On the Tightness of the Standard Lower Bound in the Two-Machine Routing Open Shop

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Open shop scheduling problem $Om||C_{\max}$

Input

- Machines $\{M_1, \ldots, M_m\}$
- Jobs $\{J_1, \ldots, J_n\}$
- Operations $\{O_{ji} | 1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n\}$.
- Processing times $\{p_{ji} | 1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n\}$.

Constraints

- Operations of each machine can be processed in arbitrary order.
- Operations of the same job or performed by the same machine can not overlap in time

Schedule

$$S = \{s_{ji}\}, \text{ operation } O_{ji} \text{ is process within } [s_{ji}, C_{ji}],$$

$$C_{ji} = s_{ji} + p_{ji}.$$

Goal

Compose a feasible schedule with minimum makespan ($C_{\max} = \max_{j,i} c_{ji}$).

Open shop

Example

How to shoe horses? Each horse (job) needs to have each leg shod, and a horse cannot stand on less than three legs. Four blacksmiths (machines) are specialists in shoeing a specific leg (one — the right front, another – the left back, etc.). The time for each such operation is known in advance.
How fast can n horses be shod?

Another example

Medical examination. A group of patients (jobs) needs to be examined by a set of doctors (machines). The time of each patient's appointment with each doctor is known in advance. The goal is to make a schedule in which the completion time of the last appointment is minimized. Delays in the transfer of patients from one doctor's office to another can be neglected.

Complexity

- $O2||C_{\max}$ is solvable in $O(n)$ by
  1. T. Gonzalez, S. Sahni 1976
  2. M. Pinedo, L. Schrage 1982
  3. D. de Werra 1989
  4. A. Soper 2013
  5. A. Khramova, Ch 2021
- $O3||C_{\max}$ is NP-hard [Gonzalez, Sahni 1976]

Standard lower bound

	J_1	J_2	$\dots$	J_n	
M_1	p_{11}	p_{21}	$\dots$	p_{n1}	
M_2	p_{12}	p_{22}	$\dots$	p_{n2}	
$\vdots$	$\vdots$			$\vdots$	
M_m	p_{1m}	p_{2m}	$\dots$	p_{nm}	

Standard lower bound

	J_1	J_2	$\dots$	J_n	Σ
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Σ	d_1	d_2	$\dots$	d_n	

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M_m	p_{1m}	p_{2m}	$\dots$	p_{nm}	ℓ_m
Σ	d_1	d_2	$\dots$	d_n	

Standard lower bound

$$C^*_{\max} \geqslant \bar{C} \doteq \max_{i,j}\{\ell_i, d_j\}.$$

How tight is $\bar{C}$?

$m = 2$

For $O2||C_{\max}$

$$C^*_{\max} = \bar{C}.$$

An instance for $m = 3$

$$J_1 = (1, 1, 1), J_2 = (2, 0, 0),$$

$$J_3 = (0, 2, 0), J_4 = (0, 0, 2).$$

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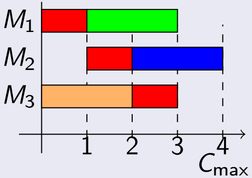

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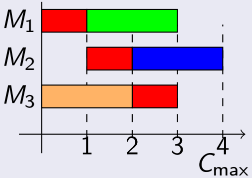
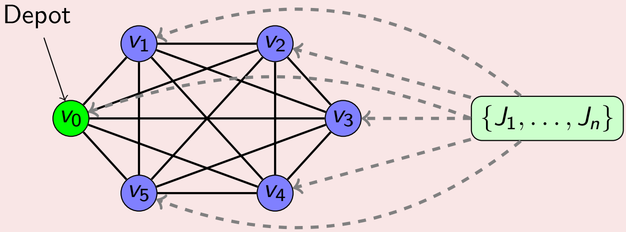


