The complexity of deciding primitivity

joint work with Stefan Kiefer

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Powers of a nonnegative matrix

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- Consider a matrix $A \in \mathbb{R}^{n \times n}_{\geq 0}$ with nonnegative rational coefficients.
- A classical question in combinatorial matrix theory is whether some its power is positive, that is, has only strictly positive entries.
- How does one determine it? And what does it have to do with graph theory?

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- This is the definition of the Boolean semiring $\mathbb{B} = \{0, 1\}$, and the described operation is a homomorphism (that is, agrees with multiplication). That is, $\chi(A^n) = \chi(A)^n$.
- Denote this homomorphism by $\chi\colon \mathbb{R}_{\geq 0}^{n\times n} \to \mathbb{B}^{n\times n}$.

The structure of finite semigroups

$$\chi(A) \qquad \chi(A)^2 \qquad \chi(A)^3 \qquad \chi(A)^4$$

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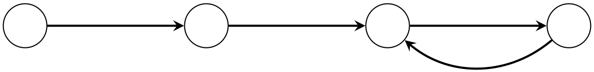

A lasso

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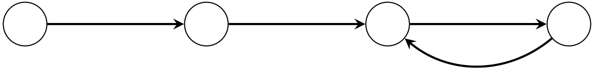

A lasso

- The length of the cycle in the lasso is called *the period* of a matrix.

From matrices over $\mathbb{B}$ to digraphs

- We can interpret a matrix $A \in \mathbb{B}^{n \times n}$ as a digraph G with n vertices:

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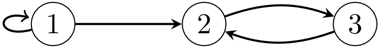

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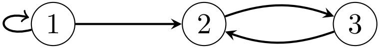

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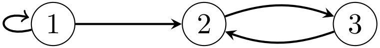

- Multiplication of such matrices corresponds to concatenating paths in G.
- In particular, A^k is the matrix of paths of length k in G.

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Theorem

A digraph is primitive if and only if the gcd of the length of all its cycles is one and it is strongly connected.

Deciding primitivity

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- Thus, so is deciding primitivity: add a self-loop to every vertex. The new digraph is primitive if and only if the original one is strongly connected.
- But what if it is promised to be strongly connected?

Main result

Theorem (Kiefer, R., 2025+)

One can decide in L if a strongly connected digraph is primitive.

Directed vs undirected reachability

- Reachability in directed graphs is NL-complete.

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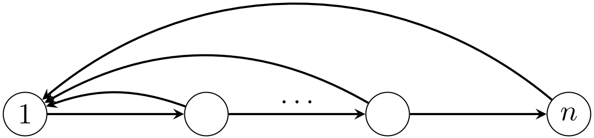
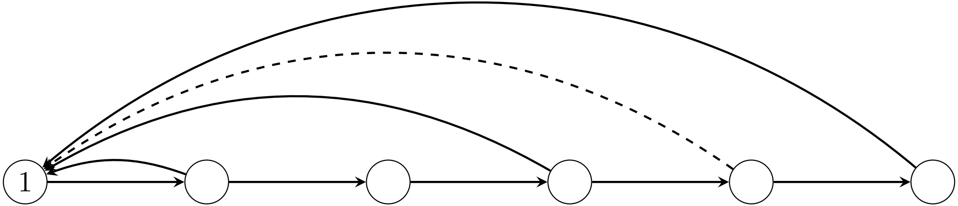
- Reachability in undirected graphs can be solved in randomised (deterministic!) logspace.



The probability of going from 1 to 4 decreases exponentially

Directed vs undirected reachability

- Similarly, a random walk forward in a digraph cannot solve computing the period in randomised logspace.



The probability of taking the dashed edge when we start from 1 decreases exponentially

Directed vs undirected reachability

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Directed vs undirected reachability

- But if we are strongly connected, we can still check compatibility with a partition in p subsets by a random walk!
- It's just that this random walk has to be symmetric.

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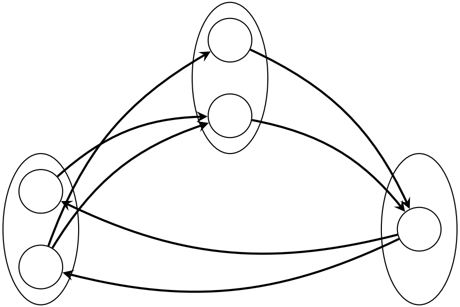

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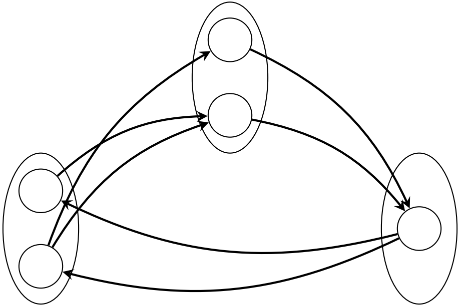

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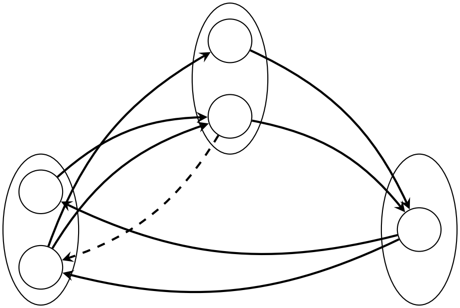
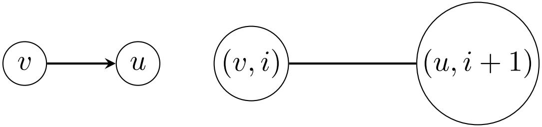


- For detecting edges that are incompatible with the partition, we can go in both directions.

The period of strongly connected digraphs

- To detect incompatibilities with a partition into p sets, it is enough to traverse the following undirected graph:



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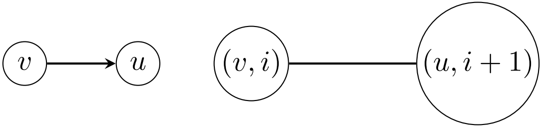

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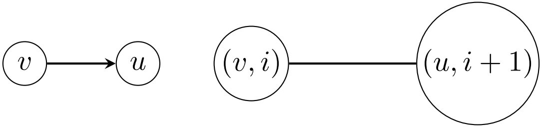

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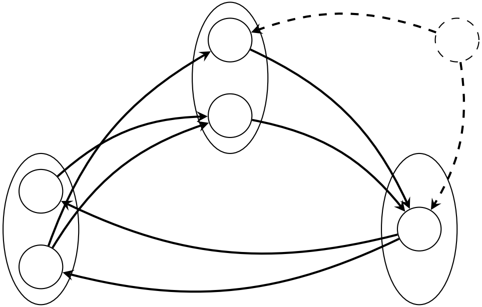
Theorem (Reingold, 2005)

Reachability in undirected graphs is in L.

The period of (non-) strongly connected digraphs

- Does not work if G is not strongly connected:



Hardness

Theorem (Kiefer, R., 2025+)

Deciding if a strongly connected digraph has period one is L-complete, even for digraphs of period at most two.

Thank you!