Julius-Maximilians-
UNIVERSITÄT
WÜRZBURG

Optimal Proof Systems for Complex Sets are Hard to Find

Fabian Egidy, Christian Glaßer

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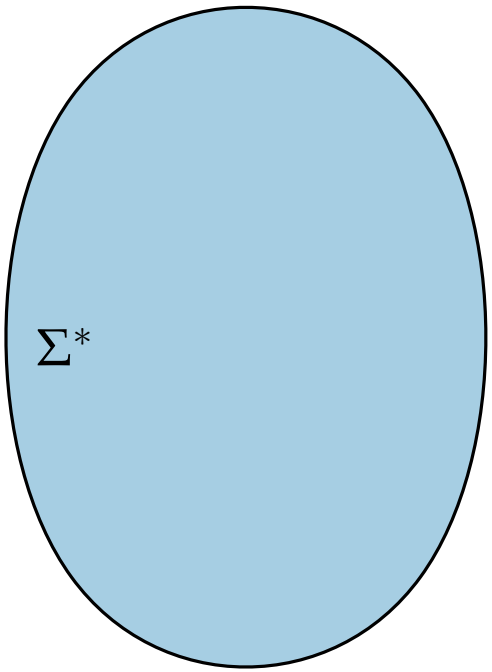

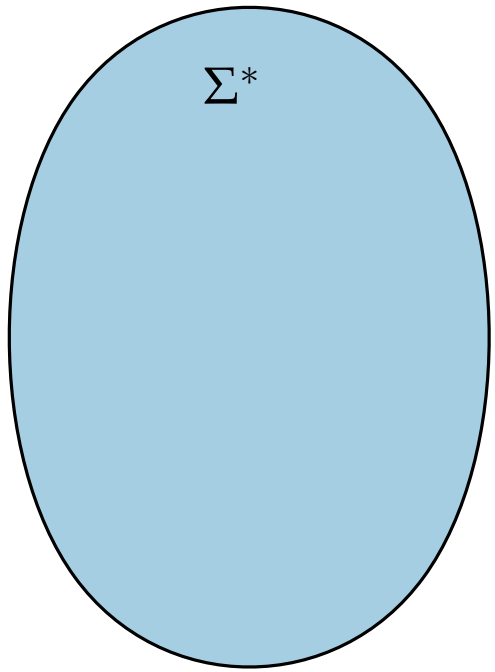

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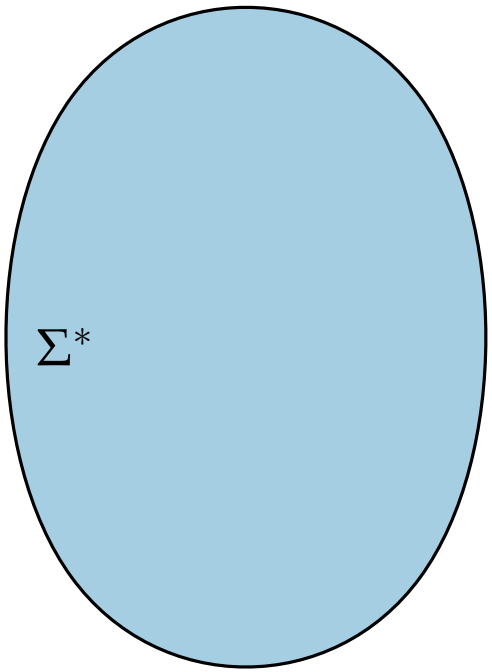

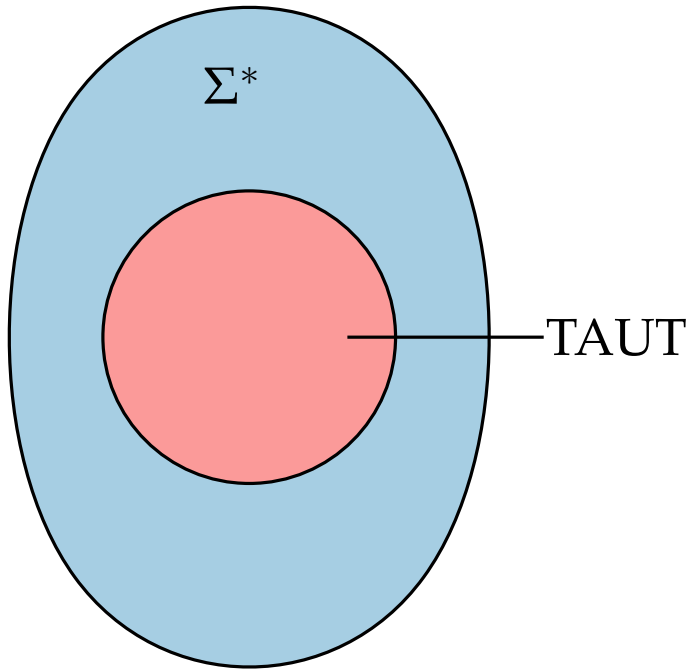

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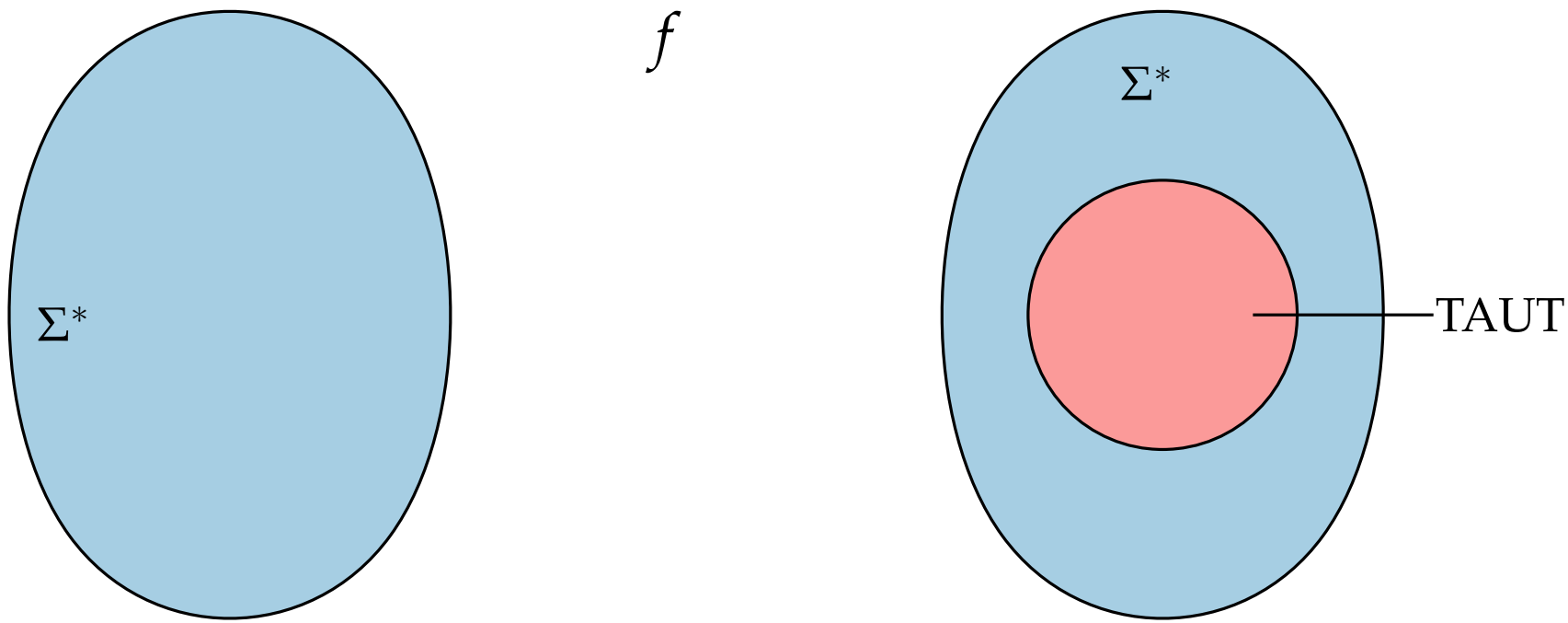