- The bound $C^*_{\max} \leqslant \frac{4}{3}\bar{C}$ is tight for $O3||C_{\max}$ [Sevastyanov, Ch 1998]
- The investigation for the tight bound of that type is called optima localization:

$$C^*_{\max} \in [\bar{C}, \rho^* \bar{C}],$$

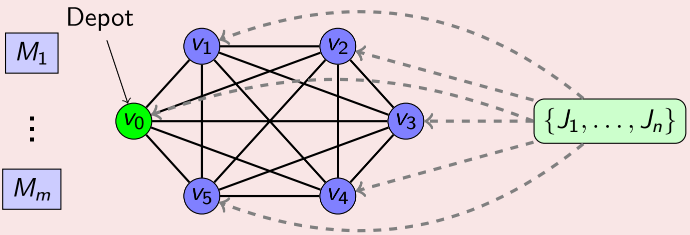
$$\rho^* = \sup \frac{C^*_{\max}(I)}{\bar{C}(I)}.$$

The combination of OPEN SHOP and Metric TSP



Informal introduction to the Routing Open Shop Problem

The combination of OPEN SHOP and Metric TSP



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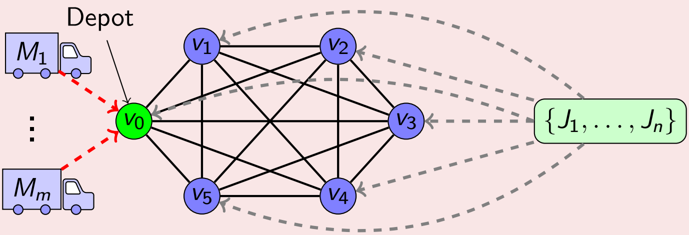

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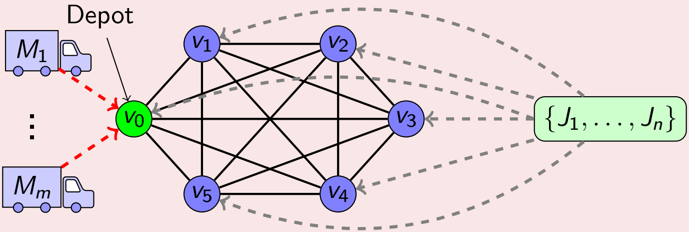

- $G = \langle V, E \rangle$ — transportation network;
- $\mathrm{dist}(u, v)$ — travel time between u and v;
- $\mathcal{J}(v)$ — subset of jobs from location v;
- $R_i(S) = \max\limits_{v} \max\limits_{J_j \in \mathcal{J}(v)} \big(C_{ji}(S) + \mathrm{dist}(v_0, v)\big)$;
- $R_{\max}(S) = \max R_i(S) \to \min_S$ — the makespan.

Complexity of the routing open shop

Notation: $ROm||R_{\max}$ or $ROm|G = X|R_{\max}$.

- Includes TSP as a special case and therefore strongly NP-hard even for $m = 1$.
- NP-hard for $m = 2$ and $G = K_2$ [I. Averbakh, O. Berman, Ch 2006]
- FPTAS for $m = 2$ and $G = K_2$ [A. Kononov 2012]
- NP-hard for $m = 2$ and $G = K_2$ and proportionate job processing time [A. Pyatkin, Ch 2022]

Standard lower bound for the routing open shop

- $\ell_{\max} = \max \ell_i$ — maximal machine load,
- $d_{\max}(v) = \max\limits_{J_j \in \mathcal{J}(v)} d_j$ — maximal length of job from v,
- T^* — length of the shortest Hamiltonian route over G (TSP optimum)

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Standard lower bound

$$\bar{R} = \max\left\{\ell_{\max} + T^*, \max_v\left(d_{\max}(v) + 2\mathrm{dist}(v_0, v)\right)\right\}$$

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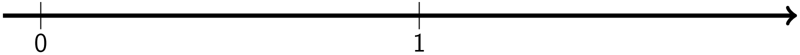
Optima localization

For $RO2||R_{\max}$,

$$R^*_{\max} \in [\bar{R}, \rho^* \bar{R}].$$

The problem: find this ρ^*.

