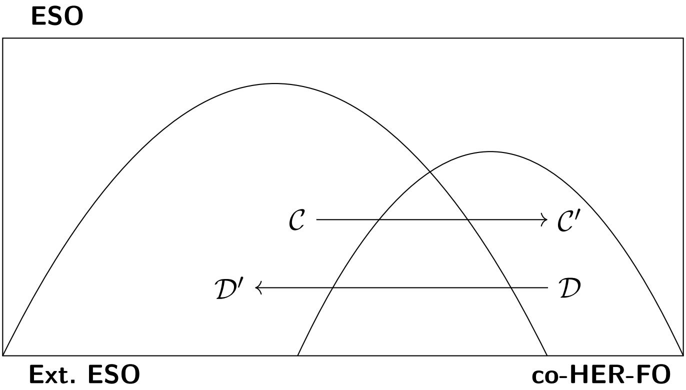
Ext. ESO: GSP(Π) is in Extensional ESO whenever Π is in FO. For instance, $\Pi := F$-free.

Extensional ESO



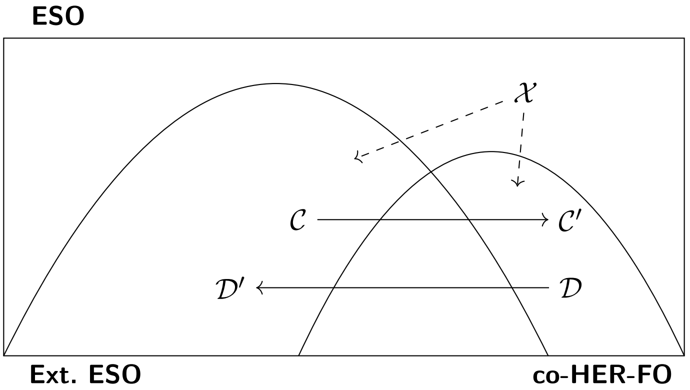
Extensional ESO

Theorem. Extensional ESO and complements of HER-FO have the same computational power (up to P-time equivalence).



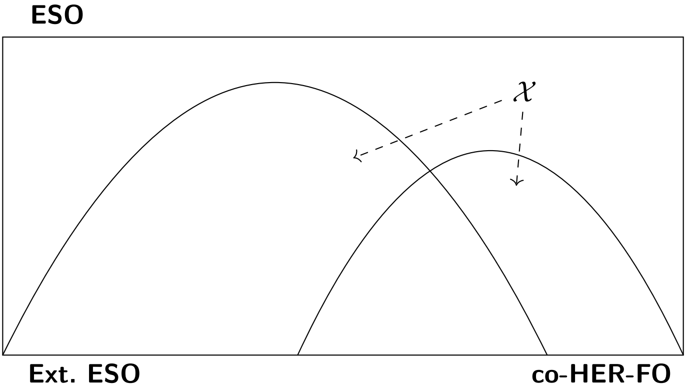
Extensional ESO

Theorem. Extensional ESO and complements of HER-FO have the same computational power (up to P-time equivalence). Moreover, these classes are not NP-rich (unless $E = NE$).



Extensional ESO

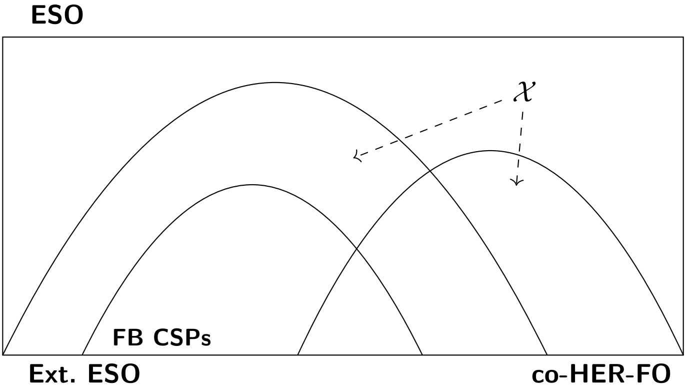
Qst 2. Are there NP-intermediate problems in (co)-HER-FO?



Extensional ESO

Qst 2. Are there NP-intermediate problems in (co)-HER-FO?

Lemma. CSP(H) is expressible in ext. ESO whenever H is finitely bounded.



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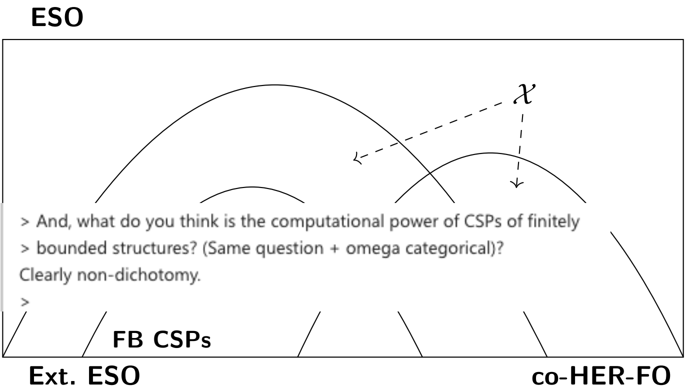

Thank you for your attention!