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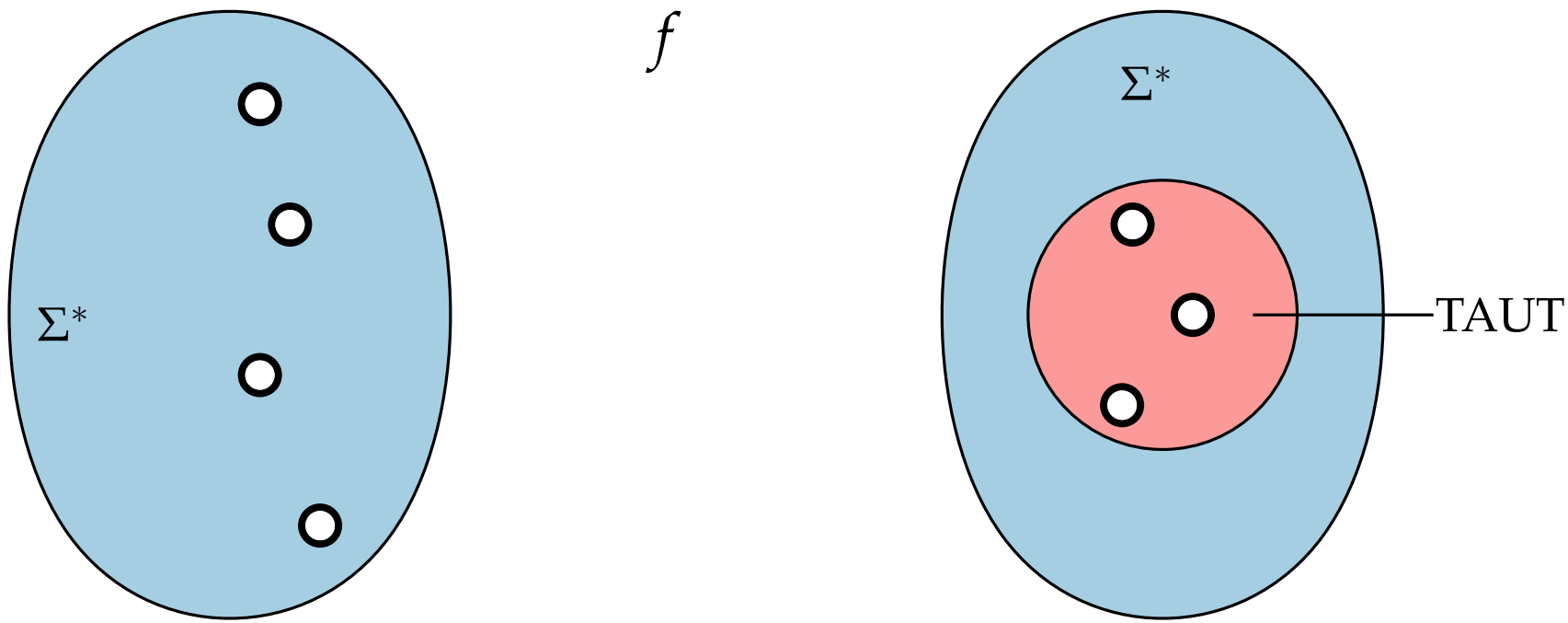

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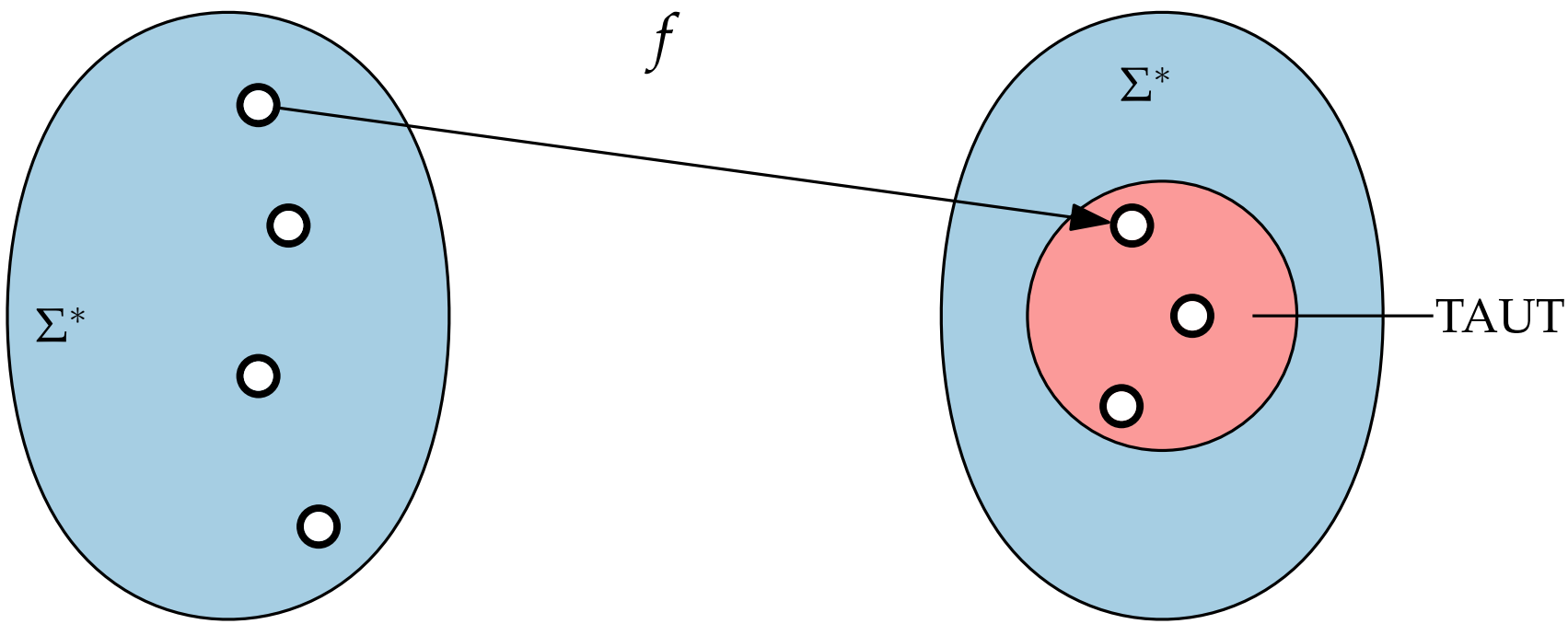

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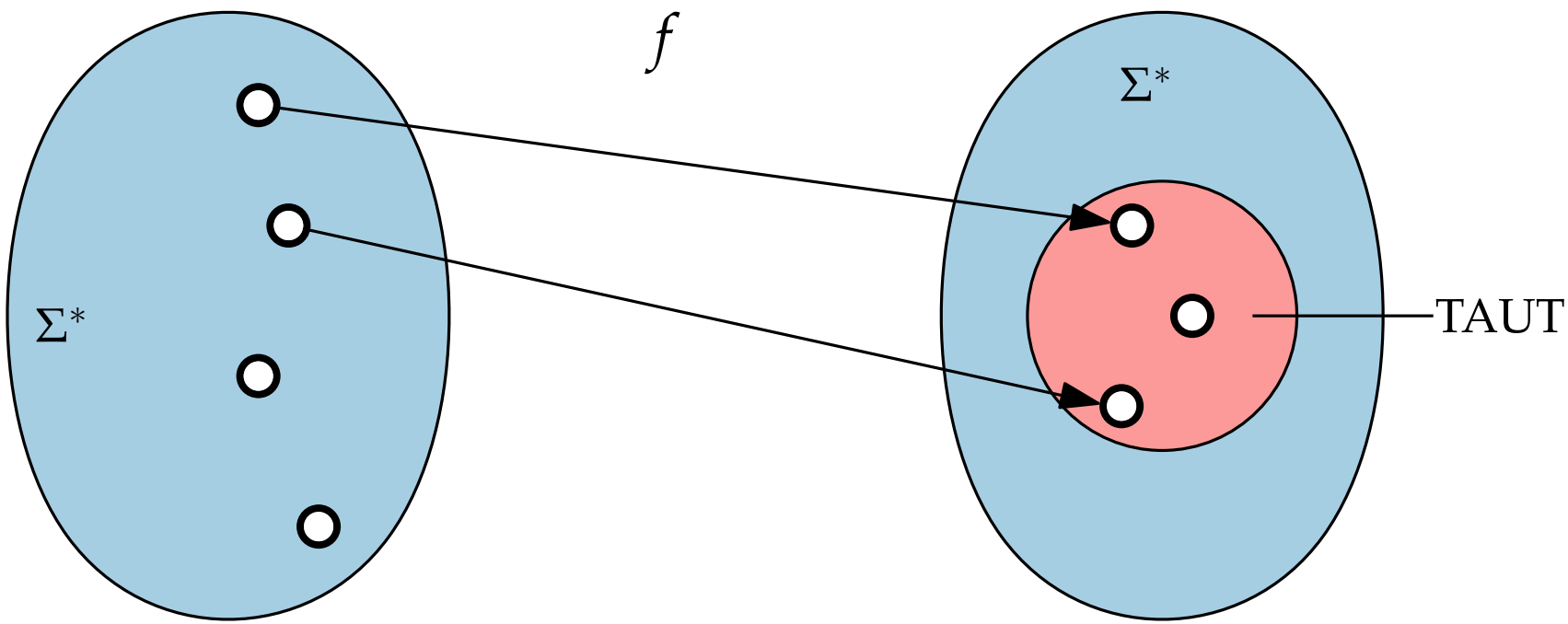