Approximation for $RO2||R_{\max}$



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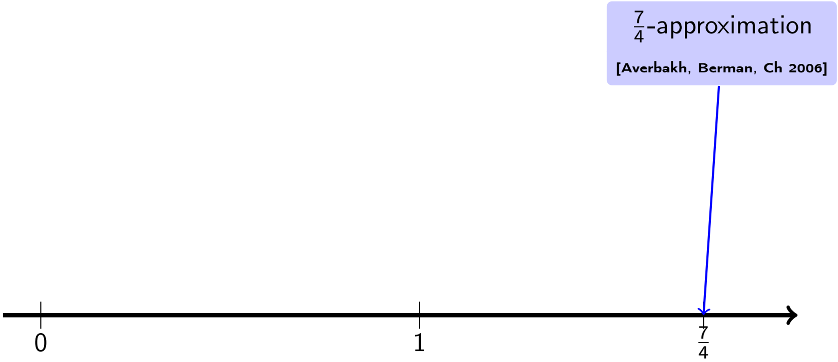

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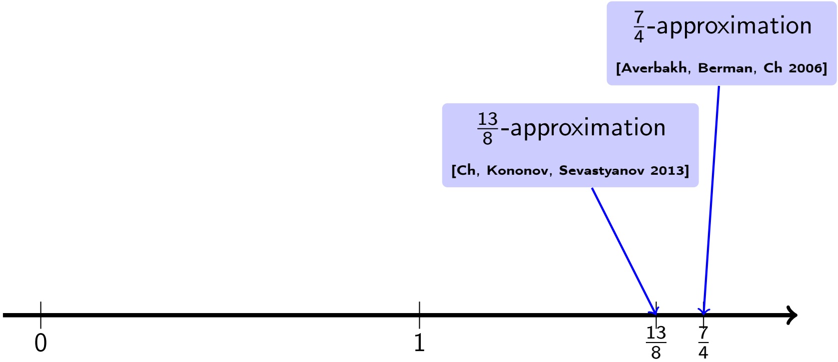

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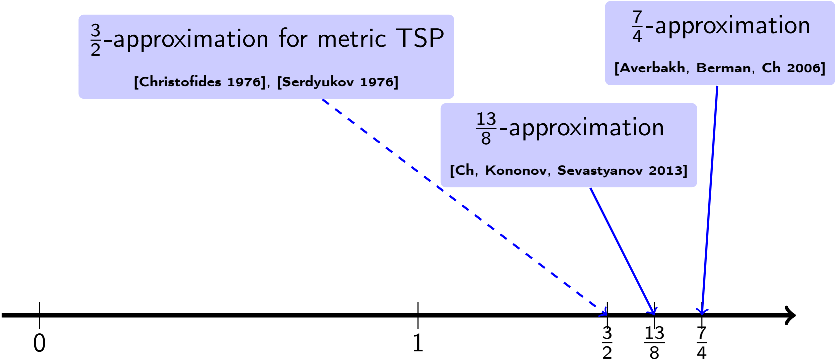

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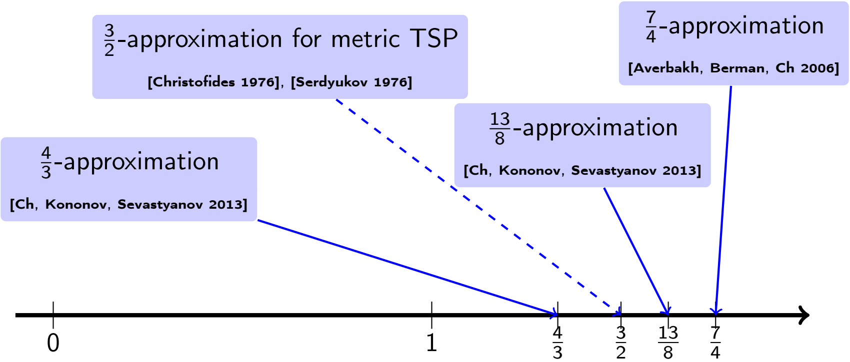

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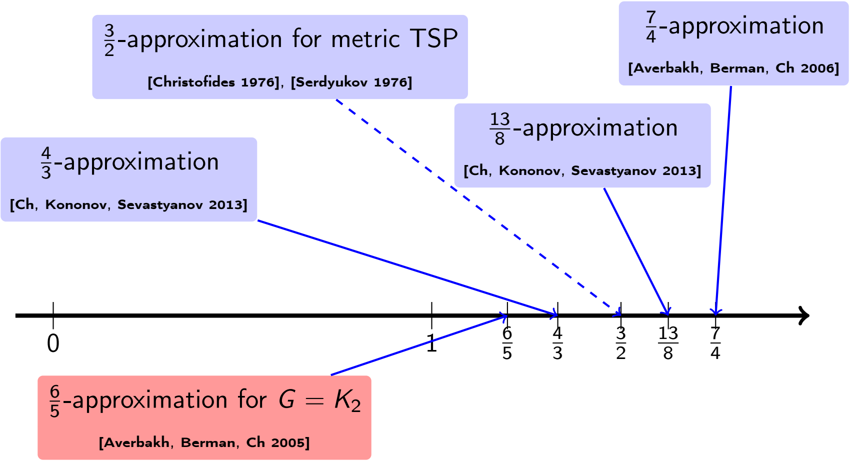