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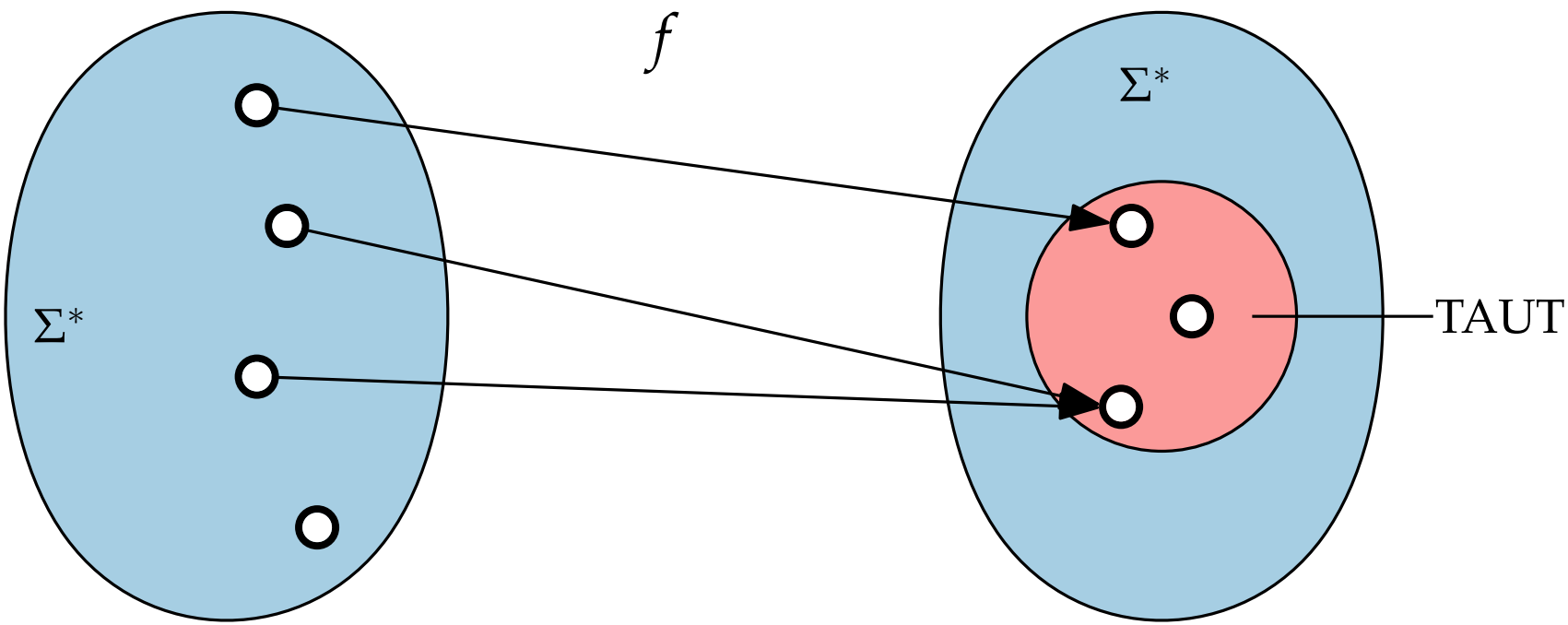

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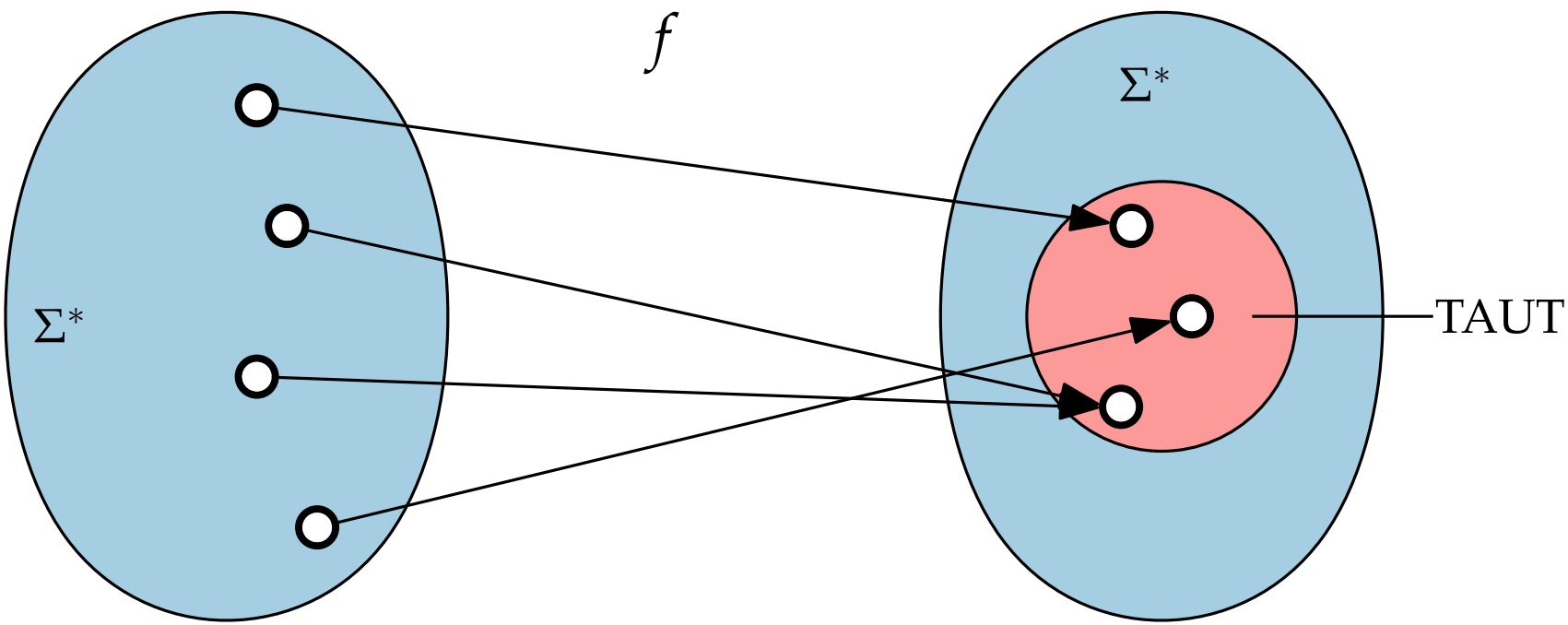

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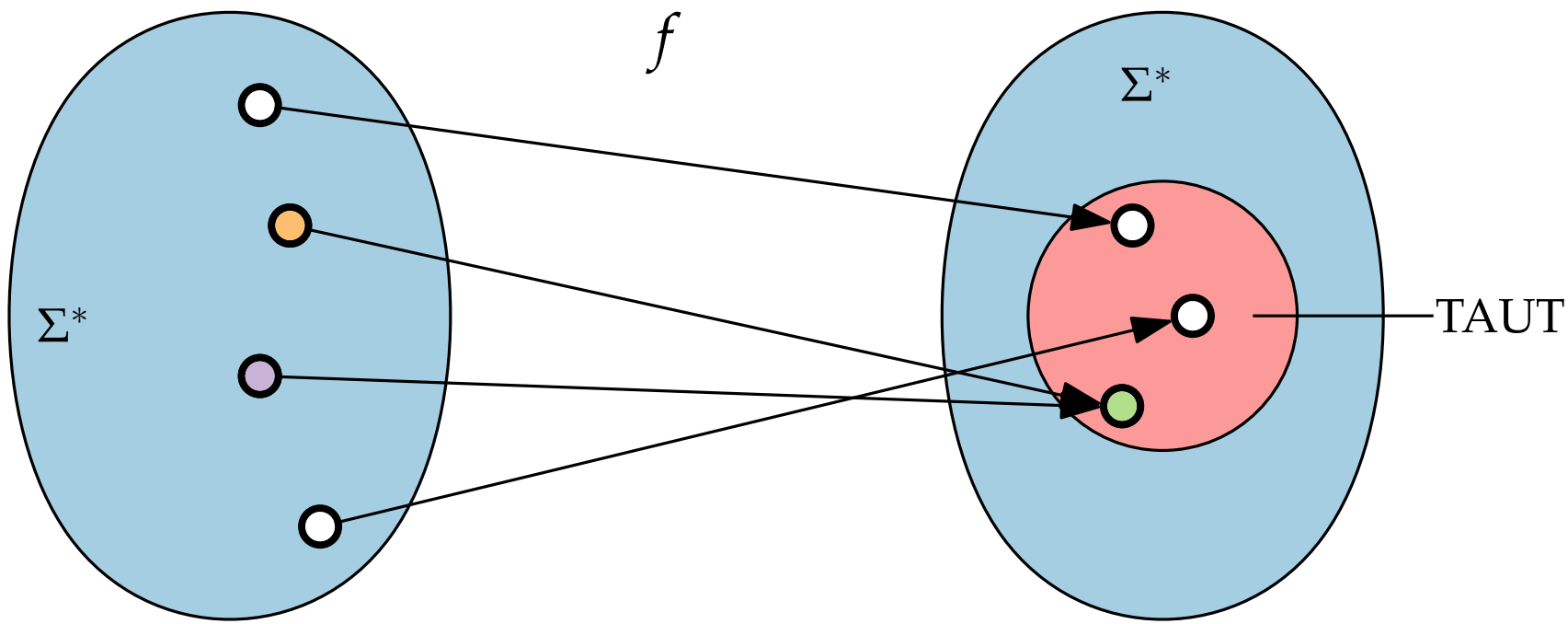