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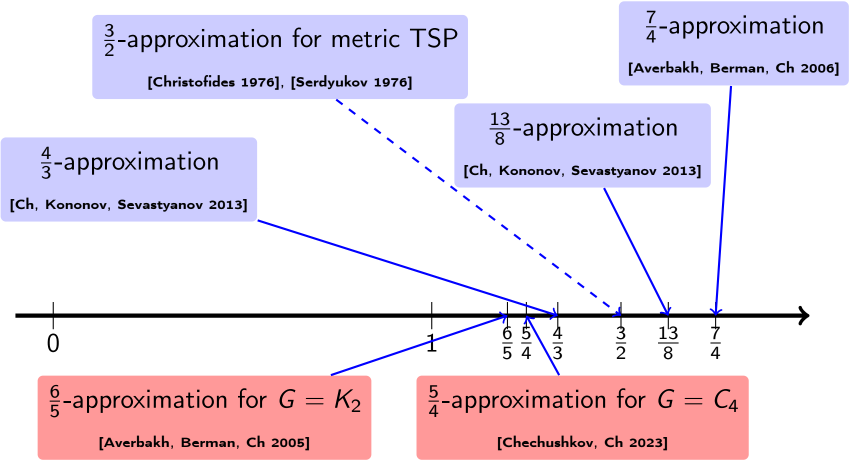

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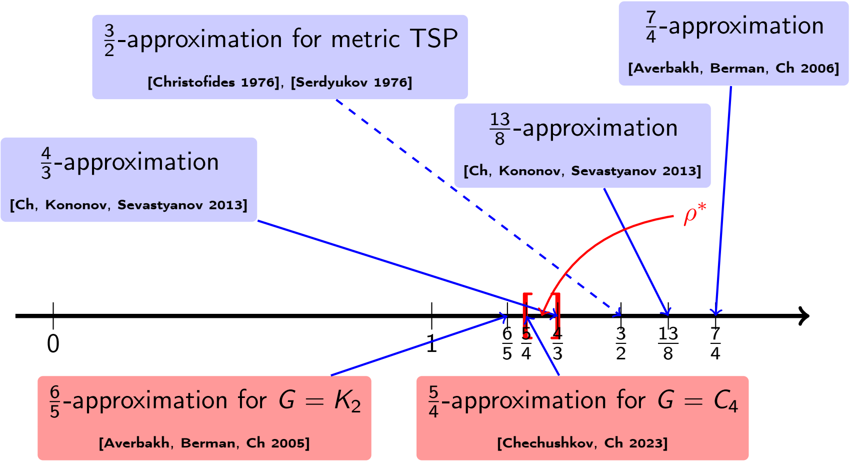
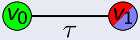
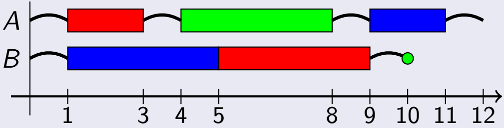


$$\rho^* \in \left[\frac{5}{4}, \frac{4}{3}\right]$$

[I. Averbakh, O. Berman, Ch 2005]

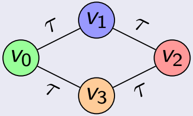


$J_1 = (4,0)$; $\tau = 1$; $J_2 = (2,4)$; $J_3 = (2,4)$.



Tight for $G = K_2$, $G = K_3$, $G = tree$.