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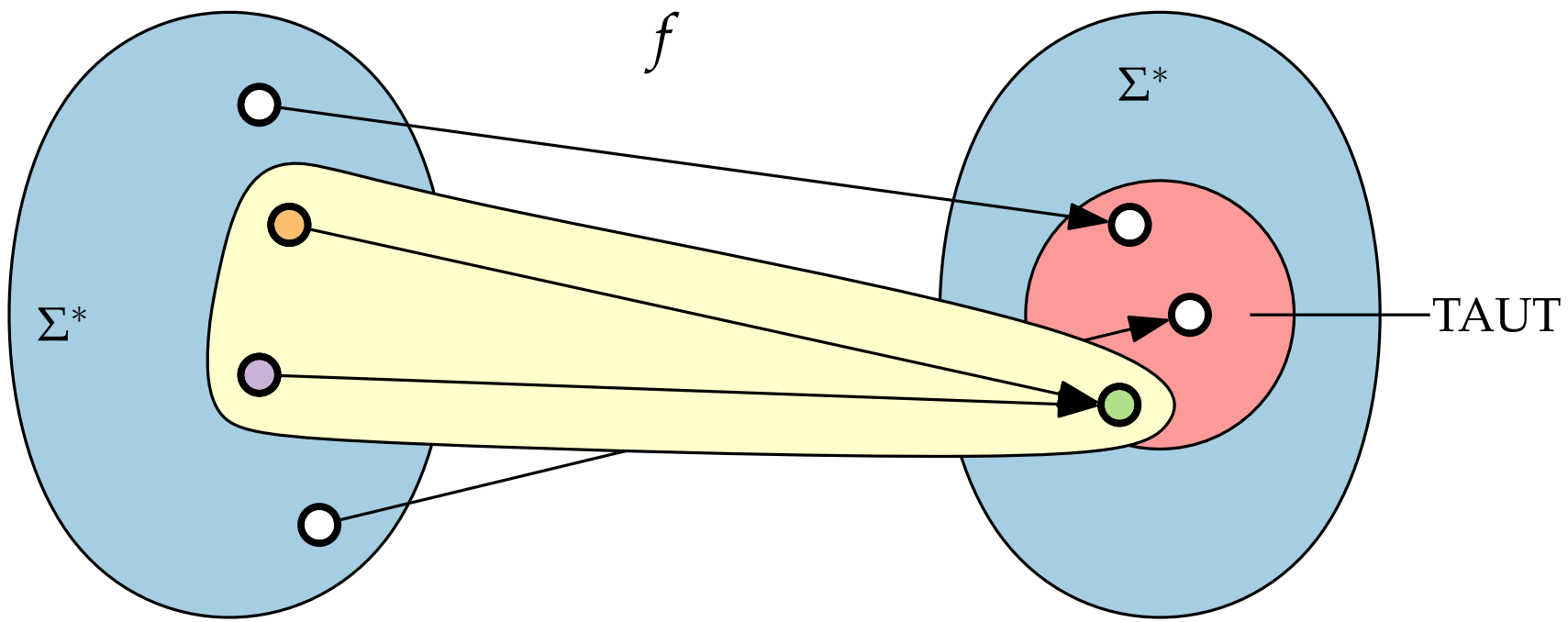

● and ● are proofs of the tautology ●

Simulation and Optimality

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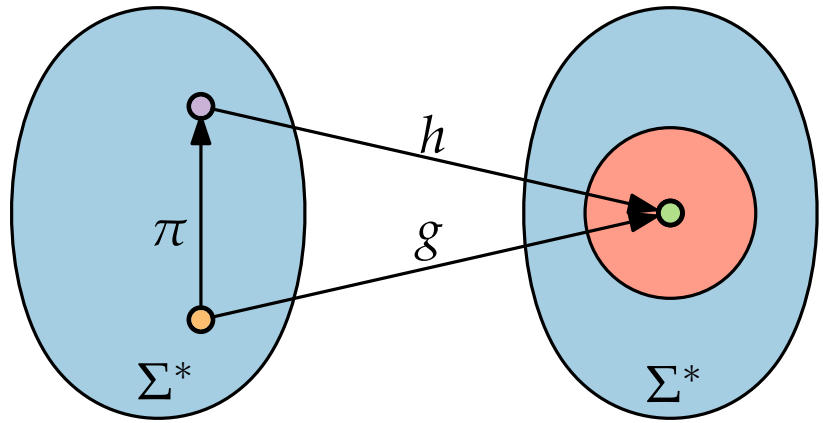

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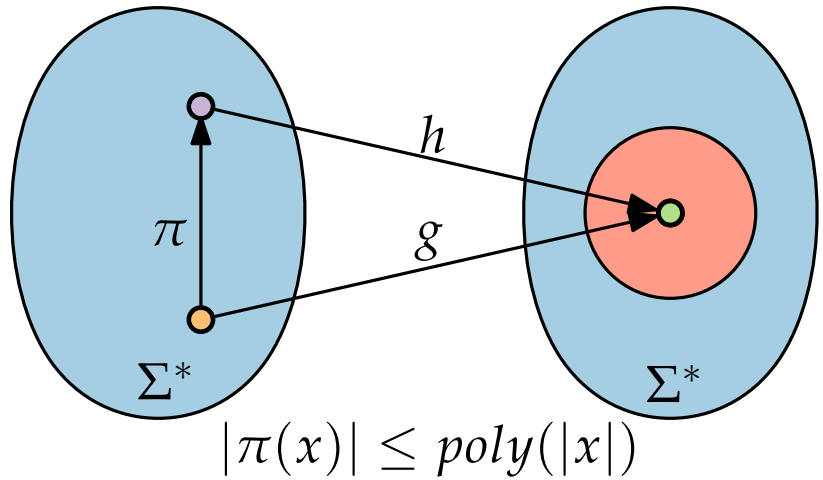

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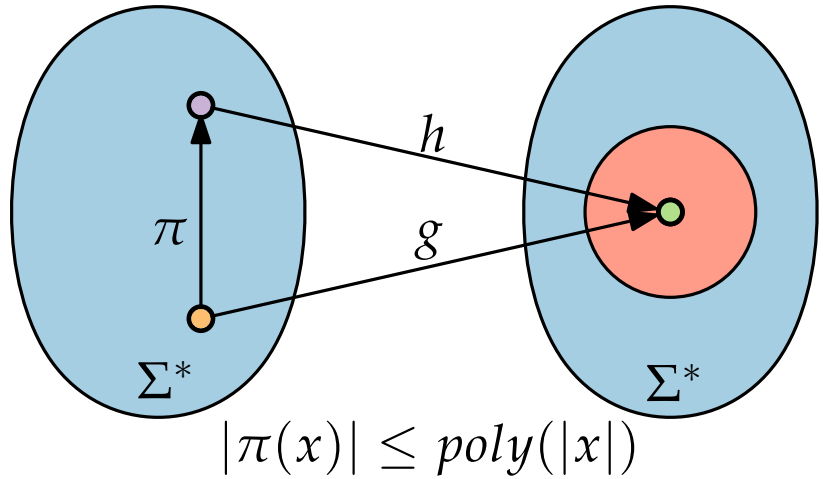

Def. A proof system for L that simulates all proof systems for L is an optimal proof system for L.

Existence of Optimal Proof Systems

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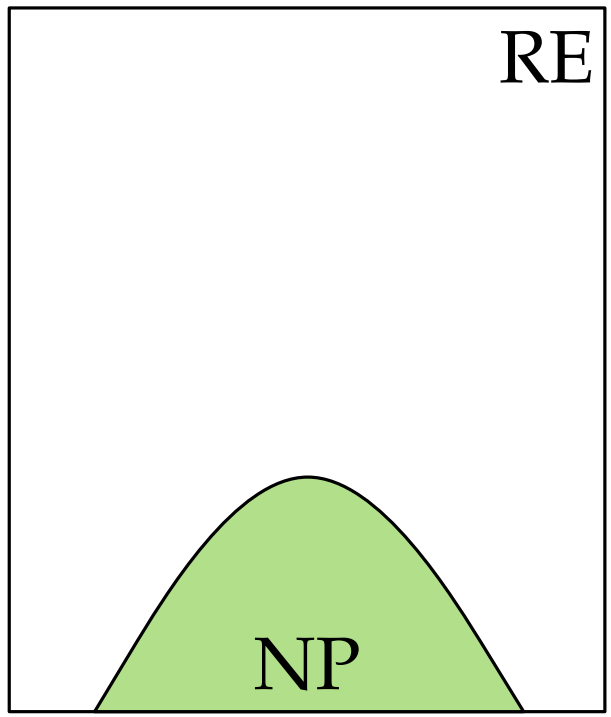