[A. Chechushkov, Ch 2023]



$\tau = 1$; $J_0 = (0,0)$; $J_1 = (0,2)$;
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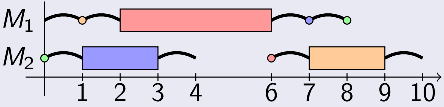
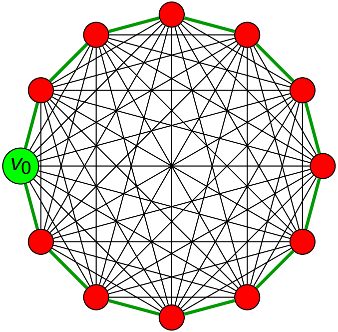


Tight for $G = C_4$.

Chain reduction: tunnels



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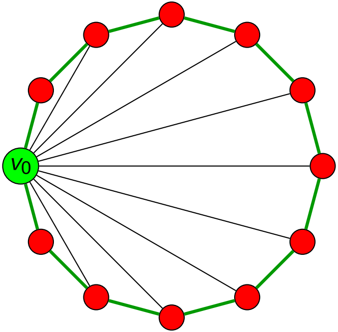

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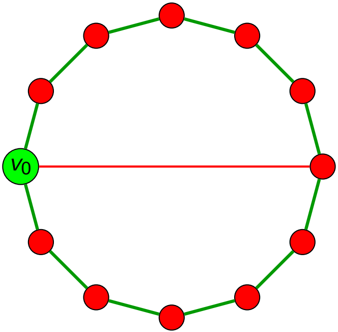

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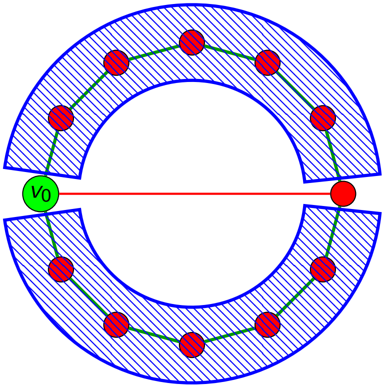

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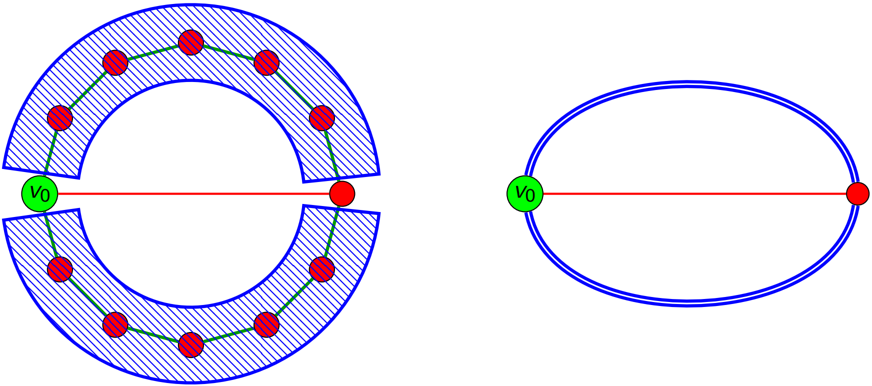
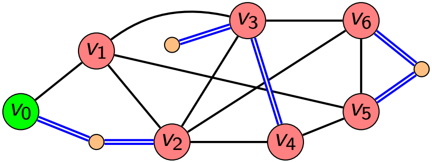


A routing open shop problem with tunnels



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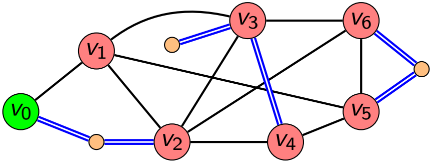

- v_0 is the depot.

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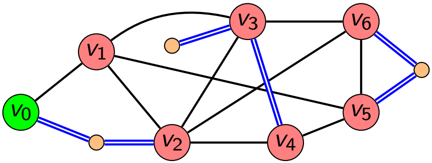

- v_0 is the depot.
- Some nodes (red ones plus depot) contain jobs (perhaps multiple jobs per node). Orange nodes do not contain jobs.