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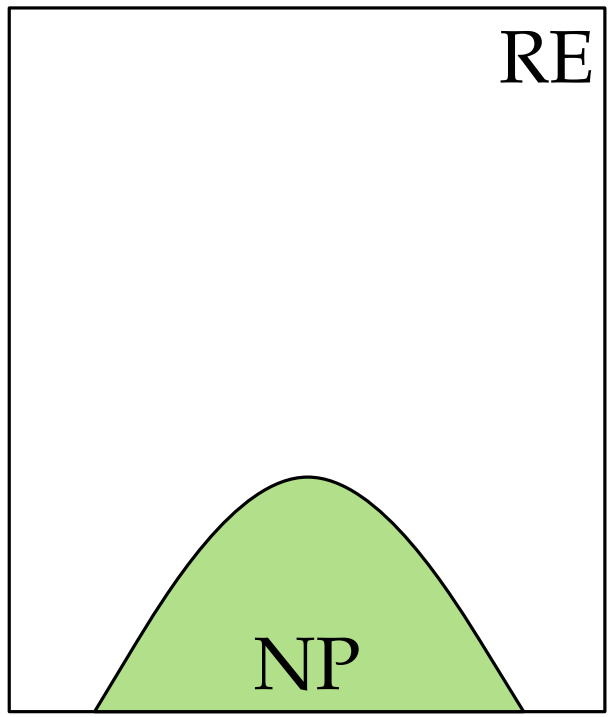

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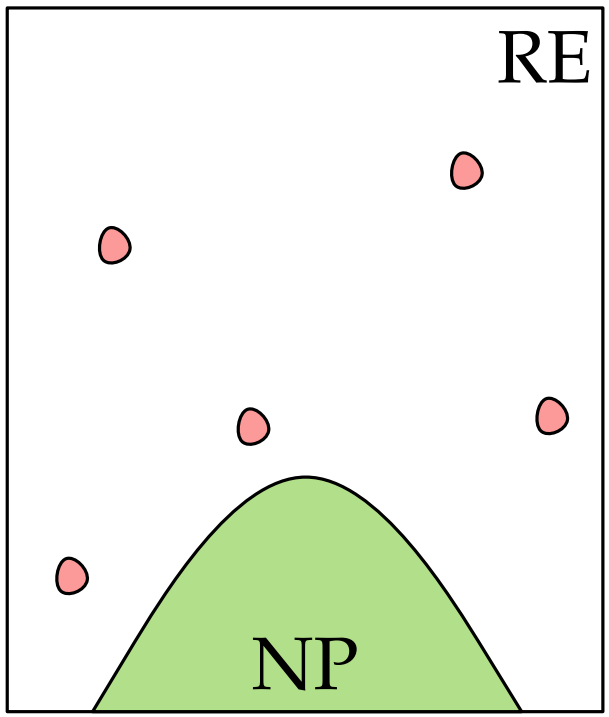

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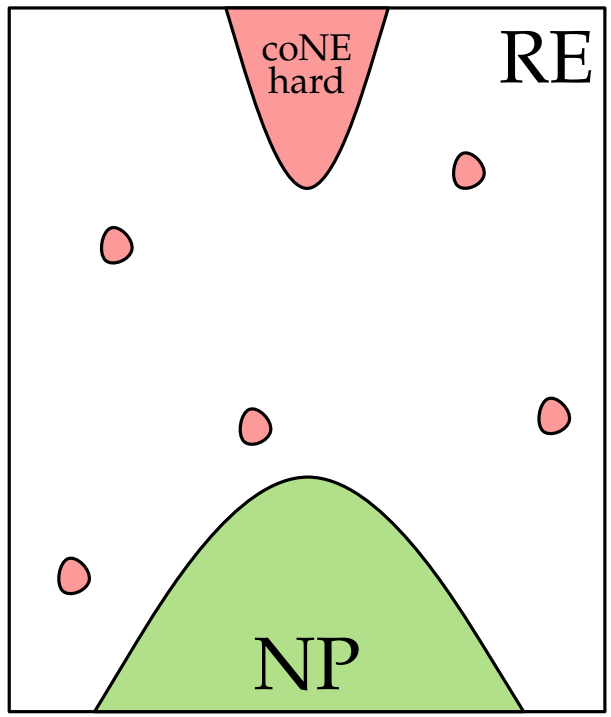

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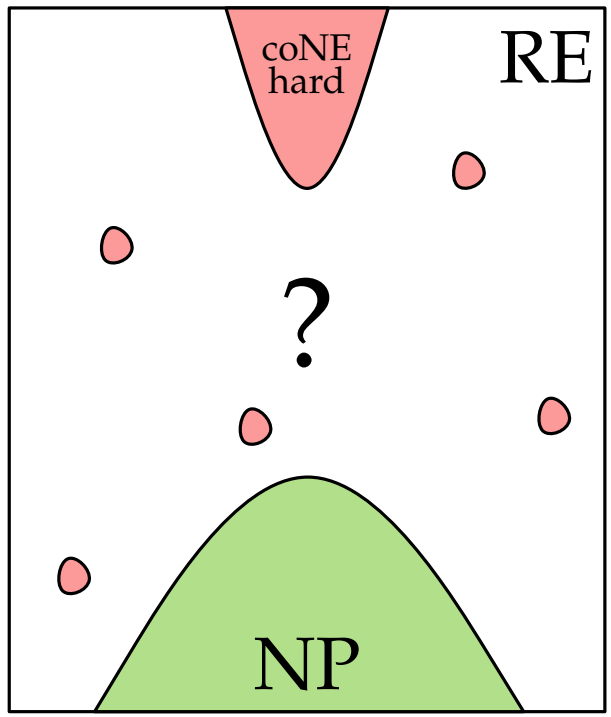

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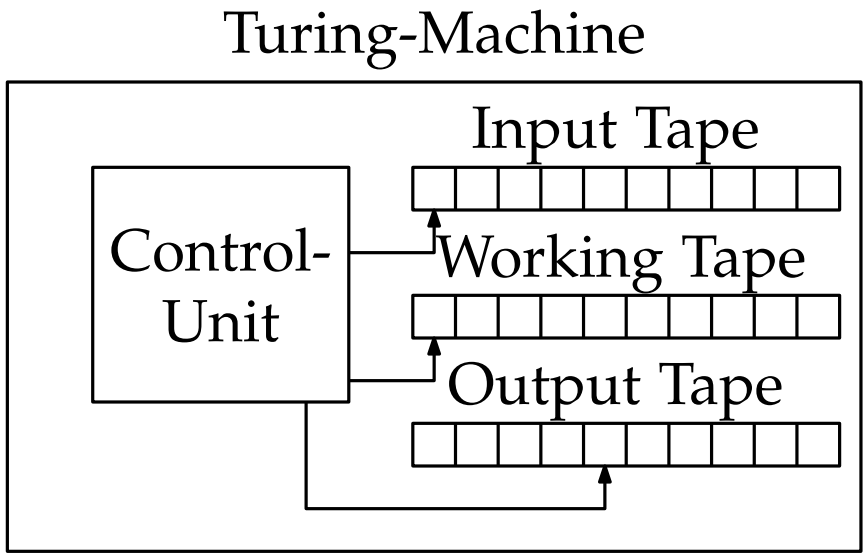
Turing-Machine
Input Tape
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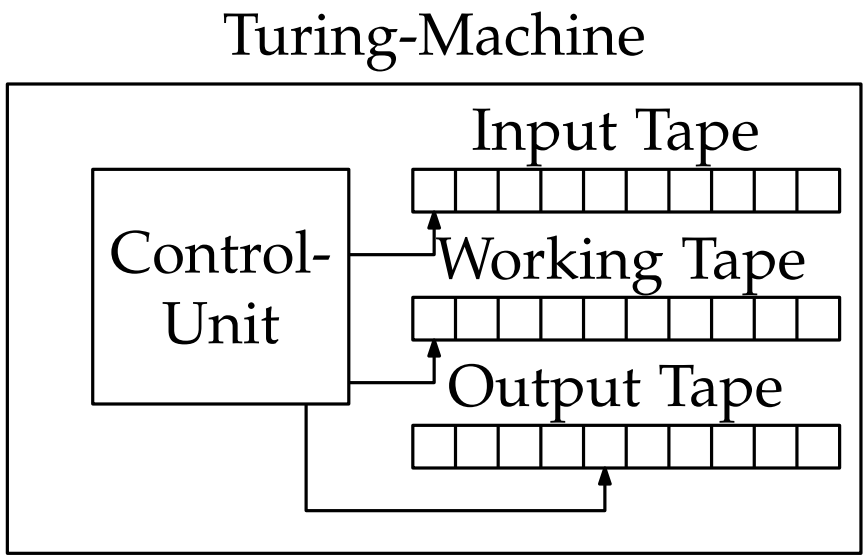
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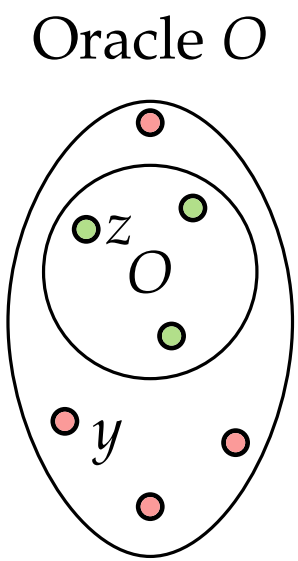
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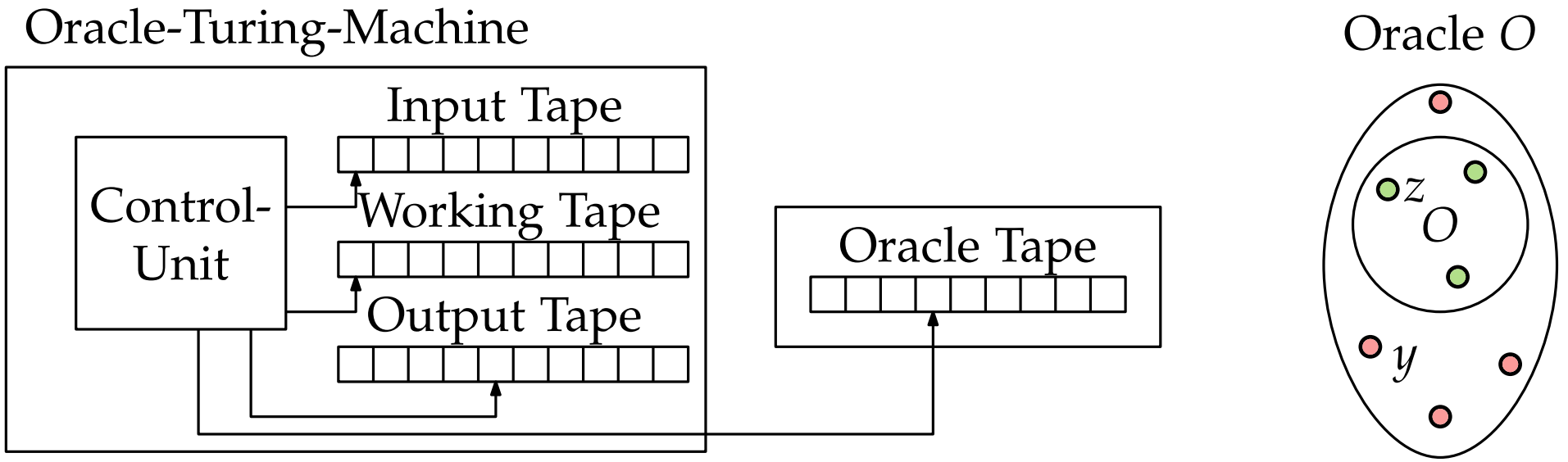
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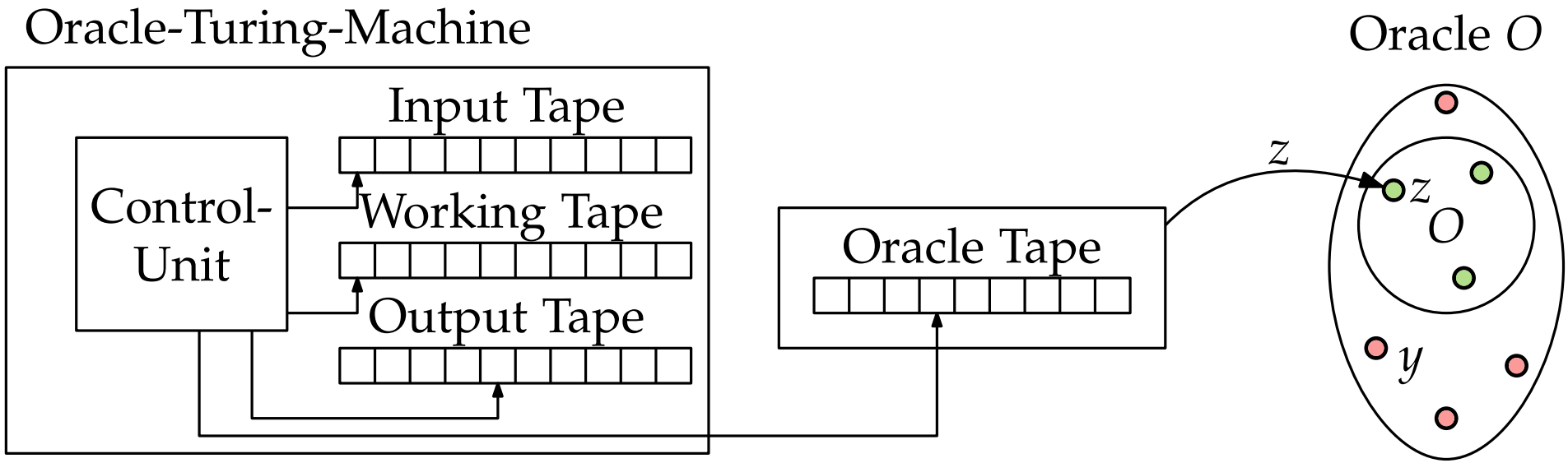
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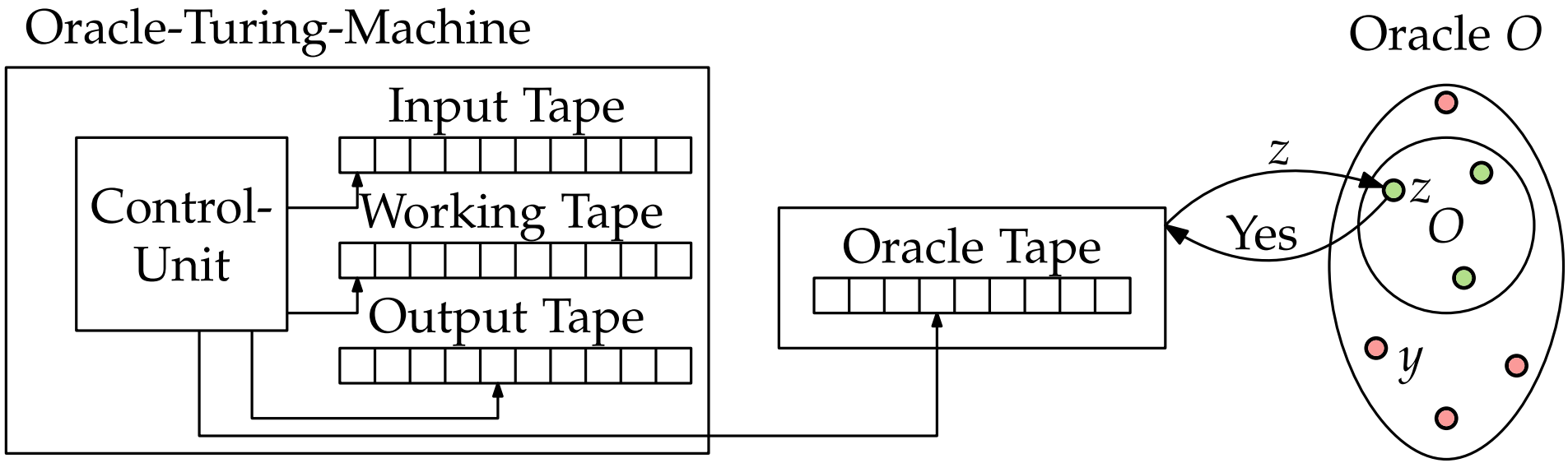
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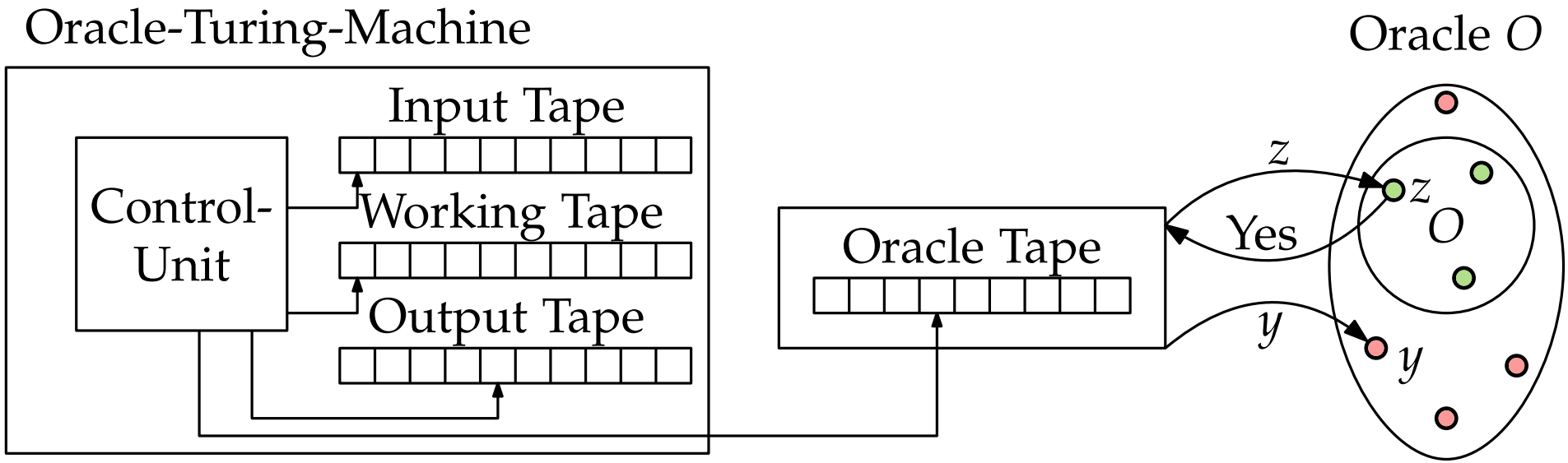
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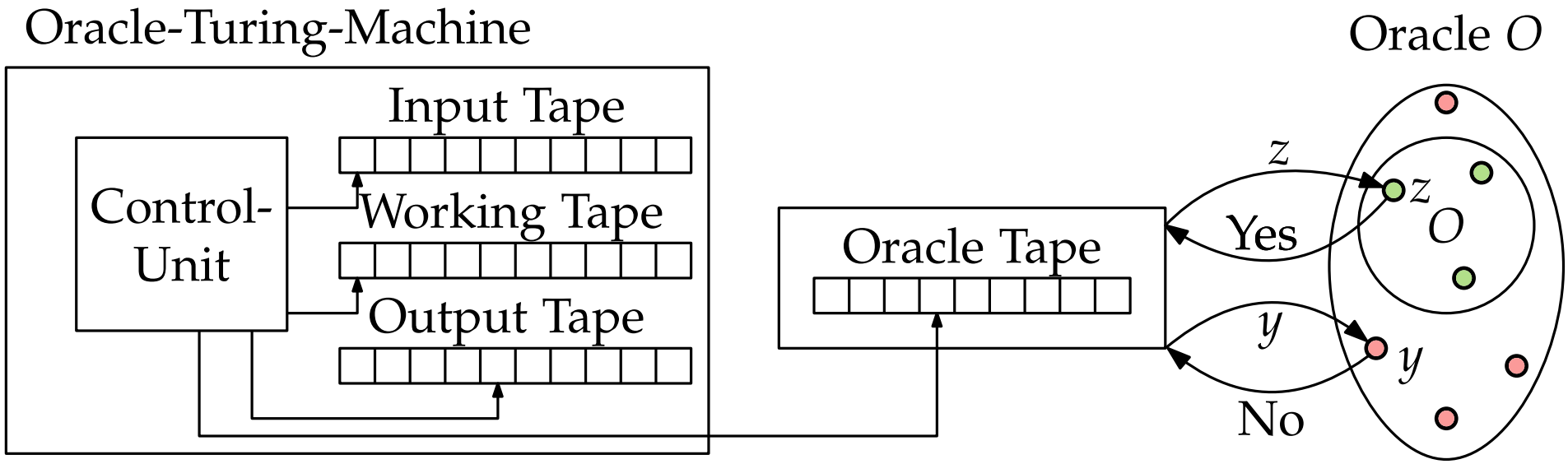
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Oracles indicate the difficulty of proving a statement