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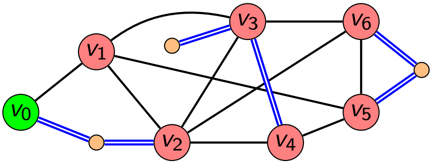

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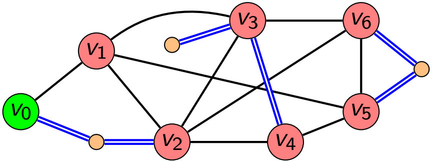

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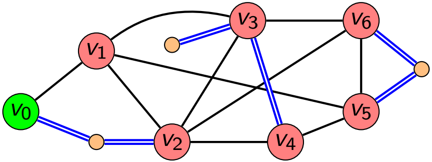

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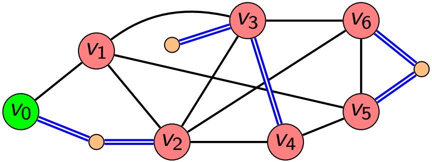

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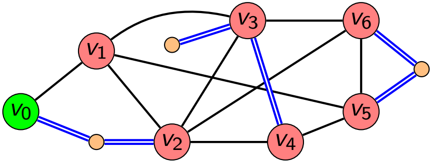

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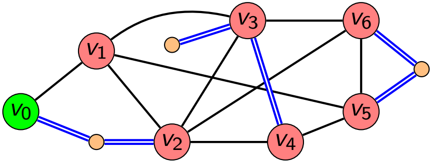

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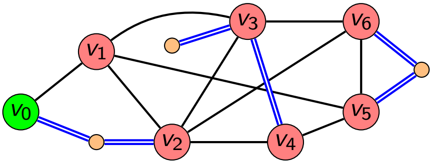

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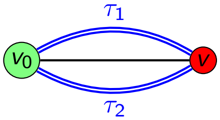
  - Machine processes the tunnel while traveling over it. Travel and processing times are combined.
- The goal is to process all jobs and to return to the depot ASAP.

Reduction of $RO2||R_{\max}$ to the problem with tunnels

Theorem [Ch 2021]

Any instance I of $RO2||R_{\max}$ can be reduced to an instance I' of the problem with tunnels $\overline{\overline{R}}O2|G = \mathcal{P}^+|R_{\max}$ preserving the standard lower bound. The instance I' contains two nodes, two parallel tunnels, one job at the depot and at most three jobs at node v.

$\mathcal{P}^+$:

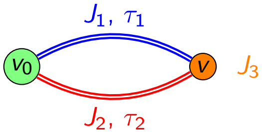


$$R^*_{\max} < \frac{4}{3}\overline{\overline{R}}?$$

Optima localization for $\overline{\overline{R}}O2|G = \mathcal{P}^+|R_{\max}$

Theorem

For any instance of $\overline{\overline{R}}O2|G = \mathcal{P}^+|R_{\max}$ the optimal makespan doesn't exceed $\frac{4}{3}\overline{\overline{R}}$, and the bound is tight.



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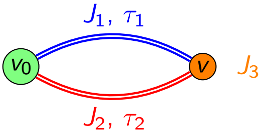
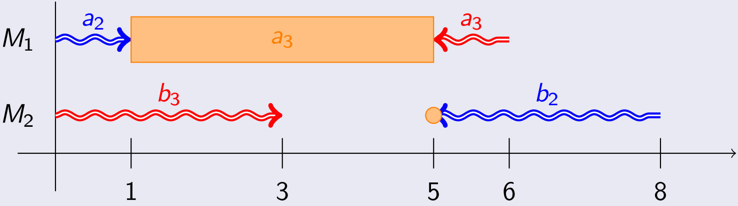


The critical instance

$J_1=(0,2),\ \tau_1=1,\ J_2=(0,2),\ \tau_2=1,\ J_3=(4,0).$



Further steps

Growing the lower bound

- Search for the new critical instances
- K_2, K_3, *tree*, C_4 have been considered.
- Question: which relatively small structure of the transportation network looks promising? (K_4? $cycle + chord$?)

Lowering the upper bound

- New approximation algorithm for $RO2||R_{\max}$ with given optimal TSP solution
- Probably possible for $RO2|G = cycle|R_{\max}$?
- Another (not so strong) way of reducing the instance.

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- New approximation algorithm for $RO2||R_{max}$ with given optimal TSP solution
- Probably possible for $RO2|G = cycle|R_{max}$?
- Another (not so strong) way of reducing the instance.

Thank you for the attention!