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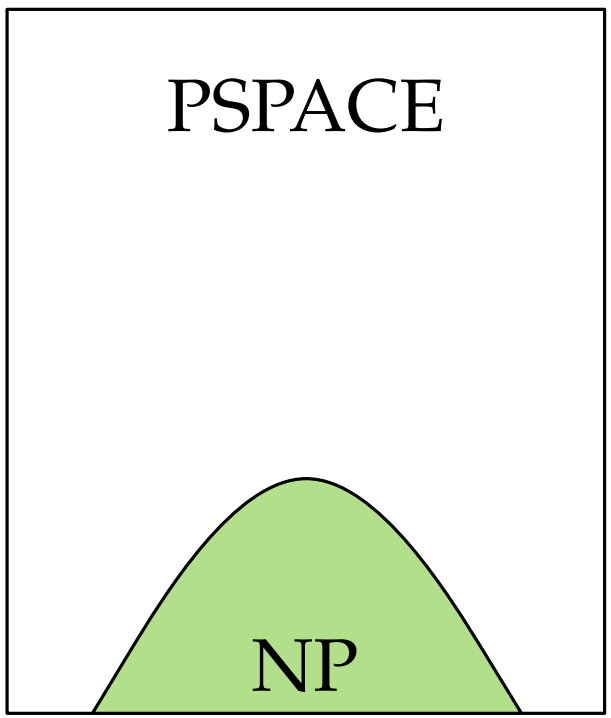

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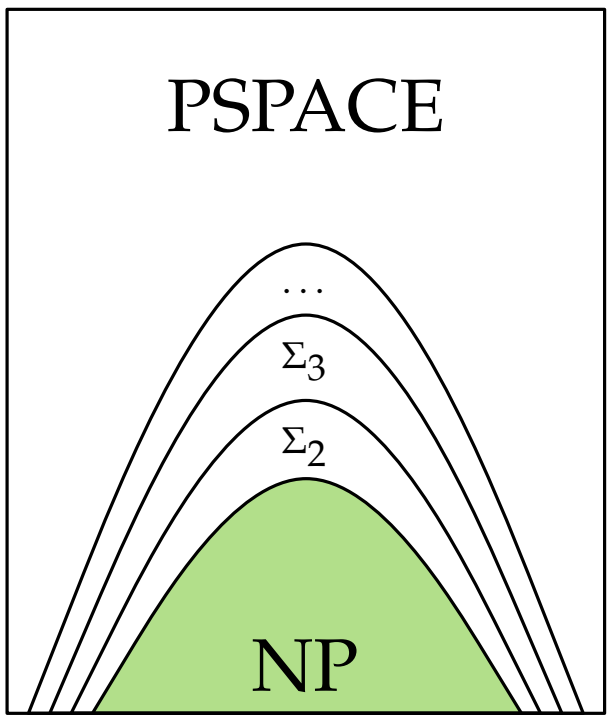

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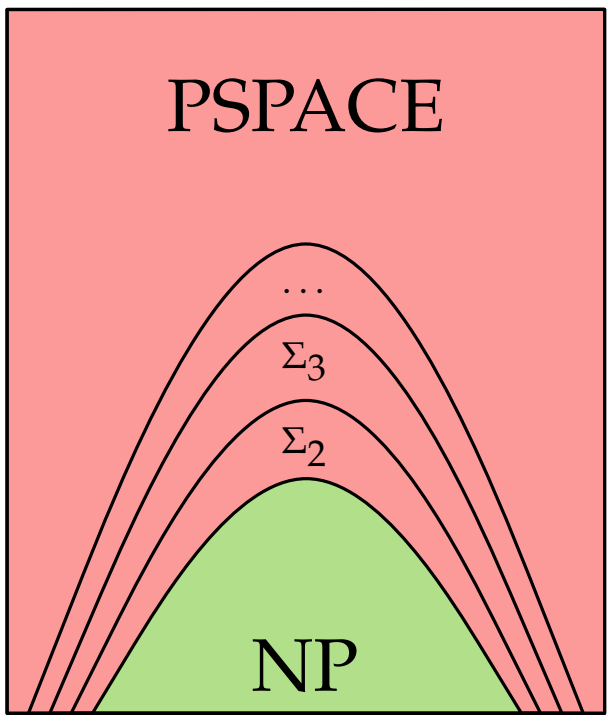

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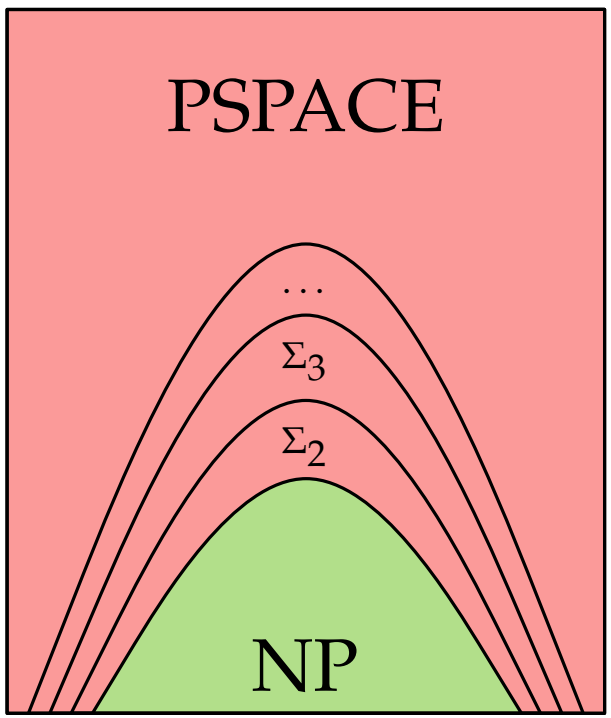

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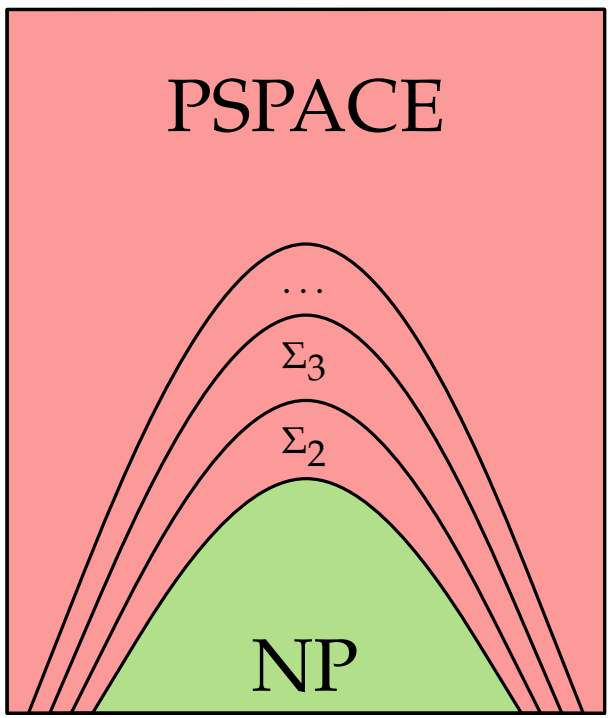

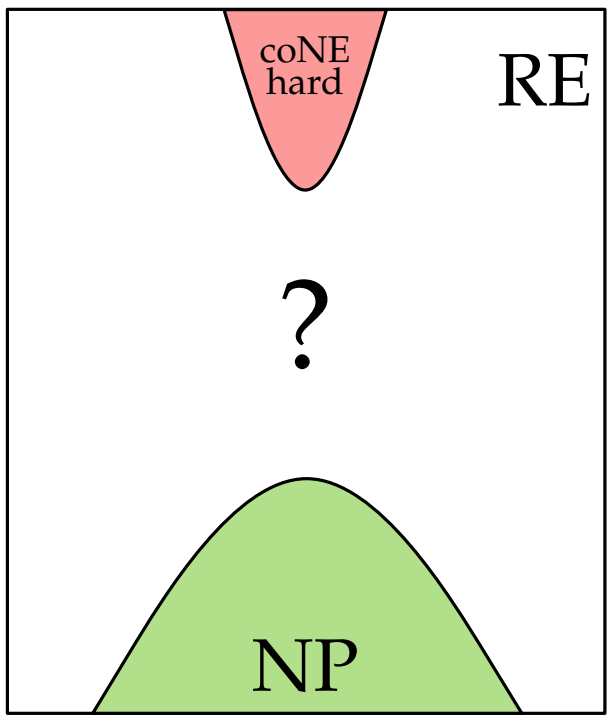

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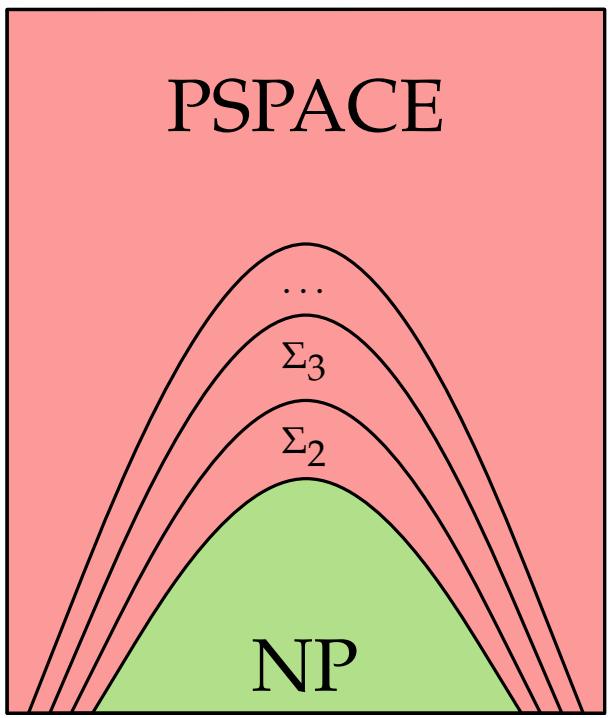

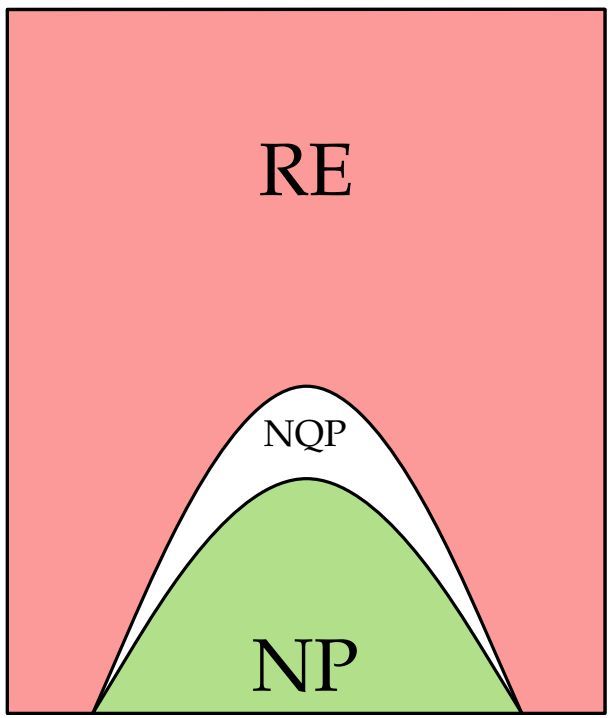

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$\Downarrow$

Feasible?

Both answers aren't known to have unlikely consequences!

Our Results

Are there sets outside NP with optimal proof systems?

Negative Answer

- implies NP $\neq$ coNP
- relativization barrier for plausible assumptions

$\Downarrow$

Provably difficult!

Positive Answer

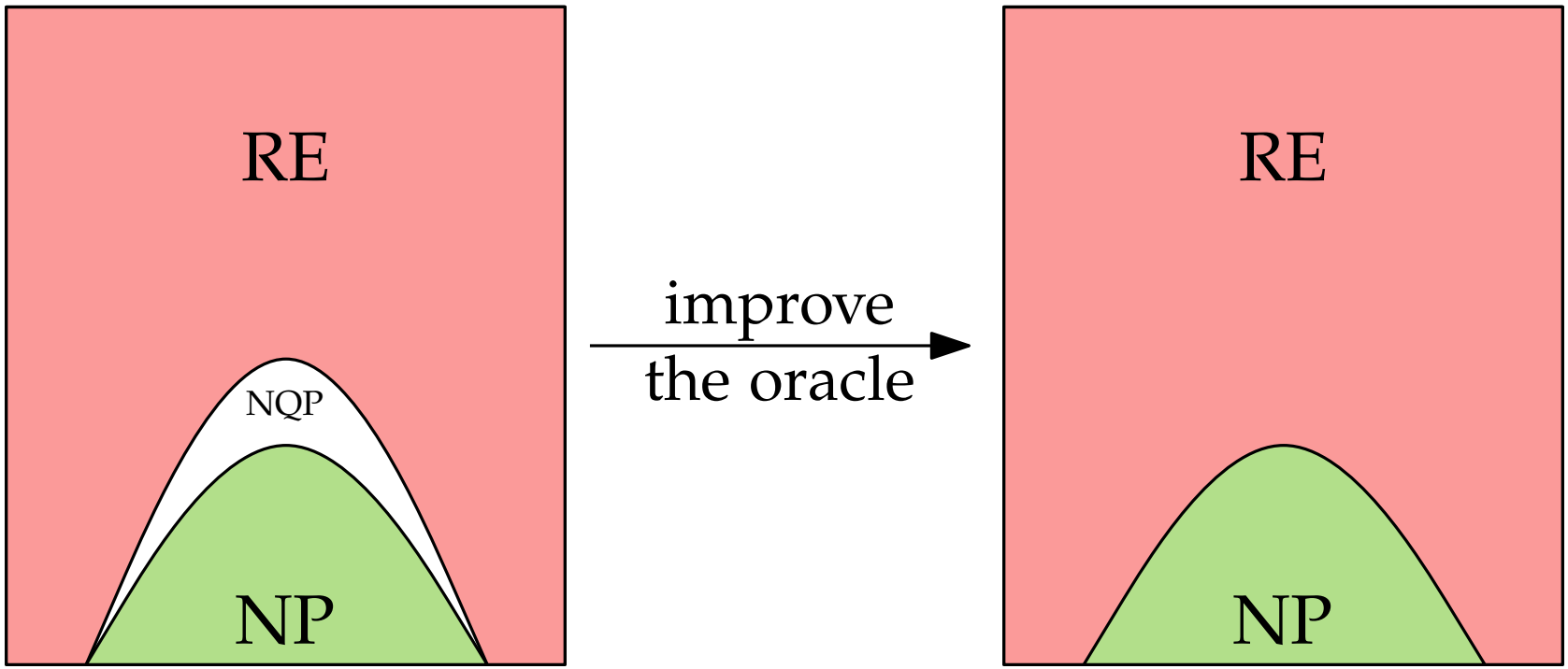
- relativization barrier for "almost all" sets

$\Downarrow$

~~Feasible?~~

Explains why no such set is known

Both answers aren't known to have unlikely consequences!

From NQP to NP?

From NQP to NP?